Contents

I. Defining the Leptonic Standard Model 1

II. $\mathcal{L}_{\text{kin}}$ and the gauge symmetry 3

III. $\mathcal{L}_{m_\psi} = 0$ 4

IV. $\mathcal{L}_\phi$ and spontaneous symmetry breaking 4

V. Back to $\mathcal{L}_{\text{kin}}(\phi)$: The vector boson spectrum 5

VI. $\mathcal{L}_{\text{Yuk}}$ and the fermion spectrum 6

VII. The Higgs boson 8

VIII. QED: Electromagnetic Interactions 8

IX. Charged weak Interactions 10

X. Neutral weak interactions 11

XI. Gauge boson self interactions 13

XII. Low energy tests 13
   A. NC in neutrino scattering 13
   B. Neutrino–Electron Scattering 14
   C. Pion Decay 14

XIII. The interaction basis and the mass basis 15

XIV. Accidental symmetries 17

References 17

I. DEFINING THE LEPTONIC STANDARD MODEL

We now have at our disposal the tools that are required in order to present the Standard Model (SM). We start with the lepton sector. Later we introduce the complete model,
including quarks. For this section the discussion in Burgess and Moore [1] (section 2) is similar to ours.

In order to define the SM, we need to provide the following three ingredients:

(i) The symmetry;

(ii) The transformation properties of the fermions and scalars;

(iii) The pattern of spontaneous symmetry breaking (SSB).

The “leptonic SM” (LSM) is defined as follows:

(i) The symmetry is a local

\[ SU(2)_L \times U(1)_Y. \]  \hspace{1cm} (1.1)

(ii) There are three fermion generations, each consisting of two different lepton representations:

\[ L_{Li}(2)_{-1/2}, \quad E_{Ri}(1)_{-1}, \quad i = 1, 2, 3. \]  \hspace{1cm} (1.2)

There is a single scalar multiplet:

\[ \phi(2)_{+1/2}. \]  \hspace{1cm} (1.3)

(iii) The pattern of spontaneous symmetry breaking is as follows:

\[ SU(2)_L \times U(1)_Y \rightarrow U(1)_{EM}, \]  \hspace{1cm} (1.4)

where \( Q_{EM} = T_3 + Y. \)

We use the notation \((N)_Y\) such that \(N\) is the irrep under \(SU(2)_L\) and \(Y\) is the hypercharge (the charge under \(U(1)_Y\)). What we mean by Eq. (1.2) is that there are nine Weyl fermion degrees of freedom that are grouped into three copies (“generations”) of the same gauge representations. The three fermionic degrees of freedom in each generation form an \(SU(2)\)-doublet (of hypercharge \(-1/2\)) and a singlet (of hypercharge \(-1\)).

The most general renormalizable Lagrangian with scalar and fermion fields can be decomposed into

\[ \mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{m_\psi} + \mathcal{L}_\phi + \mathcal{L}_{\text{Yuk}}. \]  \hspace{1cm} (1.5)

Here \(\mathcal{L}_{\text{kin}}\) describes the free propagation in spacetime, as well as the gauge interactions, \(\mathcal{L}_{m_\psi}\) gives the fermion mass terms, \(\mathcal{L}_\phi\) gives the scalar potential, and \(\mathcal{L}_{\text{Yuk}}\) describes the Yukawa interactions. It is now our task to find the specific form of the Lagrangian made of the \(L_{Li}\), \(E_{Ri}\) [Eq. (1.2)] and \(\phi\) [Eq. (1.3)] fields, subject to the gauge symmetry (1.1) and leading to the SSB of Eq. (1.4).
II. \( \mathcal{L}_{\text{kin}} \) AND THE GAUGE SYMMETRY

The gauge group is given in Eq. (1.1). It has four generators: three \( T_a \)’s that form the \( SU(2) \) algebra and a single \( Y \) that generates the \( U(1) \) algebra:

\[
[T_a, T_b] = i\epsilon_{abc}T_c, \quad [T_a, Y] = 0.
\]

Thus there are two independent coupling constants in \( \mathcal{L}_{\text{kin}} \): there is a single \( g \) for all the \( SU(2) \) couplings and a different one, \( g' \), for the \( U(1) \) coupling. The \( SU(2) \) couplings must all be the same because they mix with one another under global \( SU(2) \) rotations. But the \( U(1) \) coupling can be different because the generator \( Y \) never appears as a commutator of \( SU(2) \) generators.

The local symmetry requires four gauge bosons, three in the adjoint representation of the \( SU(2) \) and one related to the \( U(1) \) symmetry:

\[
W^\mu_a(3)_0, \quad B^\mu(1)_0.
\]

The corresponding field strengths are given by

\[
W^{\mu\nu}_a = \partial^\mu W^\nu_a - \partial^\nu W^\mu_a - g\epsilon_{abc}W^\mu_bW^\nu_c, \\
B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu.
\]

The covariant derivative is

\[
D^\mu = \partial^\mu + igW^\mu_aT^a + ig'B^\mu.
\]

\( \mathcal{L}_{\text{kin}} \) includes the kinetic terms of all the fields:

\[
\mathcal{L}_{\text{kin}} = -\frac{1}{4}W^{\mu\nu}_aW_{a\mu\nu} - \frac{1}{4}B^{\mu\nu}B_{\mu\nu} - iL_L^\dagger\psi L_L - iE_R^\dagger\psi E_R - (D^\mu\phi)^\dagger(D_\mu\phi).
\]

For the \( SU(2)_L \) doublets \( T_a = \frac{1}{2}\sigma_a \) (\( \sigma_a \) are the Pauli matrices), while for the \( SU(2)_L \) singlets, \( T_a = 0 \). Explicitly,

\[
D^\mu L_L = \left( \partial^\mu + \frac{i}{2}gW^\mu_a\sigma_a - \frac{ig'}{2}B^\mu \right) L_L, \\
D^\mu E_R = \left( \partial^\mu - ig'B^\mu \right) E_R, \\
D^\mu \phi = \left( \partial^\mu + \frac{i}{2}gW^\mu_a\sigma_a + \frac{ig'}{2}B^\mu \right) \phi.
\]

For \( SU(2)_L \) triplets, \( (T_a)_{bc} = \epsilon_{abc} \), which has already been used in writing (2.3).

We remind the reader that in \( \mathcal{L}_{\text{LSM}} \) there are no mass terms for the gauge bosons, as that would violate the gauge symmetry.

Where is QED in all of this? We defined \( Q \), the generator of \( U(1)_{\text{EM}} \), as follows:

\[
Q = T_3 + Y.
\]
Let us write explicitly the two components of $SU(2)_L$ doublets:

$$L_{L1} = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}. \quad (2.8)$$

Then

$$Q_{\nu_{eL}} = 0, \quad Q_{e_L} = -e, \quad Q_{\phi^+} = \phi^+, \quad Q_{\phi^0} = 0. \quad (2.9)$$

For now, $\nu_{eL}$ ($\phi^+$) and $e_L$ ($\phi^0$) stand for, respectively, the $T_3 = +1/2$ and $T_3 = -1/2$ components of the lepton (scalar) doublet. If $SU(2)_L \times U(1)_Y$ were an exact symmetry of Nature, there would be no way of distinguishing particles of different electric charges in the same $SU(2)_L$ multiplet. We make this choice as it will give us the correct QED after SSB as we see next.

III. $\mathcal{L}_{m_{\psi}} = 0$

There are no mass terms the fermions in the LSM. We cannot write Dirac mass terms for the fermions because they are assigned to chiral representations of the gauge symmetry. We cannot write Majorana mass terms for the fermions because they all have $Y \neq 0$.

IV. $\mathcal{L}_{\phi}$ AND SPONTANEOUS SYMMETRY BREAKING

The Higgs potential, which leads to the spontaneous symmetry breaking, is given by

$$\mathcal{L}_\phi = -\mu^2 \phi^\dagger \phi - \lambda \left( \phi^\dagger \phi \right)^2. \quad (4.1)$$

The quartic coupling $\lambda$ is dimensionless and real, and has to be positive for the potential to be bounded from below. The quadratic coupling $\mu^2$ has mass dimension 2 and is real. It can a-priori have either sign, but if the gauge symmetry is to be spontaneously broken, Eq. (1.4), then we must take $\mu^2 < 0$. Defining

$$v^2 = -\frac{\mu^2}{\lambda}, \quad (4.2)$$

we can rewrite Eq. (4.1) as follows (up to a constant term):

$$\mathcal{L}_\phi = -\lambda \left( \phi^\dagger \phi - \frac{v^2}{2} \right)^2. \quad (4.3)$$

The scalar potential (4.3) implies that the scalar field acquires a VEV, $\langle \phi \rangle = \frac{v}{\sqrt{2}}$. This VEV breaks the $SU(2) \times U(1)$ symmetry down to a $U(1)$ subgroup. We choose the unbroken subgroup to be $U(1)_{EM}$, generated by $Q$ of Eq. (2.7).
Let us denote the four real components of the scalar doublet as follows:

$$\phi(x) = \exp \left[ i \frac{\sigma_i}{2} \theta^i(x) \right] \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}. \tag{4.4}$$

The local $SU(2)_L$ symmetry of the Lagrangian allows one to rotate away any dependence on the three $\theta^i$. They represent the three would-be Goldstone bosons that are eaten by the three gauge bosons that require masses as a result of the SSB. The remaining degree of freedom, $H(x)$, represents a real scalar degree of freedom, the Higgs boson.

Note that we had to give the VEV to the $T_3 = -1/2$ component of $\phi$, because this is the electromagnetically neutral component ($Q = T_3 + Y = 0$), and we want $U(1)_{\text{EM}}$ to remain unbroken.

The main two points of this section are thus the following:

1. We have a mechanism to spontaneously break $SU(2)_L \times U(1)_Y \rightarrow U(1)_{\text{EM}}$.

2. The model predicts the existence of a single, electromagnetically neutral, real scalar field, the Higgs boson. We discuss the properties of the Higgs boson in Section VII.

V. BACK TO $\mathcal{L}_{\text{kin}}(\phi)$: THE VECTOR BOSON SPECTRUM

Since the symmetry that is related to three out of the four generators is spontaneously broken, three of the four vector bosons acquire masses, while one remains massless. To see how this happens, we write the terms in $(D_\mu \phi)^\dagger (D^\mu \phi)$ (see Eq. (2.6) for the explicit expression for $D_\mu \phi$) setting $\theta^i(x) = 0$, that is, working in the unitary (physical) gauge. The terms that are proportional to $v^2$ are given by (we omit Lorentz indices):

$$\frac{1}{8} (0 \ v) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (g W_3 + g' B \ g(W_1 - i W_2) - g(W_1 + i W_2) \ g W_3 + g' B) \begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{5.1}$$

We define

$$W^\pm = \frac{1}{\sqrt{2}} (W_1 \mp i W_2), \tag{5.2}$$

$$Z^0 = \frac{1}{\sqrt{g^2 + g'^2}} (g W_3 - g' B),$$

$$A = \frac{1}{\sqrt{g^2 + g'^2}} (g W_3 + g B).$$

Note that the $W^\pm$ are charged under electromagnetism (hence the superscripts $\pm$), while $A$ and $Z^0$ are not. In terms of the vector boson fields of Eq. (5.2), we rewrite the mass terms of Eq. (5.1) as follows:

$$\frac{1}{4} g^2 v^2 W^+ W^- + \frac{1}{8} (g^2 + g'^2) v^2 Z^0 Z^0. \tag{5.3}$$
We learn that the four states of Eq. (5.2) are the mass eigenstates, with masses
\[ m_2 = \frac{1}{4} g^2 v^2, \quad m_Z^2 = \frac{1}{4} (g^2 + g'^2) v^2, \quad m_A^2 = 0. \] (5.4)

Two points are worth emphasizing:

1. As anticipated, three vector boson acquire masses.
2. \( m_A^2 = 0 \) is not a prediction, it is a consistency check on our calculation.

We define
\[ \tan \theta_W \equiv \frac{g'}{g}. \] (5.5)

Then
\[ Z^\mu = \cos \theta_W W_3^\mu - \sin \theta_W B^\mu, \quad A^\mu = \sin \theta_W W_3^\mu + \cos \theta_W B^\mu. \] (5.6)

We learn that \( \theta_W \) represents a rotation angle from the “interaction” basis (where fields have well-defined transformation properties under the gauge symmetry), \( W_3 \) and \( B \), into the mass basis for the gauge bosons, \( Z \) and \( A \).

While \( \theta_W \) depends on the two gauge couplings, \( g \) and \( g' \), and can thus be extracted from various interaction rates, it further provides a relation between the vector boson masses:

\[ \rho \equiv \frac{m_W^2}{m_Z^2 \cos^2 \theta_W} = 1. \] (5.7)

This relation is testable. Note that the \( \rho = 1 \) relation is a consequence of the SSB by scalar doublets. (See your homework for other possibilities.) It thus tests this specific ingredient of the SM.

The weak gauge boson masses are (PDG 2011)
\[ m_W = 80.399 \pm 0.023 \text{ GeV}; \quad m_Z = 91.1876 \pm 0.0021 \text{ GeV}. \] (5.8)

The ratio is
\[ \frac{m_W}{m_Z} = 0.8817 \pm 0.0002 \implies \sin^2 \theta_W = 1 - (m_W/m_Z)^2 = 0.2226 \pm 0.0004. \] (5.9)

Below we describe the determination of \( \sin^2 \theta_W \) by various interaction rates. We will see that the \( \rho = 1 \) is indeed realized in Nature (within experimental errors, and up to calculable quantum corrections).

VI. \( \mathcal{L}_{\text{Yuk}} \) AND THE FERMION SPECTRUM

Next we see how the chiral fermions acquire their masses. The Yukawa part of the Lagrangian is given by
\[ \mathcal{L}_{\text{Yuk}} = Y_{ij}^L \overline{L_i} E_{Rj} \phi + \text{h.c.}, \] (6.1)
where \( i, j = 1, 2, 3 \) are flavor indices. The Yukawa matrix \( Y^e \) is a general complex \( 3 \times 3 \) matrix of dimensionless couplings. Without loss of generality, we can choose a basis where \( Y^e \) is diagonal and real:

\[
Y^e = \text{diag}(y_e, y_\mu, y_\tau).
\]  

(6.2)

The SSB allows us to tell the upper and lower components of the doublet. In the basis defined in Eq. (6.2), we denote these components as follows:

\[
\begin{pmatrix}
\nu_{eL} \\
\nu_{\mu L} \\
\nu_{\tau L}
\end{pmatrix}, \quad \begin{pmatrix}
e_L \\
\mu_L \\
\tau_L
\end{pmatrix}, \quad (6.3)
\]

where \( e, \mu, \tau \) are ordered by the size of \( y_{e,\mu,\tau} \) (from smallest to largest). Eq. (2.7) tells us that the neutrinos, \( \nu_{eL}, \nu_{\mu L} \) and \( \nu_{\tau L} \), have charge zero, while the charged leptons, \( e_L, \mu_L \) and \( \tau_L \), carry charge \(-1\). Similarly, the right handed fields, \( e_R, \mu_R \) and \( \tau_R \), carry charge \(-1\).

With \( \phi^0 \) acquiring a VEV, \( \langle \phi^0 \rangle = v/\sqrt{2} \), (6.1) has a piece that corresponds to the charged lepton masses:

\[
\begin{split}
-\frac{y_e v}{\sqrt{2}} e_R - \frac{y_\mu v}{\sqrt{2}} \mu_R \tau \tau_L + h.c.
\end{split}
\]  

(6.4)

namely

\[
\begin{align*}
    m_e &= \frac{y_e v}{\sqrt{2}}, &
    m_\mu &= \frac{y_\mu v}{\sqrt{2}}, &
    m_\tau &= \frac{y_\tau v}{\sqrt{2}}.
\end{align*}
\]  

(6.5)

The crucial point in this discussion is that, while the leptons are in a chiral representation of the full gauge group \( SU(2)_L \times U(1)_Y \), the charged leptons \(- e, \mu, \tau -\) are in a vectorial representation of the subgroup that is not spontaneously broken, that is \( U(1)_{\text{EM}} \). This situation is the key to opening the possibility of acquiring masses as a result of the SSB, as realized in Eq. (6.4).

These three masses have been measured:

\[
m_e = 0.510998910(13) \text{ MeV}, \quad m_\mu = 105.658367(4) \text{ MeV}, \quad m_\tau = 1776.82(16) \text{ MeV}. \quad (6.6)
\]

Note that the neutrinos are massless in this model. There are no right handed neutrinos, \( N_i(1)_0 \), in the SM so the neutrinos cannot acquire Dirac mass. A-priori, since the neutrinos have no charge under the remaining subgroup \( U(1)_{\text{EM}} \), the possibility of acquiring Majorana masses is not closed. Yet, lepton number is an accidental symmetry of the theory (see Section XIV) and thus the neutrinos do not acquire Majorana masses from renormalizable terms.

In your homework you will find that the number of Higgs representations that can give the gauge boson their masses is large, but only very few also give masses to the fermions.

We presented the details of the spectrum of the leptonic standard model. Next we further discuss the interactions of the model.
VII. THE HIGGS BOSON

Out of the four scalar degrees of freedom, three are the would-be Goldstone bosons eaten by the $W^\pm$ and $Z^0$, and one is a physical scalar $H$ called the Higgs boson.

The kinetic, gauge-interaction, self-interaction and Yukawa interaction terms of $H$ are given by

$$\mathcal{L}_H = \frac{1}{2} \partial_\mu H \partial^\mu H - \frac{1}{2} m_H^2 H^2 - \frac{m_H^2}{2v} H^3 - \frac{m_H^2}{8v^2} H^4$$

$$+ m_W^2 W^- W^{\mu +} \left( \frac{2H}{v} + \frac{H^3}{v^2} \right) + \frac{1}{2} m_Z^2 Z_\mu Z^\mu \left( \frac{2H}{v} + \frac{H^3}{v^2} \right)$$

$$- \frac{H}{v} \left( m_e e_L e_R + m_\mu \mu_L \mu_R + m_\tau \tau_L \tau_R + \text{h.c.} \right).$$

(7.1)

Note that all of the Higgs couplings can be written in terms of the masses of the particles to which it couples.

The Higgs mass is given by

$$m_H = \sqrt{2\lambda v}. \quad (7.2)$$

It determines its quartic self-coupling, $\frac{m_H^2}{v^2} = 2\lambda$, which is unchanged from the quartic coupling in (4.1), and its trilinear self-coupling, $\frac{m_H^2}{2v^2} = \lambda v$, which arises as a consequence of the SSB.

The Higgs coupling to the weak interaction gauge bosons is proportional to their mass-squared. The dimensionless $HHVV$ couplings, $\frac{m_W^2}{v^2} = \frac{g^2}{4}$ and $\frac{m_Z^2}{v^2} = \frac{g^2 + g'^2}{8}$ are unchanged from Eq. (2.5). The $HVV$ couplings, $\frac{m_W^2}{v} = \frac{g^2}{2}$ and $\frac{m_Z^2}{v} = \frac{(g^2 + g'^2)v}{4}$, arise as a consequence of the SSB.

The Yukawa couplings of the Higgs bosons to the charged leptons are proportional to their masses: the heavier the lepton, the stronger the coupling. Note that these couplings, $m_\ell/v = y_\ell/\sqrt{2}$, are unchanged from Eq. (6.1).

The Higgs boson has not been found yet. Direct searches at LEP established a lower bound on its mass:

$$m_H > 114 \text{ GeV}. \quad (7.3)$$

The present situation concerning the Higgs searches at the TeVatron and at the LHC will be discussed later.

VIII. QED: ELECTROMAGNETIC INTERACTIONS

This subsection is based in part on Ref. [2]. By construction, the local $U(1)_{\text{EM}}$ symmetry survives the SSB. Our theory has thus one massless gauge boson that we identify with the photon. Let make sure that it indeed couples like the photon. From Eq. (2.4) we learn that
the couplings of the neutral gauge fields are of the form
\[ g W T^3 + g' B Y. \] (8.1)

Inserting (5.6), we obtain
\[ A / (g \sin \theta_W T^3 + g' \cos \theta_W Y) + Z / (g \cos \theta_W T^3 - g' \sin \theta_W Y). \] (8.2)

The photon field couples to \( e Q = e (T^3 + Y) \), so we must have
\[ g = e \sin \theta_W, \quad g' = e \cos \theta_W. \] (8.3)

Thus, the electromagnetic interactions are described by the QED Lagrangian, which is the part of the SM Lagrangian that involves the \( A^0 \) photon field and the charged fermions:
\[ \mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e A_\mu \bar{\ell} i \gamma^\mu \ell, \] (8.4)
where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), and \( \ell_{1,2,3} = e, \mu, \tau \) are the Dirac fermions with \( Q = -1 \) that are formed from the \( T^3 = -1/2 \) component of a left-handed lepton doublet and a right-handed lepton singlet.

This Lagrangian gives rise to the well-known Maxwell equations:
\[ \partial_\mu F^{\mu\nu} = e J^\nu \equiv -e \ell \gamma^\nu \ell. \] (8.5)

The most stringent QED test comes from high precision measurements of the \( e \) and \( \mu \) anomalous magnetic moments, \( a_\ell \equiv (g_\ell^\gamma - 2) / 2 \), with \( \bar{\mu}_\ell \equiv g_\ell^\gamma (e/2m_\ell) \bar{S}_\ell \):
\[ a_e = (1159652180.73 \pm 0.28) \times 10^{-12}, \quad a_\mu = (11659208.9 \pm 6.3) \times 10^{-10}. \] (8.6)

To the level of experimental sensitivity, \( a_e \) arises entirely from virtual electrons and photons. These contributions are fully known to \( \mathcal{O}(\alpha^4) \), and many \( \mathcal{O}(\alpha^5) \) corrections have been computed. The impressive agreement between theory and experiment has promoted QED to the level of the best theory ever built to describe Nature. The theoretical error is dominated by the uncertainty in the input value of the QED coupling \( \alpha \equiv e^2 / (4\pi) \). Turning things around, \( a_e \) provides the most accurate determination of the fine structure constant,
\[ \alpha^{-1} = 137.035999084 \pm 0.000000051. \] (8.7)

The anomalous magnetic moment of the muon is sensitive to small corrections from heavier states; compared with \( a_e \), they scale with the mass ratio \( m_\mu^2 / m_e^2 \). Electroweak effects from virtual \( W^\pm \) and \( Z^0 \) bosons amount to a contribution of \( (15.4 \pm 0.2) \times 10^{-10} \), which is larger than the present experimental accuracy. Thus \( a_\mu \) allows one to test the entire SM. The main theoretical uncertainty comes from strong interactions. We will not enter a detailed discussion, but only mention that presently there is a discrepancy between theory and experiment at a level of above 3\( \sigma \).
IX. CHARGED WEAK INTERACTIONS

We now study the interactions that change particle identity, namely the couplings of $W_1^\mu$ and $W_2^\mu$. Inserting the explicit form of the $T_a$ matrices (Pauli matrices for doublets, 0 for singlets) in $\bar{\psi} D \psi$, we obtain the following interaction terms:

$$-\frac{g}{2} (\nu_{eL}(W_1 - iW_2)e_L^+ + e_L^-(W_1 + iW_2)\nu_{eL})$$

$$+ \bar{\nu}_{\mu L}(W_1 - iW_2)\mu_L^- + \bar{\mu}_L(W_1 + iW_2)\nu_{\mu L}$$

$$+ \bar{\nu}_{\tau L}(W_1 - iW_2)\tau_L^- + \bar{\tau}_L(W_1 + iW_2)\nu_{\tau L}).$$

In terms of the charged gauge bosons, $W^{\pm\mu} = \frac{1}{\sqrt{2}}(W_1^\mu \mp iW_2^\mu)$, the interaction term for the electron and its neutrino is

$$-\frac{g}{\sqrt{2}} \bar{\nu}_{eL} W^+ e_L^- + \text{h.c.},$$

and similarly for the muon and the tau. The interactions mediated by the $W^\pm$ vector-bosons are called charged current interactions.

Eq. (9.2) reveals some important features of the model:

1. Only left-handed particles take part in charged-current interactions. (We remind the reader that we use the term “left-handed” to denote a chirality eigenstate. These are identical to helicity eigenstates in the massless limit.)

2. Parity violation: a consequence of the previous feature is that the $W$ interactions violate parity.

3. Diagonality: the charged current interactions couple each charged lepton to a single neutrino, and each neutrino to a single charged lepton. Note that a global $SU(2)$ symmetry would allow off-diagonal couplings; It is the local symmetry that leads to diagonality.

4. Universality: the couplings of the $W$-boson to $\tau \bar{\nu}_\tau$, to $\mu \bar{\nu}_\mu$ and to $e \bar{\nu}_e$ are equal. Again, a global symmetry would have allowed an independent coupling to each lepton pair.

All of these predictions have been experimentally tested. As an example of how well universality works, consider the decay rates of the $W$-bosons to the three lepton pairs:

$$\text{BR}(W^+ \rightarrow e^+ \nu_e) = (10.75 \pm 0.13) \times 10^{-2},$$

$$\text{BR}(W^+ \rightarrow \mu^+ \nu_\mu) = (10.57 \pm 0.15) \times 10^{-2},$$

$$\text{BR}(W^+ \rightarrow \tau^+ \nu_\tau) = (11.25 \pm 0.20) \times 10^{-2}.$$

You must be impressed by the nice agreement!

The charged current interaction gives rise to all flavor changing weak decays. One example is the $\mu^- \rightarrow e^- \nu_\mu \bar{\nu}_e$ decay:
As mentioned in Section V, we can get independent determinations of $\sin^2 \theta_W$ from various interaction rates. The decay rate for $\mu^- \rightarrow e^- \bar{\nu}\nu$ is one of these. The $W$-propagator is well approximated via a four fermion coupling:

$$\frac{g^2}{m_W^2 - q^2} \approx \frac{g^2}{m_W^2} = \frac{4\pi\alpha}{\sin^2 \theta_W m_W^2} \equiv 4\sqrt{2}G_F.$$  \hfill (9.4)

The measured muon lifetime,

$$\tau_\mu = (2.197034 \pm 0.000021) \times 10^{-6} \text{ s},$$  \hfill (9.5)

determines $G_F$ via

$$\Gamma_\mu = \frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3} f(m_e^2/m_\mu^2)(1 + \delta_{RC}), \quad f(x) = 1 - 8x + 8x^3 - x^4 - 12x^2 \log x,$$  \hfill (9.6)

where $\delta_{RC}$ is a correction factor from radiative corrections, which is known to $\mathcal{O}(\alpha^2)$. One gets:

$$G_F = 1.16637(1) \times 10^{-5} \text{ GeV}^{-2}. \hfill (9.7)$$

Using $\alpha$ of Eq. (8.7), $m_W$ of Eq. (5.8) and $G_F$ of Eq. (9.7), we obtain

$$\sin^2 \theta_W = 0.215,$$  \hfill (9.8)

in good agreement with Eq. (5.9). The difference between the two is accounted for by higher order radiative corrections.

Note that $G_F$ determines also the VEV:

$$v = (\sqrt{2}G_F)^{-1/2} = 246 \text{ GeV}.$$  \hfill (9.9)

X. NEUTRAL WEAK INTERACTIONS

The $Z$ couplings to fermions, given in Eq. (8.2), can be written as follows:

$$\mathcal{L} = \frac{e}{\sin \theta_W \cos \theta_W} (T_3 - \sin^2 \theta_W Q) \bar{\psi} Z \psi,$$  \hfill (10.1)
where $T_3$ and $Q$ are specific to the fermion $\psi$. For example, for left handed electrons, $T_3 = -1/2$ and $Q = -1$. Explicitly, we find the following three types of $Z$ couplings in a lepton generation:

$$\mathcal{L} = \frac{e}{\sin \theta_W \cos \theta_W} \left[ -\left( \frac{1}{2} - \sin^2 \theta_W \right) \overline{e_L} Z e_L + \sin^2 \theta_W \overline{e_R} Z e_R + \frac{1}{2} \overline{\nu_L} Z \nu_L \right].$$  \hspace{1cm} (10.2)

Note that, unlike the photon, the $Z$ couples to neutrinos. $Z$-exchange gives rise to neutral current interactions. Eq. (10.2) reveals some further important features of the model:

1. The $Z$-boson couples to both left-handed and right-handed fields. Yet, just like the $W$-boson, these couplings are different and thus $Z$-interactions violate parity.

2. **Diagonality.** Consequently, there are no flavor changing neutral currents (FCNCs). This is a result of an accidental $U(1)^3$ symmetry of the model.

3. **Universality:** the couplings of the $Z$-boson to the different generations is universal.

The branching ratios of the $Z$-boson into charged lepton pairs,

$$\text{BR}(Z \to e^+e^-) = (3.363 \pm 0.004)\%,$$  \hspace{1cm} (10.3)

$$\text{BR}(Z \to \mu^+\mu^-) = (3.366 \pm 0.007)\%,$$

$$\text{BR}(Z \to \tau^+\tau^-) = (3.367 \pm 0.008)\%.$$

beautifully confirms universality:

$$\Gamma(\mu^+\mu^-)/\Gamma(e^+e^-) = 1.0009 \pm 0.0028,$$

$$\Gamma(\tau^+\tau^-)/\Gamma(e^+e^-) = 1.0019 \pm 0.0032.$$

Diagonality is also tested by the following experimental searches:

$$\text{BR}(Z \to e^+\mu^-) < 1.7 \times 10^{-6},$$

$$\text{BR}(Z \to e^+\tau^-) < 9.8 \times 10^{-6},$$

$$\text{BR}(Z \to \mu^+\tau^-) < 1.2 \times 10^{-5}.$$  \hspace{1cm} (10.4)

The branching ratio of $Z$ decays into invisible final states which, in our model, is interpreted as the decay into final neutrinos, is measured to be

$$\text{BR}(Z \to \nu\bar{\nu}) = (20.00 \pm 0.06)\%.$$  \hspace{1cm} (10.5)

From Eq. (10.2) we obtain

$$\frac{\text{BR}(Z \to \ell^+\ell^-)}{\text{BR}(Z \to \nu\bar{\nu})} = \frac{(1/2 - \sin^2 \theta_W)^2 + \sin^4 \theta_W}{1/4} = 1 - 4 \sin^2 \theta_W + 8 \sin^4 \theta_W.$$  \hspace{1cm} (10.6)

We can thus extract $\sin^2 \theta_W$ from the experimental data, $\sin^2 \theta_W = 0.226$, consistent with Rq. (5.9).
XI. GAUGE BOSON SELF INTERACTIONS

The gauge boson self interactions that are presently most relevant to experiments are the \( WWV \) \((V = Z, A)\) couplings, which have the following form:

\[
\mathcal{L}_{WWV} = ie \cot \theta_W \left[ (W_{\mu\nu}^a W^{a\mu} - W_{\mu}^{a\nu} W^{a\mu}) Z^\mu + W_{\mu}^{a\nu} W^{a\mu} Z^\nu \right] \\
+ ie \left[ (W_{\mu\nu}^a W^{a\mu} - W_{\mu}^{a\nu} W^{a\mu}) A^\mu + W_{\mu}^{a\nu} W^{a\mu} A^\nu \right].
\]  \(11.1\)

XII. LOW ENERGY TESTS

There are seven independent parameters in the leptonic \( SU(2) \times U(1) \) model. They can be chosen to be \( g, g', v, \lambda, y_e, y_\mu, y_\tau \). There are, however, other possible choices. A good choice of parameters would be one where the experimental errors in their determination are very small. Such a set is the following:

\[
\alpha, \ G_F, \ m_e, \ m_\mu, \ m_\tau, \ m_Z, \ m_H. \tag{12.1}
\]

Out of this list, only \( m_H \) (or, equivalently, \( \lambda \) in the previous list) is unknown, though it seems that the experiments may be closing in on \( m_H \simeq 125 \text{ GeV} \). In the following we use the other, well-measured, 6 parameters to test the model.

Nowadays, experiments produce the \( W \) and \( Z \) bosons and measure their properties directly. It is interesting to understand, however, how the SM was tested at the time before the energy in experiments became high enough for such direct production. It is not only the historical aspect that is interesting; It is also important to see how we can use low energy data to understand shorter distances.

A. NC in neutrino scattering

There are several observables that can be used to test neutral currents interactions. The first example is low energy \( \nu_\mu e^- \rightarrow \nu_\mu e^- \) scattering. Since the \( W \)-boson couples diagonally, it does not couple to a \( \nu_\mu e^- \) pair. Consequently, the elastic scattering \( \nu_\mu e \rightarrow \nu_\mu e \) is mediated purely by the \( Z \)-boson.

We can use the ratio

\[
R \equiv \frac{\sigma(\nu_\mu e \rightarrow \nu_\mu e)}{\sigma(\bar{\nu}_\mu e \rightarrow \bar{\nu}_\mu e)} \tag{12.2}
\]

to fix \( \sin \theta_W \):

\[
\sigma_{\nu_\mu} = \frac{G_F^2 m_e E_\nu}{2\pi} \left[ (g_L^*)^2 + \frac{1}{3} (g_R^*)^2 \right], \quad \sigma_{\bar{\nu}_\mu} = \frac{G_F^2 m_e E_\nu}{2\pi} \left[ (g_R^*)^2 + \frac{1}{3} (g_L^*)^2 \right], \tag{12.3}
\]

where

\[
g_L^* = -1/2 + \sin^2 \theta_W; \quad g_R^* = \sin^2 \theta_W. \tag{12.4}
\]
We also use the notations
\[ g_V^c = g_L^c + g_R^c = -1/2 + 2\sin^2\theta; \quad g_A^c = g_L^c - g_R^c = -1/2. \]  
(12.5)

From PDG00 we find (p. 101) \( g_A^c = -0.503 \pm 0.017 \) and \( g_V^c = -0.041 \pm 0.015 \). This gives
\[ \sin^2 \theta_W = 0.230 \pm 0.008. \]  
(12.6)

B. Neutrino–Electron Scattering

Let us compare the charged current contributions to the two elastic scattering processes \( \bar{\nu}_e e^- \rightarrow \bar{\nu}_e e^- \) and \( \nu_e e^- \rightarrow \nu_e e^- \) scattering. Note that both processes get contributions also from Z-boson exchange, which must be taken into account in a full calculation. We consider scattering with a center-of-mass energy much higher than the electron and muon masses, so we can consider the leptons massless.

We define \( \theta \) to be the angle between the incoming (anti)neutrino and the outgoing electron. Then \( \cos \theta = 1 \) corresponds to backward scattering of the beam particle. \( \bar{\nu}_L \) and \( \ell_L \) have positive and negative helicities, respectively. Thus, in the center of mass frame, their spins are in the same direction. Therefore \( (J_z)_i = +1 \). When they are scattered backwards, their momenta change to the opposite directions, and so do their helicities: \( (J_z)_f = -1 \). Therefore, backward \( \bar{\nu} \ell \) scattering is forbidden by angular momentum conservation. In fact, the process \( \bar{\nu}_e e^- \rightarrow \bar{\nu}_e \mu^- \) proceeds entirely in a \( J = 1 \) state with net helicity +1. That is, only one of the three states is allowed. In contrast, in \( \nu_e e^- \rightarrow \nu_e e^- \), backward scattering has \( (J_Z)_i = (J_Z)_f = 0 \) and all helicity states are allowed. The full \( SU(2) \times U(1) \) calculation yields:
\[ \frac{d\sigma(\nu_e e^-)}{d\Omega} = \frac{G_F^2 s}{4\pi^2}; \quad \frac{d\sigma(\bar{\nu}_e e^-)}{d\Omega} = \frac{G_F^2 s}{16\pi^2}(1 - \cos \theta)^2. \]  
(12.7)
\[ \sigma(\nu_e e^-) = \frac{G_F^2 s}{\pi}; \quad \sigma(\bar{\nu}_e e^-) = \frac{G_F^2 s}{3\pi}. \]  
(12.8)

Let us mention that the calculation of the charged current contribution of \( \nu_e e^- \rightarrow \nu_e e^- \) scattering rate provides the full calculation of the \( \nu_\mu e^- \rightarrow \nu_\mu e^- \) scattering rate. Since the Z-boson couples diagonally, it cannot mediate the latter process which is, therefore, purely charged current.

C. Pion Decay

Charged pion decay is mediated by the \( W \). We have
\[ \pi^- \rightarrow \mu^- + \bar{\nu}_\mu, \quad \pi^- \rightarrow e^- + \bar{\nu}_e. \]  
(12.9)
Taking into consideration that
\[ m_\pi = 140 \text{ MeV}; \quad m_\mu = 106 \text{ MeV}; \quad m_e = 0.5 \text{ MeV}, \quad (12.10) \]
one would naively expect that the branching ratio into \( e \) would be larger than that into \( \mu \), because phase space is much larger. However, the SM features of the \( W \)-couplings 1 (left-handedness) and 4 (universality) turn the naive expectation around.

The \( \pi^- \) is spinless (\( J = 0 \)), and so, by the conservation of angular momentum, the outgoing lepton pair (\( \ell^- \bar{\nu}_\ell \)) must have \( J = 0 \). As the \( \bar{\nu}_\ell \) has positive helicity, the \( \ell^- \) is also forced into a positive helicity state. But recall that in the limit \( m(\ell) = 0 \), \( \ell_L \) has negative helicity! In this limit the weak current only couples negative helicity \( \ell^- \). The positive helicity is highly suppressed \( \propto m_\ell \). Thus a pion decay into \( \mu^- \) is much more likely than into \( e^- \).

An exact calculation in the \( SU(2) \times U(1) \) framework, taking into account the equality of couplings, gives:
\[
\frac{\Gamma(\pi^- \rightarrow e^- \bar{\nu}_e)}{\Gamma(\pi^- \rightarrow \mu^- \bar{\nu}_\mu)} = \left( \frac{m_e}{m_\mu} \right)^2 \left( \frac{m_\mu^2 - m_e^2}{m_\mu^2 - m_\pi^2} \right)^2 = (1.235 \pm 0.001) \times 10^{-4}, \quad (12.11)
\]
(where the numerical value includes radiative corrections). The most recent experimental results [PDG 2011] give
\[
\frac{\Gamma(\pi^- \rightarrow e^- \bar{\nu}_e)}{\Gamma(\pi^- \rightarrow \mu^- \bar{\nu}_\mu)} = (1.230 \pm 0.004) \times 10^{-4}. \quad (12.12)
\]
Assuming the \( V - A \) structure of the interactions, one could bound by this result deviations from universality to be \( \leq 0.001 \).

**XIII. THE INTERACTION BASIS AND THE MASS BASIS**

The interaction basis is the one where all fields have well-defined transformation properties under the symmetries of the Lagrangian. In particular, in this basis, the gauge interactions are universal.

If there are several fields with the same quantum numbers, then the interaction basis is not unique. The kinetic and gauge terms are invariant under a global unitary transformation among these fields. On the other hand, the Yukawa terms and the fermion mass terms are, in general, not invariant under a unitary transformation among fermion fields with the same quantum numbers, \( f_i \rightarrow U^f_{ji} f_i \), while the Yukawa terms and scalar potential are, in general, not invariant under a unitary transformation among scalar fields with the same quantum numbers, \( s_i \rightarrow U^s_{ji} s_i \). Thus, by performing such a transformation, we are changing the interaction basis.

In the LSM, there are three copies of \((2)_{-1/2}\) fermions and three copies of \((1)_{-1}\) fermions. Transforming the first by a unitary transformation \( U_L \), and the latter by an independent
unitary transformation $U_R$, the Yukawa matrix $Y^e$ is transformed into $U_L Y^e U_R^\dagger$. The matrix $Y^e$ is a $3 \times 3$ complex matrix and thus has, in general, nine complex parameters. We can always find a bi-unitary transformation that would make $Y^e$ real and diagonal, and thus depend on only three real parameters:

$$Y^e \rightarrow U_L Y^e U_R^\dagger = Y^e_{\text{diag}} = \text{diag}(y_e, y_\mu, y_\tau). \quad (13.1)$$

Often one chooses a basis where the number of Lagrangian parameters is minimal, as is the case with the diagonal basis of Eq. (13.1). One could work in any other interaction basis. However, when calculating physical observables, only the eigenvalues of $Y^e Y^e$ would play a role. Using the diagonal basis just provides a shortcut to this result.

The mass basis is the one where all fields have well defined transformation properties under the symmetries that are not spontaneously broken and are mass eigenstates. The fields in this basis correspond to the particles that are eigenstates of free propagation in spacetime. The Lagrangian parameters in this basis correspond directly to physical observables.

For the LSM, the interaction eigenstates have well defined transformation properties under the $SU(2)_L \times U(1)_Y$ symmetry:

$$W_a(3)_0, \quad B(1)_0, \quad L_{L1,2,3}(2)_{-1/2}, \quad E_{R1,2,3}(1)_{-1}, \quad \phi(2)_{+1/2}. \quad (13.2)$$

The mass eigenstates have well defined electromagnetic charge and mass:

$$W^\pm, \quad Z^0, \quad A^0, \quad e^-, \quad \mu^-, \quad \tau^-, \quad \nu_e, \quad \nu_\mu, \quad \nu_\tau, \quad H^0. \quad (13.3)$$

The number of degrees of freedom is the same in both bases. To verify this statement one has to take into account the following features:

1. $W_a$ and $B^\mu$ have only transverse components, while $W^\pm$ and $Z^0$ have also a longitudinal one.

2. $L_L$ and $E_R$ are Weyl fermions, while $e, \mu, \tau$ are Dirac fermions.

3. $\phi$ is a complex scalar, while $H$ is a real one.

The three electromagnetically neutral neutrino states are, at the renormalizable level, massless and, in particular, degenerate. Thus, there is freedom in choosing the basis for the neutrinos. We choose the basis where the $W^\pm$ couplings to the charged lepton mass eigenstates are diagonal. Later we will see that non-renormalizable terms provide the neutrinos with (non-degenerate) masses, and then the mass basis becomes unique.
XIV. ACCIDENTAL SYMMETRIES

If we set the Yukawa couplings to zero, $\mathcal{L}_\text{Yuk} = 0$, the leptonic SM (LSM) gains a large accidental global symmetry:

$$G_{\text{global}}^{\text{LSM}}(Y^e = 0) = U(3)_L \times U(3)_E,$$

where $U(3)_L$ has $(L_{L1}, L_{L2}, L_{L3})$ transforming as an $SU(3)_L$ triplet, and all other fields singlets, while $U(3)_E$ has $(E_{R1}, E_{R2}, E_{R3})$ transforming as an $SU(3)_E$ triplet, and all other fields singlets. The Yukawa couplings break this symmetry into the following subgroup:

$$G_{\text{global}}^{\text{LSM}} = U(1)_e \times U(1)_\mu \times U(1)_\tau,$$

where $\ell_L, \ell_R$ and $\nu_\ell$ carry charge +1 under $U(1)_\ell$. Total lepton number is a subgroup of $G_{\text{LSM}}^{\text{global}}$ and is thus conserved.

Thus, electron number, muon number, tau number, and total lepton number are accidental symmetries of the SM. They are, however, all broken by nonrenormalizable terms of the form $(1/\Lambda) L_L L_L \phi \phi$. If the scale $\Lambda$ is high enough, these effects are very small.

Another interesting point is that the breaking of the symmetry (14.1) into (14.2) is by the Yukawa couplings $y_e, y_\mu, y_\tau$ which are small, of $\mathcal{O}(10^{-6}, 10^{-3}, 10^{-2})$, respectively. Thus, the full $[SU(3)]^2$ remains an approximate symmetry of the SM.

The symmetry structure can also be used to count the number of independent parameters. In a generic interaction basis, $\mathcal{L}_\text{Yuk}$ has 9 real and 9 imaginary parameters. Since this part of the Lagrangian induces the breaking of the symmetry, $[U(3)]^2 \rightarrow [U(1)]^3$, we can affect the form of $\mathcal{L}_\text{Yuk}$ (change basis) with two $3 \times 3$ unitary matrices except for three phase transformations which constitute an invariance of the Lagrangian, namely reduce the number of parameters by $2 \times (3_R + 6_I) - 3_I = 6_R + 9_I$, where sub-index $R/I$ stands for real (imaginary) parameters. Thus we are left with three real and no imaginary parameters. This confirms that our choice of a basis where $Y^e = \text{diag}(y_e, y_\mu, y_\tau)$ indeed corresponds to the minimal number of parameters.