THE STANDARD MODEL

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Chapter 1

Formal background

1.1 Lagrangians

The most fundamental postulate in physics is the principle of minimal action in its quantum interpretation. Thus, the fundamental laws of physics can be encoded in the action, S. In Quantum Field Theory (QFT) the action is an integral over spacetime of the "Lagrange density" or Lagrangian, $\mathcal{L}$, for short. For most of our purposes, we need to consider just the Lagrangian. In this section we explain how we "construct" Lagrangians. Later we discuss how we determine their parameters, and how we test whether they describe Nature correctly.

In QFT, the equivalent of the generalized coordinates of classical mechanics are the fields. The action is given by

$$S = \int d^4x \; \mathcal{L}[\phi_i(x), \partial_\mu \phi(x)] \,, \tag{1.1}$$

where $d^4x = dx^0 dx^1 dx^2 dx^3$ is the integration measure in four-dimensional Minkowski space. The index i runs from 1 to the number of fields. We denote a generic field by $\phi(x)$. Later, we use $\phi(x)$ for a scalar field, $\psi(x)$ for a fermion field, and $V(x)$ for a vector field.

The action S has units of ML^2T^{-1} or, equivalently, of $\hbar$. In a natural unit system, where $\hbar = 1$, S is taken to be "dimensionless." Then in four dimensions $\mathcal{L}$ has natural dimensions of $L^{-4} = M^4$. The requirement of the variation of the action with respect to variation of the fields vanishes, $\delta S = 0$, leads to the equations of motion (EoM):

$$\frac{\delta \mathcal{L}}{\delta \phi} = \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \right) \,, \tag{1.2}$$

where the x dependence of ϕ is omitted. When there are several fields, the above equation should be satisfied for each of them.

In general, we require the following properties for the Lagrangian:

(i) It is a function of the fields and their derivatives only, so as to ensure translational invariance.

(ii) It depends on the fields taken at one space-time point x^μ only, leading to a local field theory.

(iii) It is real in order to have the total probability conserved.

(iv) It is invariant under the Poincaré group.

(v) It is an analytic function in the fields. This is not a general requirement, but it is common to all field theories that are solved via perturbation theory. In all of these, we expand around a minimum, and this expansion means that we consider a Lagrangian that is a polynomial in the fields.

(vi) It is invariant under certain internal symmetry groups. The symmetries of S (or of $\mathcal{L}$) are in correspondence with conserved quantities and therefore reflect the basic symmetries of the physical system.

Often, we add two other requirements:

(vii) Naturalness: Every term in the Lagrangian that is not forbidden by a symmetry should appear.

(viii) Renormalizability: Only terms that are of (mass) dimension less of or equal to four in the fields and their derivatives are included.

The issue of renormalizability deserves further discussion. If the full theory of Nature is described by QFT, its Lagrangian should be renormalizable. Renormalizability ensures that the Lagrangian contains at most two ∂_μ operations, so it leads to classical equations of motion that are no higher than second order derivatives. The theories that we consider and, in particular, the Standard Model, are, however, only low energy effective theories, valid up to some energy scale Λ. Therefore, we must include also non-renormalizable terms. These terms have coefficients with inverse mass dimensions, $1/\Lambda^n$, $n = 1, 2, \ldots$. For most purposes, however, the renormalizable terms constitute the leading terms in an expansion in E/Λ, where E is the energy scale of the physical processes under study. Therefore, the renormalizable part of the Lagrangian is a good starting point for our study.

Properties (i)-(v) are not the subject of this book. You must be familiar with them from your QFT course(s). We do, however, deal intensively with the other requirements. Actually, the most important message that we would like to convey in this book is the following: *(Almost) all experimental data for elementary particles and their interactions are explained by the standard model of a spontaneously broken $SU(3) \times SU(2) \times U(1)$ gauge symmetry.*[1]

We next present a few simple examples of Lagrangians.

[1]Actually, the great hope of all high-energy physics community is to prove this statement wrong!

1.1.1 Scalars

The renormalizable Lagrangian for a real scalar field ϕ is given by

$$\mathcal{L}_S = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\mu}{2\sqrt{2}}\phi^3 - \frac{\lambda}{4}\phi^4\,. \tag{1.3}$$

We work in the "canonically normalized" basis where the coefficient of the kinetic term in the EOM is one. (To get that the kinetic term as well as the mass term of a real field has a factor of a half in front. This factor is not there for a complex field.) In this basis, all terms have a clear interpretation. The Lagrangian $\mathcal{L}_S$ of Eq. (1.3) is the most general renormalizable $\mathcal{L}(\phi)$ we can write, so it satisfies the naturalness principle. We emphasize the following points:

1. The first term of the Lagrangian is necessary if we want ϕ to be a *dynamical* field, namely to be able to describe propagation in spacetime.

2. We do not write a constant term since it does not enter the equation of motion for ϕ.

3. In principle we could write a linear term but it is not physical, that is, we can always redefine the field such that the linear term vanishes.

4. The quadratic term (ϕ^2) is a mass-squared term.

5. Additional terms, namely the trilinear (ϕ^3) and quartic (ϕ^4) terms, describe interactions.

1.1.2 Fermions

The Lagrangian for a Dirac fermion field ψ is given by

$$\mathcal{L}_F = \bar{\psi}(i\partial\!\!\!/ - m)\psi\,. \tag{1.4}$$

Again, we work in the canonically normalized basis. The Lagrangian $\mathcal{L}_F$ of Eq. (1.4) is the most general renormalizable $\mathcal{L}(\psi)$ we can write, so it satisfies the naturalness principle. (There is a subtlety involved in this statement. By saying that the fermion in question is of the Dirac type, we are implicitly imposing a symmetry that forbids Majorana mass terms. We discuss this issue later.) We treat ψ and $\bar{\psi}$ as independent fields. The reason is that a fermion field is complex, and it is more convenient to deal with ψ and $\bar{\psi}$ than with $\mathcal{R}e(\phi)$ and $\mathcal{I}m(\psi)$. We emphasize the following points:

1. Terms with an odd number of fermion fields violate Lorentz symmetry, and so they are forbidden.

2. The quadratic term ($\bar{\psi}\psi$) is a mass term.

3. Terms with four or more fermions are non-renormalizable.

1.1.3 Fermions and scalars

The renormalizable Lagrangian for a single Dirac fermion and a single real scalar field includes, in addition to the terms written in Eqs. (1.3) and (1.4), the following term:

$$\mathcal{L}_{\text{Yuk}} = -Y\overline{\psi_L}\psi_R\phi + \text{h.c.}. \tag{1.5}$$

Such a term is called a Yukawa interaction and Y is the dimensionless Yukawa coupling. The most general Lagrangian for a real scalar field and a Dirac fermion is thus

$$\mathcal{L}(\phi, \psi) = \mathcal{L}_S + \mathcal{L}_F + \mathcal{L}_{\text{Yuk}}, \tag{1.6}$$

where $\mathcal{L}_S$ is given in (1.3), $\mathcal{L}_F$ is given in (1.4), and $\mathcal{L}_{\text{Yuk}}$ in (1.5).

1.2 Symmetries

1.2.1 Introduction

Particle physicists seek deeper reasons for the rules they have discovered. A major role in these answers in modern theories is played by symmetries. The term *symmetry* refers to an invariance of the equations that describe a physical system. The fact that a symmetry and an invariance are related concepts is obvious enough — a smooth ball has a spherical symmetry and its appearance is invariant under rotation.

Symmetries are built into physics as invariance properties of the Lagrangian. If we construct our theories to encode various empirical facts and, in particular, the observed conservation laws, then the equations turn out to exhibit certain invariance properties. For example, if we want the theory to have energy conservation, then the Lagrangian cannot depend explicitly on time. In this view point the conservation law is the input and the symmetry is the output.

Conversely, if we take the symmetries to be the fundamental rules that determine the theory we can write, then various observed features of particles and their interactions are a necessary consequence of the symmetry principle. In this sense, symmetries provide an explanation of these features. In modern particle physics we often take that former view point in which symmetries are the input.

In the following we will discuss different types of symmetries. In particular, We obtain the consequences of *imposing* a symmetry on a Lagrangian. This is the starting point of model building in modern particle physics: one defines the basic symmetries and the particle content, and then obtains the predictions that follow from these imposed symmetries.

In the way we contract our Lagrangian there are two types of symmetries. Those that we impost and those that are there without being imposed. Such symmetries are called *accidental* symmetries. They are a result as outputs of the theory rather than as external constraints. In particular, they come from the structure imposed by renormalizability and gauge invariance. These are broken explicitly by non-renormalizable terms, but since these terms are small, and we often just irnore them, one can often make use of these symmetries.

1.2.2 Spacestime vs internal symmetries

There are several types and aspect related to symmetries that we discuss next. First, we distinguish between *spacetime* and *internal* symmetries. Spacetime symmetries include the Poincaré group of translations, rotations and boosts. They give us the energy–momentum and angular momentum conservation laws. In additional they also include the space inversion, P, time-reversal, T, and charge conjugation, C operators. (While C is not trully a spacetiem symmetry, the way it acts on fermions and the CPT theorm, make it simpler to include them in the same class of opertaors.) We will not discuss these symmetries much.

Internal symmetries act on the fields, not directly on spacetime. That is, they work in mathematical spaces that are generated by the fields. It is these kind of symmetries that we will discuss in length. The Noether's theorem relates internal global continuous symmetries to conserved charges. The proof is provided in Appendix 1.A and here we just state the result: For any invariant generator T^a there is a corresponding conserved current that is given by

$$J_\mu^a = i\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}T^a\phi. \tag{1.7}$$

Below we discuss the impliction of imposing internal symmetries

1.2.3 Global Discrete Symmetries

We start with a simple example of an internal discrete global Z_2 symmetry.

Consider a real scalar field ϕ. The most general Lagrangian we can write is given in Eq. (1.3). We now impose a symmetry: we demand that $\mathcal{L}$ is invariant under a Z_2 symmetry, $\phi \to -\phi$, namely

$$\mathcal{L}(\phi) = \mathcal{L}(-\phi)\,. \tag{1.8}$$

$\mathcal{L}$ is invariant under this symmetry if $\mu = 0$. Thus, by imposing the symmetry we force $\mu = 0$: The most general $\mathcal{L}(\phi)$ that we can write that also respects the Z_2 symmetry is

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4\,. \tag{1.9}$$

What conservation law corresponds to this symmetry? We can call it ϕ parity. The number of particles in a system can change, but always by an even number. Therefore, if we define parity as $(-1)^n$, where n is the number of particles in the system, we see that this parity is conserved. When we do not impose the symmetry and $\mu \neq 0$, the number of particle can change by any integer and ϕ parity is not conserved. When μ is very small (in the appropriate units), ϕ parity is an approximate symmetry.

While this is a simple example, it is a useful exercise to describe it in terms of group theory. Recall that Z_2 has two elements that we call even (+) and odd (−). The multiplication table is very simple:

$$(+)\cdot(+) = (-)\cdot(-) = (+), \qquad (+)\cdot(-) = (-)\cdot(+) = (-). \tag{1.10}$$

When we say that we impose a Z_2 symmetry on $\mathcal{L}$, we mean that $\mathcal{L}$ belongs to the even representation of Z_2. By saying that $\phi \to -\phi$ we mean that ϕ belongs to the odd representation of Z_2. Since $\mathcal{L}$ is even, all terms in $\mathcal{L}$ must be even. The field ϕ, however, is odd. Thus, we can keep only terms with even powers of ϕ. Then we can construct the most general $\mathcal{L}$ and it is given by Eq. (1.9).

1.2.4 Global Continuous symmetries

We now extend our "model building" ideas to continuous symmetries. The idea is that we demand that $\mathcal{L}$ is invariant under rotation in some internal space. That is, while (some of) the fields are not invariant under rotation in that space, the combinations that appear in the Lagrangian are invariant.

Our first example ios the case of two scalars. Consider a Lagrangian that depends on two real scalar fields, $\mathcal{L}(\phi_1, \phi_2)$:

$$\mathcal{L} = \frac{1}{2}\delta_{ij}\partial^\mu\phi_i\partial_\mu\phi_j - \frac{m_{ij}^2}{2}\phi_i\phi_j - \frac{\mu_{ijk}}{2\sqrt{2}}\phi_i\phi_j\phi_k - \frac{\lambda_{ijk\ell}}{4}\phi_i\phi_j\phi_k\phi_\ell\,, \tag{1.11}$$

with m^2, μ and λ real.[2] We can always choose a basis where m^2 is diagonal. We impose an SO(2) symmetry under which the scalars transform as follows:

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to O\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{1.12}$$

where O is a general orthonormal matrix. Imposing this symmetry leads to a much simpler Lagrangian:

$$\mathcal{L} = \frac{1}{2}\delta_{ij}\partial^\mu\phi_i\partial_\mu\phi_j - \frac{m^2}{2}\delta_{ij}\phi_i\phi_j - \frac{\lambda}{4}\left(\phi_1^4 + \phi_2^4 + 2\phi_1^2\phi_2^2\right). \tag{1.13}$$

It can be written in an even simpler way by taking advantage of the fact that SO(2) and U(1) are equivalent. Then instead of considering two real scalar fields, we can consider a single complex scalar field

$$\phi \equiv \frac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right), \tag{1.14}$$

[2]In order for the potential to be bounded from below, we require that some combinations of the λ are positive. For simplicity, we will take *all* the parameters as positive.

with the following $U(1)$ transformation:

$$\phi \to \exp(2\pi i\theta)\phi, \qquad \phi^\dagger \to \exp(-2\pi i\theta)\phi^*. \tag{1.15}$$

Then we rewrite (1.13) as

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2. \tag{1.16}$$

We would like to emphasize the following points regarding Eq. (1.16):

- All three terms that appear in this equation and, in particular, the mass term, do not violate any internal symmetry. Thus, there is no way to forbid them by imposing a symmetry.
- The conserved charge is very similar in nature to an electric charge. We can think of ϕ as a charged field that carries a positive charge and then $\phi^\dagger$ carries negative charge. This is the source of the statement that only complex fields can be charged.
- The normalization of a $U(1)$ charge is arbitrary.

Let us next consider a model with four real scalar field. We group them into two complex fields $\phi_1(+1)$ and $\phi_2(+3)$ where the number inside the parantesis is the charge under a $U(1)$ symmetry. Then the most general $U(1)$-symmetric Lagrangian is

$$\mathcal{L} = \partial^\mu \phi_i \partial_\mu \phi_i^* - m_1^2 \phi_1 \phi_1^\dagger - m_2^2 \phi_2 \phi_2^\dagger - \lambda_{ij}(\phi_i \phi_i^\dagger)(\phi_j \phi_j^\dagger) - (\eta \phi_1^3 \phi_2^\dagger + \text{h.c.}). \tag{1.17}$$

We now examine the symmetry properties of the various terms of $\mathcal{L}$. The symmetry is largest for the kinetic term, it become smaller when the mass terms are included, and even smaller with interaction terms added. Explicitly, the kinetic term has an SO(4) symmetry. The mass (m^2) and the quartic interaction (λ) terms have a $U(1)^2$ symmetry. The other interaction (η) term reduces the symmetry to a single $U(1)$. In case that $|\eta| \ll 1$, the Lagrangian has an *approximate* $U(1)^2$ symmetry. In particular, for $|\lambda_{1112}| \ll |\lambda_{ij}|$ [where the λ_{ij}'s are defined in Eq. (1.17)], the processes that break the $U(1)^2$ symmetry, such as $\phi_1 + \phi_1 \to \phi_1^\dagger + \phi_2$, have much lower cross section than those that do not, such as $\phi_1 + \phi_1^\dagger \to \phi_2 + \phi_2^\dagger$.

Consider a similar model, but now we assign ϕ_2 charge of 4. The renormalizable terms in the Lagrangian have a U$(1)^2$ symmetry. This $U(1)^2$ is our first example of an accidental symmetry: We did not impose it, we get it as a consequence of the $U(1)$ symmetry and particle content (the charge assignments of the scalar fields). This Accidental symmetry is broken by non-renormalizable terms. In our case, the dimension-5 term of the form $\phi_1^4 \phi_2^\dagger$ breaks the symmetry down to the one we imposed. In the full UV model these non-renormalizable operators arise by adding other fields.

1.2.5 Fermions

Next we discuss fermions. The introduction of ψ_L and ψ_R allows yet another classification of symmetries. A *chiral symmetry* is defined as a symmetry where the LH fermion transforms differently from the RH fermion. A *vectorial symmetry* is one under which ψ_L and ψ_R transform in the same way. Denoting the charge under a $U(1)$ symmetry as Q, we thus define

$$\begin{aligned} \text{vectorial symmetry :} \quad & Q(\psi_L) = Q(\psi_R), \\ \text{chiral symmetry :} \quad & Q(\psi_L) \neq Q(\psi_R). \end{aligned} \tag{1.18}$$

There are two possible mass terms for fermions: Dirac and Majorana. Dirac masses couple left- and right-handed fields,

$$m_D \overline{\psi_L} \psi_R + \text{h.c.}. \tag{1.19}$$

Here m_D is the Dirac mass.

Majorana masses couple a left-handed or a right-handed field to itself. Consider ψ_R, a right-handed field that carries no charge. Defining

$$\psi^c = C\,\overline{\psi}^T, \tag{1.20}$$

where C is the charge conjugation matrix, a Majorana mass term reads

$$m_M \overline{\psi_R^c}\, \psi_R, \tag{1.21}$$

where m_M is the Majorana mass. Note that ψ_R and $\overline{\psi_R^c}$ transform in the same way under all symmetries. A similar expression holds for left handed fields.

We emphasize the following points regarding Eqs. (1.19) and (1.21):

- Since ψ_L and ψ_R are different fields, there are four degrees of freedom with the same Dirac mass, m_D. In contrast, since only one Weyl fermion field is needed in order to generate a Majorana mass term, there are only two degrees of freedom that have the same Majorana mass, m_M.

- Consider a theory with one or more exact $U(1)$ symmetries. To allow a Dirac mass, the charges of $\overline{\psi_L}$ and ψ_R under these symmetries must be opposite. In particular, the two fields can carry electric charge as long as $Q(\psi_L) = Q(\psi_R)$. Thus, to have a Dirac mass term, the fermion has to be in a *vector representation* of the symmetry group.

- The additive quantum numbers of $\overline{\psi_R^c}$ and ψ_R are the same. Thus, a fermion field can have a Majorana mass only if it is neutral under all unbroken local and global $U(1)$ symmetries. In particular, fields that carry electric charges cannot acquire Majorana masses. If we include any non-Abelian group the condition is that the fermion cannot be in a complex representation.

- When there are m left-handed fields and n right-handed fields with the same quantum numbers, the Dirac mass terms for these fields form an $m \times n$ general complex matrix m_D:

$$(m_D)_{ij} (\overline{\psi_L})_i (\psi_R)_j + \text{h.c.}. \tag{1.22}$$

Table 1.1: Dirac and Majorana masses

	Dirac	Majorana
# of degrees of freedom	4	2
Representation	vector	neutral
Mass matrix	$m \times n$, general	$n \times n$, symmetric
SM fermions	quarks, charged leptons	neutrinos (?)

In the SM, fermion fields are present in three copies with the same quantum numbers, and the Dirac mass matrices are 3×3. In general, however, m_D does not have to be a square matrix.

- When there are n neutral fermion fields, the Majorana mass terms form an $n \times n$ symmetric, complex matrix m_M:

$$(m_M)_{ij}(\overline{\psi_R^c})_i(\psi_R)_j. \tag{1.23}$$

  In the SM, neutrinos are the only neutral fermions. If they have Majorana masses, then their mass matrix is 3×3.

We summarize these differences between Dirac and Majorana masses in Table 1.1.

The main lesson that we can draw from these observations is the following: *Charged fermions in a chiral representation are massless.* In other words, if we encounter massless fermions in Nature, there is a way to explain their masslessness from symmetry principles.

We now discuss the case of many Dirac fields and their accidental symmetries. Consider N Dirac fermions charged under a $U(1)$ fermion number. If we give the left- and right-handed fields different charges under the $U(1)$ symmetry, the mass terms are forbidden and all we have is a theory of free massless fermions, and an accidental symmetry of $[U(N)]^2$. To allow masses, we assign left- and right-handed fields the same charge under $U(1)$. Then we get the mass terms of Eq. (1.126), and an accidental symmetry of $U(1)^N$. Finally, add to this model a single scalar field of charge zero:

$$\mathcal{L} = \bar{\psi}_i[i\partial\!\!\!/\delta_{ij} - m_{ij} - Y_{ij}\phi]\psi_j + \mathcal{L}_S\,, \tag{1.24}$$

where $\mathcal{L}_S$ includes the kinetic term for the scalar field and Y_{ij} are the Yukawa couplings. In general we can diagonalize only m or only Y but not both. We see that the accidental symmetry is even smaller. The only exact symmetry is $U(1)$, which is the fermion number symmetry. As we will see later, this is the case in the SM for the quarks, where the only exact, at the renormalizable level, global symmetry is baryon number.

1.2.6 Local Symmetries

So far we discussed global symmetries, that is, symmetries that transform the field in the same way over all space-time. Now we discuss local symmetries, that is, symmetries where the transformation can be different in different space-time points. The space-time dependence of the phase of charged fields should not be observable. Therefore, we would now let the infinitesimal parameter ϵ_a depend on x.

Before proceeding, we introduce the following notation:

$$\tilde{O} \equiv T_a O_a. \tag{1.25}$$

$\tilde{O}$ is an $N \times N$ matrix. Knowing $\tilde{O}$ allows us to easily recover the O_a's. Take the T_a's to be orthogonal:

$$\mathrm{tr}(T_a T_b) = \delta_{ab}. \tag{1.26}$$

Then

$$O_a = \mathrm{tr}(T_a \tilde{O}). \tag{1.27}$$

Consider the effect of a local transformation,

$$\phi(x) \to e^{i\tilde{\epsilon}(x)}\phi(x) \quad \Longrightarrow \quad \delta\phi(x) = i\tilde{\epsilon}(x)\phi(x) \tag{1.28}$$

on a Lagrangian

$$\mathcal{L}(\phi, \partial_\mu \phi). \tag{1.29}$$

Note that a global transformation is a special case of the local transformation. However, when we apply the local transformation on a globally invariant $\mathcal{L}$, we encounter a problem with the derivative term:

$$\delta\partial^\mu\phi = \partial^\mu\delta\phi = i\tilde{\epsilon}\partial^\mu\phi + i(\partial^\mu\tilde{\epsilon}(x))\phi\,. \tag{1.30}$$

The second term breaks the local symmetry. Take, for example, free massless fermions:

$$\delta\mathcal{L} = \delta\bar{\psi}\frac{\delta\mathcal{L}}{\delta\bar{\psi}} + \frac{\delta\mathcal{L}}{\delta\partial\!\!\!/\psi}\delta\partial\!\!\!/\psi\,. \tag{1.31}$$

We have, as before,

$$\delta\bar{\psi}\frac{\delta\mathcal{L}}{\delta\bar{\psi}} = (-i\bar{\psi}\epsilon_a T^a)(i\partial\!\!\!/\psi) = \bar{\psi}\tilde{\epsilon}\partial\!\!\!/\psi, \tag{1.32}$$

but now

$$\frac{\delta\mathcal{L}}{\delta\partial\!\!\!/\psi}\delta\partial\!\!\!/\psi = (i\bar{\psi})\partial\!\!\!/(i\epsilon_a T^a\psi) = -\bar{\psi}\tilde{\epsilon}\partial\!\!\!/\psi - \bar{\psi}(\partial\!\!\!/\tilde{\epsilon})\psi\,. \tag{1.33}$$

Thus, the symmetry is violated:

$$\delta\mathcal{L} = -\bar{\psi}(\partial\!\!\!/\tilde{\epsilon})\psi \neq 0. \tag{1.34}$$

We learn that in a theory that includes only scalars and fermions, a local symmetry acting on these scalar and fermionic fields would forbid the kinetic terms. Can we still have a theory of dynamical scalars and fermions that is invariant under a local symmetry?

To do that, we have to "correct" for the extra term in Eq. (1.34). For the global symmetry case, $\delta\mathcal{L}$ vanishes since ϕ and $\partial_\mu\phi$ transform in the same way, and we constructed all the terms in $\mathcal{L}$ as products of ϕ and $\phi^\dagger$ or their derivatives. (Recall, ϕ and $\phi^\dagger$ transform in the opposite way). The way to solve the situation for the local case is to generalize the derivative, such that its generalized form transforms as the field: We need to replace $\partial^\mu\phi$ with a so-called "covariant" derivative $D^\mu\phi$ such that

$$\delta D^\mu\phi = i\tilde{\epsilon}D^\mu\phi. \tag{1.35}$$

The D^μ should have a term which cancels the $\partial^\mu\tilde{\epsilon}$ piece in (1.34). This is the case if D^μ transforms as

$$D^\mu \to e^{i\tilde{\epsilon}(x)}D^\mu e^{-i\tilde{\epsilon}(x)}\,. \tag{1.36}$$

Let us try

$$D^\mu = \partial^\mu + ig\tilde{A}^\mu\,, \tag{1.37}$$

where g is a fixed constant called "the coupling constant" and the transformation of A^μ_a is designed to cancel the extra piece in (1.34).

The construction that leads to a non-trivial local symmetry is to take A^μ_a to be a set of adjoint vector fields. We do not give here the full proof but only a brief explanation. Note that T_a are the generators of the symmetry group. Thus, the index a runs from 1 to the dimension of the group. For example, for $SU(N)$ the index a runs from 1 to N^2-1. Namely, there are N^2-1 copies of A^μ. This suggest that A^μ belongs to the adjoint representation.

The transformation law for A_a is directly obtained from Eq. (1.36):

$$\delta(\partial^\mu + ig\tilde{A}^\mu) = (1+i\tilde{\epsilon})(\partial^\mu + ig\tilde{A}^\mu)(1-i\tilde{\epsilon}) - (\partial^\mu + ig\tilde{A}^\mu) = ig\left(i[\tilde{\epsilon},\tilde{A}] - \frac{1}{g}\partial^\mu\tilde{\epsilon}\right). \tag{1.38}$$

Thus, $\tilde{A}^\mu$ transforms as follows:

$$\delta\tilde{A}^\mu = i[\tilde{\epsilon},\tilde{A}^\mu] - \frac{1}{g}\partial^\mu\tilde{\epsilon}. \tag{1.39}$$

Using the algebra of the group,

$$[T_a, T_b] = if_{abc}T_c \tag{1.40}$$

we can rewrite Eq. (1.39) as

$$\delta A^\mu_a = -f_{abc}\epsilon_b A^\mu_c - \frac{1}{g}\partial^\mu\epsilon_a. \tag{1.41}$$

Now we can check that our "guess" (1.37) indeed works. Remember:

$$\delta\phi = i\tilde{\epsilon}\phi\,. \tag{1.42}$$

Then

$$\begin{aligned}\delta D^\mu\phi &= \partial^\mu(\delta\phi) + ig\delta(\tilde{A}^\mu\phi) \qquad (1.43)\\ &= i\tilde{\epsilon}\partial^\mu\phi + i(\partial^\mu\tilde{\epsilon})\phi + ig\tilde{A}^\mu i\tilde{\epsilon}\phi + ig\{i[\tilde{\epsilon},\tilde{A}^\mu]\phi - \frac{1}{g}(\partial^\mu\tilde{\epsilon})\phi\} = i\tilde{\epsilon}D^\mu\phi.\end{aligned}$$

The covariant derivative of a field transforms in the same way as the field. We conclude that taking a Lagrangian $\mathcal{L}$ that is invariant under a global symmetry , and replacing ∂^μ with D^μ, makes $\mathcal{L}$ invariant under the corresponding local symmetry.

The field A^μ is called a gauge field. The constant g is the gauge coupling constant. To promote A^μ to a dynamical field, we must find the kinetic term of A^μ. We define

$$[D^\mu, D^\nu] = ig\tilde{F}^{\mu\nu}. \tag{1.44}$$

Then

$$\tilde{F}^{\mu\nu} = \partial^\mu \tilde{A}^\nu - \partial^\nu \tilde{A}^\mu + ig[\tilde{A}^\mu, \tilde{A}^\nu]. \tag{1.45}$$

Using the algebra, we can rewrite Eq. (1.45) as follows:

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g f_{abc} A_b^\mu A_c^\nu\,. \tag{1.46}$$

Using the transformation law of A^μ (1.39) we find the transformation law for $F^{\mu\nu}$:

$$\delta(\tilde{F}^{\mu\nu}) = i[\tilde{\epsilon}, \tilde{F}^{\mu\nu}]\,. \tag{1.47}$$

This transformation law implies that $F^{\mu\nu}$ belongs to the adjoint representation. We can thus obtain a singlet by multiplying it with $F_{\mu\nu}$. Since this is also a Lorentz singlet, we get the locally invariant kinetic term,

$$-\frac{1}{4} F_a^{\mu\nu} F_{a\mu\nu}, \tag{1.48}$$

where the $-1/4$ factor is a normalization factor. While a kinetic term is gauge invariant, a mass term $\frac{1}{2}m^2 A_a^\mu A_{a\mu}$ is not. You will prove it in your homework. Here we just emphasize the result: *Local invariance implies massless gauge fields.* These gauge bosons have only two degree of freedom and they transform as the adjoin repersentation of their corresponding group.

If the symmetry decomposes into several commuting factors, each factor has its own independent coupling constant. For example, if the symmetry is $SU(2) \times U(1)$, we have two independent coupling constants that we can denote as g for the $SU(2)$ and g' for the $U(1)$.

We now move to few examples.

1.2.7 QED

As our first example consider QED. This theory has an Abelian local symmetry, that is $U(1)$. This is the simplest case as $\tilde{\epsilon}$ is a commuting number and $\tilde{A}$ is a commuting field. Actually, A^μ is the photon field, and

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{1.49}$$

is the familiar field strength tensor of EM. The Lagrangian for free photon fields is then

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}. \tag{1.50}$$

Using the Euler–Lagrange equation, $\mathcal{L}$ gives the Maxwell equations.

Adding a charged fermion to the theory, we have

$$\mathcal{L}_{\rm QED} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{1.51}$$

where

$$D^\mu = \partial^\mu + ieqA^\mu\,. \tag{1.52}$$

Note that we identify the coupling constant $g = eq$, where q the electric charge of the fermions in units of the positron charge. For the electron $q = -1$. That is, in the units of the positron charge the "representation" of the electron under $U(1)_{\rm EM}$ is -1.

Expanding D^μ, we obtain the photon–fermion interaction term:

$$\mathcal{L}_{\rm int} = -eq\bar{\psi}\not{A}\psi \tag{1.53}$$

We learn that the coupling is proportional to the fermion charge and that the interaction is vector-like.

We return to QED later in the book.

QCD

For non-Abelian symmetries the situation is more complicated. The gauge bosons have self-interactions, namely, they are charged under the symmetry group. In QCD the gauge group is $SU(3)$. The gluon field G_a^μ is in the adjoint (octet) representation of the group, and

$$F_a^{\mu\nu} = \partial^\mu G_a^\nu - \partial^\nu G_a^\mu - g_s f_{abc} G_b^\mu G_c^\nu\,. \tag{1.54}$$

where g_s is the strong interaction constant. Note the extra term compared to the photon case, Eq. (1.49). This term gives rise to self interactions of the gluons. To see this, we inspect the kinetic term:

$$\mathcal{L}_{\rm kin} = -\frac{1}{4}F_a^{\mu\nu}F_{a\mu\nu} = \mathcal{L}_0 + g_s f_{abc}(\partial^\mu G_a^\nu)G_b^\mu G_c^\nu \; + g_s^2(f_{abc}G_b^\mu G_c^\nu)(f_{ade}G_d^\mu G_e^\nu)\,, \tag{1.55}$$

where $\mathcal{L}_0$ is the free field Lagrangian. The last two terms are the 3-point and 4-point gluon self interactions.

Adding a fermion to the theory we have

$$\mathcal{L}_{\rm QCD} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_a^{\mu\nu}F_{a\mu\nu}, \tag{1.56}$$

where

$$D^\mu = \partial^\mu + ig_s T^a G_a^\mu\,. \tag{1.57}$$

Expanding D^μ we obtain the gluon–fermion interaction terms:

$$\mathcal{L}_{\rm int} = -g_s\bar{\psi}T_a\not{G}^a\psi. \tag{1.58}$$

We learn that the coupling is proportional to the fermion representation, T_a, and that the strong interaction is a vector-like interaction. Note that fermions that are singlets under $SU(3)_C$ have $T_a = 0$ and thus they do not interact with the gluons.

We return to QCD later in the book.

1.3 Spontaneous Symmetry Breaking

Symmetries can be broken explicitly or spontaneously. By explicit breaking we refer to breaking by terms in the Lagrangian that is characterized by a small parameter (either a small dimensionless coupling, or small ratio between mass scales), so the symmetry is approximate.[3] Spontaneous breaking, however, refers to the case where the Lagrangian is symmetric, but the vacuum state is not. Before we get to the formal discussion, let us first explain this concept in more detail.

Symmetries of interactions are determined by the symmetry of the Lagrangian. The states, however, do not have to obey these symmetries. Consider, for example, the hydrogen atom. While the Lagrangian is invariant under rotations, an eigenstate does not have to be. Any state with a finite m quantum number is not invariant under rotation around the z axis. This is a general case when we have degenerate states. We can always find a basis of states that preserve the symmetry but there is the possibility to have another set that does not.

In perturbative QFT we always expand around the lowest energy state. This lowest state is called the "vacuum" state. When the vacuum state is degenerate, we can end up expanding around a state that does not conserve the initial symmetry of the theory. Then, it may seem that the symmetry is not there. Yet, there are features that testify to the fact that the symmetry is only spontaneously broken.

The name "spontaneously broken" indicates that there is no preference as to which of the states is chosen. The classical example is that of the hungry donkey. A donkey is in exactly the middle between two stacks of hay. Symmetry tells us that it costs the same energy to go to either stack. Thus, the donkey cannot choose and would not go anywhere! Yet, a real donkey would arbitrarily choose one side and go there to eat. We say that the donkey spontaneously breaks the symmetry between the two sides.

1.3.1 Global Discrete symmetries

Consider the following Lagrangian for a single real scalar field:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4. \tag{1.59}$$

[3]There is no sense in talking about a symmetry that is broken by an $\mathcal{O}(1)$ parameter, as such a situation is equivalent to the situation where there is no symmetry at all.

It is invariant under the transformation

$$\phi \to -\phi. \tag{1.60}$$

This symmetry would have been broken if we had a ϕ^3 term. The potential should be physically relevant, so we take $\lambda > 0$. But we can still have either $\mu^2 > 0$ or $\mu^2 < 0$. (μ^2 should be real for hermiticity of $\mathcal{L}$.) For $\mu^2 > 0$ we have an ordinary ϕ^4 theory with $|\mu|$ is the mass of ϕ. The case of interest for our purposes is

$$\mu^2 < 0. \tag{1.61}$$

The potential has two minima. They satisfy

$$0 = \frac{\partial V}{\partial \phi} = \phi(\mu^2 + \lambda\phi^2). \tag{1.62}$$

The solutions are

$$\phi_\pm = \pm\sqrt{\frac{-\mu^2}{\lambda}} \equiv \pm v. \tag{1.63}$$

The classical solution would be either ϕ_+ or ϕ_-. We say that ϕ acquires a *vacuum expectation value* (VEV):

$$\langle\phi\rangle \equiv \langle 0|\phi|0\rangle \neq 0. \tag{1.64}$$

Perturbative calculations should involve expansions around the classical minimum. Let us choose ϕ_+ (the two solutions are physically equivalent). Define a field ϕ' with a vanishing VEV:

$$\phi' = \phi - v. \tag{1.65}$$

In terms of ϕ', the Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi')(\partial^\mu\phi') - \frac{1}{2}(2\lambda v^2)\phi'^2 - \lambda v\phi'^3 - \frac{1}{4}\lambda\phi'^4, \tag{1.66}$$

where we used $\mu^2 = -\lambda v^2$ and discarded a constant term. Let us make several points:

a. The $\phi \to -\phi$ symmetry is *hidden*. It is *spontaneously broken* by our choice of the ground state $\langle\phi\rangle = +v$.

b. The theory is still described by two parameters only. The two parameters can be μ^2 and λ or v and λ.

c. The field ϕ' corresponds to a massive scalar field of mass $\sqrt{2}|\mu|$.

The fact that the three terms — the mass term, the trilinear terms and the quartic term — depend on only two parameters means that there is a relation between the three couplings. This relation is the clue that the symmetry is spontaneously, rather than explicitly, broken.

1.3.2 Global Continuous Symmetries

Consider a Lagrangian for a complex scalar field ϕ that is invariant under $U(1)$ transformations

$$\phi \to e^{i\theta}\phi. \tag{1.67}$$

It is given by

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2. \tag{1.68}$$

We can rewrite it in terms of two real scalar fields, π and σ, such that

$$\phi = (\sigma + i\pi)/\sqrt{2}. \tag{1.69}$$

Then

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \sigma)(\partial^\mu \sigma) + (\partial_\mu \pi)(\partial^\mu \pi)] - \frac{\mu^2}{2}(\sigma^2 + \pi^2) - \frac{\lambda}{4}(\sigma^2 + \pi^2)^2 \tag{1.70}$$

In term of the two real fields, the invariance is under $SO(2)$ transformations:

$$\begin{pmatrix} \sigma \\ \pi \end{pmatrix} \to \begin{pmatrix} \sigma \\ \pi \end{pmatrix}' = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sigma \\ \pi \end{pmatrix}. \tag{1.71}$$

The symmetry would have been broken if we had *e.g.* a $\sigma(\sigma^2 + \pi^2)$ term. Again, we take $\mu^2 < 0$. In the (σ, π) plane, there is a circle of radius v of minima of the potential:

$$\langle \sigma^2 + \pi^2 \rangle = v^2 = -\frac{\mu^2}{\lambda}. \tag{1.72}$$

Without loss of generality, we choose

$$\langle \sigma \rangle = v, \qquad \langle \pi \rangle = 0. \tag{1.73}$$

In terms of

$$\sigma' = \sigma - v, \qquad \pi' = \pi, \tag{1.74}$$

the scalar is written as

$$\phi = (\sigma' + v + i\pi')/\sqrt{2}. \tag{1.75}$$

The Lagrangian is then

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \sigma')(\partial^\mu \sigma') + (\partial_\mu \pi')(\partial^\mu \pi')] - \lambda v^2 \sigma'^2 - \lambda v \sigma'(\sigma'^2 + \pi'^2) - \frac{1}{4}\lambda(\sigma'^2 + \pi'^2)^2. \tag{1.76}$$

We used $\mu^2 = -\lambda v^2$ and discarded a constant term.

Note the following points:

a. The $SO(2)$ symmetry is spontaneously broken.

b. The Lagrangian describes one massive scalar σ' and one massless scalar π'.

c. In the symmetry limit we could not tell the two components of the complex scalar field. After the breaking they are different. For example, they have different masses.

d. The spontaneous breaking of a continuous global symmetry is always accompanied by the appearance of a massless scalar called *Goldstone Boson*. We discuss it in more detauils in appendix 1.B.

e. We chose a basis by assigning the VEV to the real component of the field. This is an arbitrary choice. We made it since it is convenient.

The Lagrangian (1.76) is not the most general Lagrangian without an $SO(2)$ symmetry. The three couplings obey a relation that signals spontaneous symmetry breaking.

1.3.3 Fermion Masses

Spontaneous symmetry breaking can give masses to chiral fermions, provided that these fermions are in a vector-like representation of the unbroken subgroup. Consider a model with a $U(1)$ symmetry. The particle content consists of two chiral fermions and a complex scalar with the following $U(1)$ charges:

$$q\,(\psi_L) = 1, \qquad q\,(\psi_R) = 2, \qquad q\,(\phi) = 1. \tag{1.77}$$

The most general Lagrangian we can write is

$$\mathcal{L} = \mathcal{L}_{\text{kin}} - V(\phi) - Y\phi\overline{\psi_R}\psi_L + \text{h.c.}, \tag{1.78}$$

where $V(\phi)$ is the scalar potential that describes the mass and self interaction terms of the scalar. In particular, since the fermions are charged and chiral, we cannot write mass terms for them.

We assume that the scalar potential is such that $\langle\phi\rangle = v \neq 0$, and define

$$\phi = (h + v + i\xi)/\sqrt{2}, \tag{1.79}$$

so that h and ξ do not acquire VEVs. Expanding around the vacuum we find

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + V(h) - \frac{Yv}{\sqrt{2}}\overline{\psi_R}\psi_L - \frac{Y(h + i\xi)}{\sqrt{2}}\overline{\psi_R}\psi_L + \text{h.c.}. \tag{1.80}$$

Note the following points:

a. The fermion acquires a Dirac mass, $m_\psi = Yv/\sqrt{2}$.

b. The two real scalar fields, h and ξ, couple to the fermion in the same way. The Yukawa coupling is proportional to the fermion mass.

1.3.4 Local symmetries: the Higgs mechanism

In this subsection we discuss spontaneous breaking of local symmetries. We demonstrate it by studying a $U(1)$ gauge symmetry. We will find out that breaking of a local symmetry results in mass terms for the gauge bosons that correspond to the broken generators. It is a somewhat surprising result, since the spontaneous breaking of a global symmetry gives massless Goldstone boson. In the case of a local symmetry, these would-be Goldstone bosons are "eaten" by the gauge bosons such that the gauge bosons have longitudinal components.

Consider the following Lagrangian for a single complex scalar field ϕ:

$$\mathcal{L} = [(\partial_\mu - igV_\mu)\phi^\dagger][(\partial^\mu + igV^\mu)\phi] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2. \tag{1.81}$$

This Lagrangian is invariant under a local $U(1)$ symmetry,

$$\phi \to e^{i\epsilon(x)}\phi, \qquad V_\mu \to V_\mu - \frac{1}{g}\partial_\mu\epsilon(x). \tag{1.82}$$

Both λ and μ^2 are real, with $\lambda > 0$ and $\mu^2 < 0$. Consequently, ϕ acquires a VEV,

$$\langle\phi\rangle = \frac{v}{\sqrt{2}}, \qquad v^2 = -\frac{\mu^2}{\lambda}. \tag{1.83}$$

Up to a constant term, the scalar potential can be written as follows:

$$V = \lambda\left(\phi^\dagger\phi - \frac{v^2}{2}\right)^2. \tag{1.84}$$

We choose the real component of ϕ to carry the VEV, $\langle\mathcal{I}m\ \phi\rangle = 0$, and define

$$\phi = \frac{1}{\sqrt{2}}(v + \eta + i\zeta) \tag{1.85}$$

with

$$\langle\eta\rangle = \langle\zeta\rangle = 0. \tag{1.86}$$

Furthermore, it is convenient to choose a gauge $\epsilon(x) = -\zeta(x)/v$. Since the symmetry is broken, a gauge choice does change the way we write the Lagrangian. It is this gauge choice that is best suited for our purposes. In this gauge

$$\phi \to \phi' = \frac{1}{\sqrt{2}}(\eta + v), \qquad V_\mu \to V'_\mu = V_\mu + \frac{1}{gv}\partial_\mu\zeta. \tag{1.87}$$

Then

$$\begin{aligned}\mathcal{L} \ = \ & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \frac{1}{2}(g^2v^2)V'_\mu V'^\mu - \frac{1}{2}(2\lambda v^2)\eta^2 \\ & + \frac{1}{2}g^2V'_\mu V'^\mu\eta(2v+\eta) - \lambda v\eta^3 - \frac{1}{4}\lambda\eta^4.\end{aligned} \tag{1.88}$$

Note the following points concerning this specific model:

1. The $U(1)$ symmetry is spontaneously broken.

2. The Lagrangian describes a massive vector boson with $m_V = gv$. In the limit $g \to 0$ we have $m_V \to 0$. That is, the longitudinal component is the Goldstone boson as expected.

3. The ζ field was "eaten" in order to give mass to the gauge boson. The number of degrees of freedom did not change: instead of the scalar ζ, we have the longitudinal component of a massive vector boson.

4. η is a massive scalar with $m_\eta = \sqrt{2\lambda}\, v$. We call it "a Higgs boson".

The following lessons are generic to all cases of spontaneous breaking of a local symmetry:

1. Spontaneous symmetry breaking gives masses to the gauge bosons related to the broken generators.

2. Gauge bosons related to an unbroken subgroup will remain massless, because their masslessness is protected by the symmetry.

3. The Brout-Englert-Higgs (BEH) field, that is the field that acquires a VEV, must be a scalar field. Otherwise its VEV would break Lorentz invariance.

4. Spontaneous breaking of local symmetry can give masses also to fermions, as is the case for global symmetry.

5. In the physical gauge, the coupling of the longitudinal part of the gauge boson to the fermion is proportional to the mass, while that of the transverse component is proportional to the gauge coupling.

1.4 Summary

The main consequences of the various types of symmetries are summarized in Table 1.2.

1.5 Model building

Now we are ready to explain how we are constract a Lagrangian. We first provide the following three ingredients:

(i) The symmetry;

(ii) The transformation properties of the fermions and scalars;

(iii) The pattern of spontaneous symmetry breaking (SSB).

Table 1.2: Symmetries

Type	Consequences
Spacetime	Conservation of energy, momentum, angular momentum
Discrete	Selection rules
Global (exact)	Conserved charges
Global (spon. broken)	Massless scalars
Local (exact)	Interactions, massless spin-1 mediators
Local (spon. broken)	Interactions, massive spin-1 mediators

Then we write the most general renormalizable Lagrangian that is invariant under the symmetry.

Such a Lagrangian has a finite number of parameters that we need to determined by experiment. In priciple in a theory with N parameters we need to perform N different measurments to determine the parametrs and from the $N+1$ one we can test the theory. In the following we will give example of models that are all based on the above principle.

1.5.1 Parameters counting

Before we go on to study the SM in detail, we explain how to identify the number of physical parameter in any model. The point is that in general there are parameters in a theory that are not physical. That is, there is a basis where they are identically zero. Of course, it is important to identify the physical parameters in any model in order to probe and check it. Below we explain hwo to determined the number of physical parameter.

We start with a very simple example. Consider a hydrogen atom in a uniform magnetic field. Before turning on the magnetic field, the hydrogen atom is invariant under spatial rotations, which are described by the $SO(3)$ group. Furthermore, there is an energy eigenvalue degeneracy of the Hamiltonian: states with different angular momenta have the same energy. This degeneracy is a consequence of the symmetry of the system.

When magnetic field is added to the system, it is conventional to pick a direction for the magnetic field without a loss of generality. Usually, we define the positive z direction to be the direction of the magnetic field. Consider this choice more carefully. A generic uniform magnetic field would be described by three real numbers: the three components of the magnetic field. The magnetic field breaks the $SO(3)$ symmetry of the hydrogen atom system down to an $SO(2)$ symmetry of rotations in the plane perpendicular to the magnetic field. The one generator of the $SO(2)$ symmetry is the only valid symmetry generator now; the remaining two $SO(3)$ generators in the orthogonal planes are broken. These broken symmetry generators allow us to rotate the system such that the magnetic field points in the z direction:

$$O_{xz}O_{yz}(B_x, B_y, B_z) = (0, 0, B'_z), \tag{1.89}$$

where O_{xz} and O_{yz} are rotations in the xz and yz planes respectively. The two broken generators were used to rotate away two unphysical parameters, leaving us with one physical parameter, the magnitude of the magnetic field. That is, when turning on the magnetic field, all measurable quantities in the system depend only on one new parameter, rather than the naïve three.

The results described above are more generally applicable. Particularly, they are useful in studying the flavor physics of quantum field theories. Consider a gauge theory with matter content. This theory always has kinetic and gauge terms, which have a certain global symmetry, G_f, on their own. In adding a potential that respect the imposed gauge symmetries, the global symmetry may be broken down to a smaller symmetry group. In breaking the global symmetry, there is an added freedom to rotate away unphysical parameters, as when a magnetic field is added to the hydrogen atom system.

In order to analyze this process, we define a few quantities. The added potential has coefficients that can be described by N_{general} parameters in a general basis. The global symmetry of the entire model, H_f, has fewer generators than G_f and we call the difference in the number of generators N_{broken}. Finally, the quantity that we would ultimately like to determine is the number of parameters affecting physical measurements, N_{phys}. These numbers are related by

$$N_{\text{phys}} = N_{\text{general}} - N_{\text{broken}}. \tag{1.90}$$

Furthermore, the rule in (**??**) applies separately for both real parameters (masses and mixing angles) and phases. A general, $n \times n$ complex matrix can be parametrized by n^2 real parameters and n^2 phases. Imposing restrictions like Hermiticity or unitarity reduces the number of parameters required to describe the matrix. A Hermitian matrix can be described by $n(n+1)/2$ real parameters and $n(n-1)/2$ phases, while a unitary matrix can be described by $n(n-1)/2$ real parameters and $n(n+1)/2$ phases.

While at this point the above may be a bit absract, we will use it in specific examples below.

Appendix

1.A Noether's theorem

Let $\phi_i(x)$ be a set of fields, $i = 1, 2, \ldots, N$, on which the Lagrangian $\mathcal{L}(\phi)$ depends. Consider an infinitesimal change $\delta\phi_i$ in the fields. This is a symmetry if

$$\mathcal{L}(\phi + \delta\phi) = \mathcal{L}(\phi). \tag{1.91}$$

Since $\mathcal{L}$ depends only on ϕ and $\partial_\mu\phi$, we have

$$\delta\mathcal{L}(\phi) = \mathcal{L}(\phi + \delta\phi) - \mathcal{L}(\phi) = \frac{\delta\mathcal{L}}{\delta\phi_j}\delta\phi_j + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_j)}\delta(\partial_\mu\phi_j). \tag{1.92}$$

The relation between symmetries and conserved quantities is expressed by Noether's theorem: *To every symmetry in the Lagrangian there corresponds a conserved current.* To prove the theorem, one uses the equation of motion:

$$\partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_j)} = \frac{\delta\mathcal{L}}{\delta\phi_j}. \tag{1.93}$$

The condition for a symmetry is then

$$\partial_\mu\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_j)}\right]\delta\phi_j + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_j)}\delta(\partial_\mu\phi_j) = \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_j)}\delta\phi_j\right] = 0. \tag{1.94}$$

Thus, the conserved current ($\partial_\mu J^\mu = 0$) is

$$J^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_j)}\delta\phi_j. \tag{1.95}$$

The conserved charge ($\dot{Q} = 0$) is given by

$$Q = \int d^3x\ J_0(x). \tag{1.96}$$

We are interested in unitary transformations,

$$\phi \to \phi' = U\phi, \qquad UU^\dagger = \mathbf{1}. \tag{1.97}$$

Here ϕ is a vector with N components, U is an $N \times N$ unitary matrix, and $\mathbf{1}$ stands for the $N \times N$ unit matrix. The reason that we are interested in unitary transformation is that they keep the canonical form of the kinetic terms. A unitary matrix can always be written as

$$U = e^{i\epsilon_a T^a}, \qquad T^{a\dagger} = T^a, \tag{1.98}$$

where ϵ_a are numbers and T^a are $N \times N$ *hermitian* matrices. For infinitesimal transformation ($\epsilon_a \ll 1$),

$$\phi' \approx (1 + i\epsilon_a T^a)\phi \implies \delta\phi = i\epsilon_a T^a \phi. \tag{1.99}$$

A *global* symmetry is defined by $\epsilon_a = \text{const}(x)$. For *internal* symmetry, $\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi)$. Thus, for an internal global symmetry,

$$\delta(\partial_\mu \phi) = i\epsilon_a T^a \partial_\mu \phi. \tag{1.100}$$

In the physics jargon, we say that $\partial_\mu \phi$ transforms like ϕ. The conserved current is

$$J^a_\mu = i\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} T^a \phi. \tag{1.101}$$

The matrices T^a form am algebra of the symmetry group,

$$[T^a, T^b] = i f^{abc} T^c. \tag{1.102}$$

The charges that are associated with these symmetry also satisfy the algebra:

$$[Q^a, Q^b] = i f^{abc} Q^c. \tag{1.103}$$

Note that T^a are $N \times N$ matrices, while Q^a are operators in the Hilbert space where the theory lives.

1.A.1 Free massless scalars

Consider N real, free, massless scalar fields ϕ_j:

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi). \tag{1.104}$$

The index j is called flavor index, and here and in what follows the summation over it is implicit. The theory is invariant under the group of orthogonal $N \times N$ matrices, which is the group of rotations in an N-dimensional real vector space. This group is called $SO(N)$. The generators T^a_{jk} are the $N(N-1)/2$ independent antisymmetric $N \times N$ imaginary matrices, that is

$$\delta\phi = i\epsilon_a T^a \phi, \tag{1.105}$$

with T^a antisymmetric and imaginary. (It must be imaginary so that $\delta\phi$ is real.) For an internal global symmetry, the spacetime and the internal symmetry group are unrelated and thus $\partial_\mu \phi$ transforms like ϕ, that is

$$\delta(\partial_\mu \phi) = i\epsilon_a T^a (\partial_\mu \phi). \tag{1.106}$$

Then,

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi) = 0 + (\partial_\mu\phi)\, i\epsilon_a T^a(\partial_\mu\phi) = 0, \tag{1.107}$$

where in the last step we used the antisymmetry of T^a in its flavor indeces. The associated conserved current is then

$$J^a_\mu = i(\partial_\mu\phi)_j T^a\phi. \tag{1.108}$$

The $SO(N)$ group that we have found is the largest possible symmetry for a Lagrangian involving N real scalar fields. In general, mass and interaction terms will reduce the symmetry to a subgroup of $SO(N)$. In the presence of such $SO(N)$ breaking terms, an $SO(N)$ transformation means a change of basis where the Lagrangian is not invariant under. The $SO(N)$ groups have no important role in the SM. We will mention $SO(4)$ when we discuss the Higgs mechanism. In a more advanced course, you may encounter $SO(10)$ as a grand unifying group.

1.A.2 Free massless Dirac fermions

Consider N free, massless, spin-$\frac{1}{2}$, four-component fermion fields ψ_j:

$$\mathcal{L}(\psi) = i\overline{\psi}\partial\!\!\!/\psi \tag{1.109}$$

The ψ's are necessarily complex because of the Dirac structure. The theory is invariant under the group of unitary $N \times N$ matrices. This group is called $U(N) = SU(N) \times U(1)$. The generators are the independent N^2 Hermitian matrices, where the $N^2 - 1$ traceless ones generate the $SU(N)$ group:

$$\delta\psi = i\epsilon_a T^a\psi \tag{1.110}$$

where T^a is a general Hermitian matrix. The transformation law of $\bar{\psi}$ is as follows:

$$\delta\overline{\psi} = \delta(\psi^\dagger\gamma^0) = (i\epsilon_a T^a\psi)^\dagger\gamma^0 = \psi^\dagger(-i)\epsilon_a^* T^{a\dagger}\gamma^0 = -i\psi^\dagger\gamma^0\epsilon_a T^a = -i\bar{\psi}\epsilon_a T^a \tag{1.111}$$

because T^a are Hermitian. The T^a matrices and the γ^μ matrices commute because they act on different spaces. We also need to derive the transformation property of the derivative. For an internal symmetry,

$$\delta\partial\!\!\!/\psi = \partial\!\!\!/\delta\psi. \tag{1.112}$$

For an internal global symmetry (ϵ_a independent of x)

$$\delta\partial\!\!\!/\psi = i\epsilon_a T^a\partial\!\!\!/\psi. \tag{1.113}$$

We then have

$$\delta\mathcal{L} = \delta\bar{\psi}\frac{\delta\mathcal{L}}{\delta\bar{\psi}} + \delta\partial\!\!\!/\bar{\psi}\frac{\delta\mathcal{L}}{\delta\partial\!\!\!/\bar{\psi}} + \frac{\delta\mathcal{L}}{\delta\psi}\delta\psi + \frac{\delta\mathcal{L}}{\delta\partial\!\!\!/\psi}\delta\partial\!\!\!/\psi\,. \tag{1.114}$$

We use

$$
\begin{aligned}
\delta\bar{\psi}\frac{\delta\mathcal{L}}{\delta\bar{\psi}} &= (-i\bar{\psi}\epsilon_a T^a)(i\partial\!\!\!/\psi) = \bar{\psi}\epsilon_a T^a \partial\!\!\!/\psi, \\
\delta\partial\!\!\!/\bar{\psi}\frac{\delta\mathcal{L}}{\delta\partial\!\!\!/\bar{\psi}} &= 0, \\
\frac{\delta\mathcal{L}}{\delta\psi}\delta\psi &= 0, \\
\frac{\delta\mathcal{L}}{\delta\partial\!\!\!/\psi}\delta\partial\!\!\!/\psi &= (i\bar{\psi})(i\epsilon_a T^a \partial\!\!\!/\psi) = -\bar{\psi}\epsilon_a T^a \partial\!\!\!/\psi,
\end{aligned}
\tag{1.115}
$$

and find that $\delta\mathcal{L} = 0$. The corresponding conserved current is

$$J^a_\mu = \bar{\psi}\gamma_\mu T^a \psi. \tag{1.116}$$

The charge associated with $U(1)$, $\int d^3x\psi^\dagger\psi$, is the fermion number operator.

In fact, the symmetry of N free massless Dirac fermions is larger: $[U(N)]^2$, rather than just a single $U(N)$. To understand this point, let us define the following projection operators:

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5). \tag{1.117}$$

The four-component Dirac fermion can be decomposed to a left-handed and a right-handed (L and R) Weyl spinor fields,

$$\psi_L \;=\; P_-\psi, \quad \psi_R = P_+\psi, \quad \overline{\psi_L} = \overline{\psi}P_+, \quad \overline{\psi_R} = \overline{\psi}P_-. \tag{1.118}$$

The Lagrangian of Eq. (1.109) can be written as follows:

$$\mathcal{L}(\psi) = i\overline{\psi_{Lj}}\partial\!\!\!/\psi_{Lj} + i\overline{\psi_{Rj}}\partial\!\!\!/\psi_{Rj}. \tag{1.119}$$

It follows straightforwardly that this Lagrangian has a symmetry under independent rotations of the left-chirality and right-chirality fields, namely

$$[SU(N) \times U(1)]_L \times [SU(N) \times U(1)]_R. \tag{1.120}$$

The conserved currents are

$$J^a_{L\mu} = \overline{\psi_L}\gamma_\mu T^a_L \psi_L, \quad J^a_{R\mu} = \overline{\psi_R}\gamma_\mu T^a_R \psi_R. \tag{1.121}$$

The symmetry (1.120) is the largest possible symmetry group for a Lagrangian of N Dirac fermions. In general, mass and interaction terms break the symmetry to a subgroup of (1.120). In the presence of such symmetry breaking terms, an $[SU(N)\times U(1)]_L \times [SU(N)\times U(1)]_R$ transformation means a change of basis.

The ψ_L and ψ_R are eigenstates of the *chirality* operator γ_5 (with eigenvalues -1 and $+1$, respectively). For massless fields, they are also *helicity* eigenstates. To see that, consider a plane wave traveling in the z direction, $p^0 = p^3$, and $p^1 = p^2 = 0$. The Dirac equation in momentum space is $\not{p}\psi = 0$, so $p(\gamma^0 - \gamma^3)\psi = 0$, or

$$\gamma^0\psi = \gamma^3\psi. \tag{1.122}$$

The spin angular momentum in the z direction is

$$J^3 = \sigma^{12}/2 = i\gamma^1\gamma^2/2. \tag{1.123}$$

Then

$$\begin{aligned} J^3\psi_L &= \frac{i}{2}\gamma^1\gamma^2\psi_L = \frac{i}{2}\gamma^0\gamma^0\gamma^1\gamma^2\psi_L = \frac{i}{2}\gamma^0\gamma^1\gamma^2\gamma^0\psi_L = \frac{i}{2}\gamma^0\gamma^1\gamma^2\gamma^3\psi_L = \\ &= \frac{1}{2}\gamma_5\psi_L = -\frac{1}{2}\psi_L. \end{aligned} \tag{1.124}$$

We learn that ψ_L describes a massless particle with helicity $-1/2$. Similarly, ψ_R describes a massless particle with helicity $+1/2$.

1.A.3 Free massive Dirac fermions

Consider N free spin-$\frac{1}{2}$, four-component fermion fields ψ_i with *universal* mass m:

$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi = i\bar{\psi}_L\not{\partial}\psi_L + i\bar{\psi}_R\not{\partial}\psi_R - m\bar{\psi}_L\psi_R - m\bar{\psi}_R\psi_L, \tag{1.125}$$

where the "flavor" index j is omitted. This Lagrangian is invariant under the symmetry in which the L and R fields rotate together, $U(N) = SU(N) \times U(1)$. We learn that a universal mass term breaks $[U(N)]^2 \to U(N)$. This $U(N)$ symmetry is actually the one identified in Eq. (1.110), with the conserved current of Eq. (1.116).

For a general, non-universal, mass term the symmetry is smaller. By performing an $[SU(N) \times U(1)]_L \times [SU(N) \times U(1)]_R$ transformation ($M \to V_L M V_R^\dagger$, where M is the mass matrix and $V_{L,R}$ are unitary matrices), We can always choose a basis where the mass matrix is diagonal and real:

$$\mathcal{L} = i\bar{\psi}_L\not{\partial}\psi_L + i\bar{\psi}_R\not{\partial}\psi_R - m_i(\bar{\psi}_{Li}\psi_{Ri} + \bar{\psi}_{Ri}\psi_{Li}) = \bar{\psi}_i(i\not{\partial} - m_i)\psi_i. \tag{1.126}$$

The symmetry is $[U(1)]^N$. The conserved currents are

$$J_{\mu i} = \bar{\psi}_i\gamma_\mu\psi_i, \tag{1.127}$$

and the corresponding conserved charges are simply the fermion number for each fermion separately,

$$Q_i = \int d^3x\psi_i^\dagger\psi_i. \tag{1.128}$$

Within the SM, this is the case of lepton flavor symmetry, which ensures that the flavor of the leptons (namely, e, μ and τ) is conserved. This is also the approximate flavor symmetry of the quark sector that is conserved by the strong and the EM forces.

The Lagrangian of Eq. (1.126) is the most general renormalizable Lagrangian that includes only Direc fermion fields [see Eq. (1.4)]. Thus, in the absence of Yukawa interactions, any renormalizable Lagrangian that has N Dirac-fermions fields has an accidental $[U(1)]^N$ symmetry.

1.B The Goldstone Theorem

The spontaneous breaking of a global continuous symmetry is accompanied by massless scalars. Their number and QN's equal those of the broken generators.

Consider the Lagrangian

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) \tag{1.129}$$

where ϕ is some multiplet of scalar fields, and $\mathcal{L}(\phi)$ is invariant under some symmetry group:

$$\delta\phi = i\epsilon_a T^a \phi, \tag{1.130}$$

where the T^a are imaginary antisymmetric matrices.

We want to perturb around a minimum of the potential $V(\phi)$. We expect the ϕ field to have a VEV, $\langle\phi\rangle = v$, which minimizes V. We define

$$V_{j_1 \cdots j_n}(\phi) = \frac{\partial^n}{\partial\phi_{j_1} \cdots \partial\phi_{j_n}} V(\phi). \tag{1.131}$$

The condition that v is an *extremum* of $V(\phi)$ reads

$$V_j(v) = 0. \tag{1.132}$$

The condition for a *minimum* at v is, in addition to (1.132),

$$V_{jk}(v) \geq 0. \tag{1.133}$$

The second derivative matrix $V_{jk}(v)$ is the scalar mass-squared matrix. We can see that by expanding $V(\phi)$ in a Taylor series in the shifted fields $\phi' = \phi - v$ and noting that the mass term is $\frac{1}{2}V_{jk}(v)\phi'_j\phi'_k$.

Now we check for the behavior of the VEV v under the transformation (1.130). There are two cases. If

$$T_a v = 0 \tag{1.134}$$

for all a, the symmetry is not broken. This is certainly what happens if $v = 0$. But (1.134) is the more general statement that the vacuum does not carry the charge T_a, so the charge cannot disappear into the vacuum. However, it is also possible that

$$T_a v \neq 0 \quad \text{for some } a. \tag{1.135}$$

Then the charge T_a can disappear into the vacuum even though the associated current is conserved. This is spontaneous symmetry breaking.

Often there are some generators of the original symmetry that are spontaneously broken while others are not. The set of generators satisfying (1.134) is closed under commutation (because

$T_a v = 0$ and $T_b v = 0 \Longrightarrow [T_a, T_b]v = 0$) and generates the unbroken subgroup of the original symmetry group.

Because V is invariant under (1.130), we can write

$$V(\phi + \delta\phi) - V(\phi) = iV_k(\phi)\epsilon_a(T^a)_{kl}\phi_l = 0. \tag{1.136}$$

If we differentiate with respect to ϕ_j, we get

$$V_{jk}(\phi)(T^a)_{kl}\phi_l + V_k(\phi)(T^a)_{kj} = 0. \tag{1.137}$$

Setting $\phi = v$ in (1.137), we find that the second term drops out because of (1.132), and we obtain

$$V_{jk}(v)(T^a)_{kl}v_l = 0. \tag{1.138}$$

But $V_{jk}(v)$ is the mass-squared matrix M^2_{jk} for the scalar fields, so we can rewrite (1.138) in a matrix form as

$$M^2 T^a v = 0. \tag{1.139}$$

For T^a in the unbroken subgroup, (1.139) is trivially satisfied. But if $T^a v \neq 0$, (1.139) requires that $T^a v$ is an eigenvector of M^2 with eigenvalue zero. It corresponds to a massless boson field given by

$$\phi^T T^a v \tag{1.140}$$

which is called a Goldstone boson.

Chapter 2

The Leptonic Standard Model

2.1 Defining the Leptonic Standard Model (LSM)

We now have at our disposal the tools that are required in order to present the Standard Model (SM). We start with the lepton sector. Later we introduce the complete model, including quarks. For this section the discussion in Burgess and Moore [1] (section 2) is similar to ours.

In order to define the SM, we need to provide the following three ingredients:

(*i*) The symmetry;

(*ii*) The transformation properties of the fermions and scalars;

(*iii*) The pattern of spontaneous symmetry breaking (SSB).

The "leptonic SM" (LSM) is defined as follows:

(*i*) The symmetry is a local

$$SU(2)_L \times U(1)_Y. \tag{2.1}$$

(*ii*) There are three fermion generations, each consisting of two different lepton representations:

$$L_{Li}(2)_{-1/2}, \qquad E_{Ri}(1)_{-1}, \qquad i = 1, 2, 3. \tag{2.2}$$

There is a single scalar multiplet:

$$\phi(2)_{+1/2}. \tag{2.3}$$

(*iii*) The pattern of spontaneous symmetry breaking is as follows:

$$SU(2)_L \times U(1)_Y \to U(1)_{\rm EM}, \tag{2.4}$$

where $Q_{\rm EM} = T_3 + Y$.

We use the notation $(N)_Y$ such that N is the irreducible representation (irrep) under $SU(2)_L$ and Y is the hypercharge (the charge under $U(1)_Y$). What we mean by Eq. (2.2) is that there are nine Weyl fermion degrees of freedom that are grouped into three copies ("generations") of the same gauge representations. The three fermionic degrees of freedom in each generation form an $SU(2)$-doublet (of hypercharge $-1/2$) and an $SU(2)$-singlet (of hypercharge -1).

The most general renormalizable Lagrangian with scalar and fermion fields can be decomposed into

$$\mathcal{L} = \mathcal{L}_{\rm kin} + \mathcal{L}_\psi + \mathcal{L}_{\rm Yuk} + \mathcal{L}_\phi. \tag{2.5}$$

Here $\mathcal{L}_{\rm kin}$ describes the free propagation in spacetime, as well as the gauge interactions, $\mathcal{L}_\psi$ gives the fermion mass terms, $\mathcal{L}_{\rm Yuk}$ describes the Yukawa interactions, and $\mathcal{L}_\phi$ gives the scalar potential. It is now our task to find the specific form of the Lagrangian made of the L_{Li}, E_{Ri} [Eq. (2.2)] and ϕ [Eq. (2.3] fields, subject to the gauge symmetry (2.1) and leading to the SSB of Eq. (2.4).

2.2 The LSM Lagrangian

2.2.1 $\mathcal{L}_{\rm kin}$ and the gauge symmetry

The gauge group is given in Eq. (2.1). It has four generators: three T_a's that form the $SU(2)$ algebra and a single Y that generates the $U(1)$ algebra:

$$[T_a, T_b] = i\epsilon_{abc}T_c, \qquad [T_a, Y] = 0. \tag{2.6}$$

Thus there are two independent coupling constants in $\mathcal{L}_{\rm kin}$: there is a single g for all the $SU(2)$ couplings and a different one, g', for the $U(1)$ coupling. The $SU(2)$ couplings must all be the same because they mix with one another under global $SU(2)$ rotations. But the $U(1)$ coupling can be different because the generator Y never appears as a commutator of $SU(2)$ generators.

The local symmetry requires four gauge bosons, three in the adjoint representation of the $SU(2)$ and one related to the $U(1)$ symmetry:

$$W_a^\mu(3)_0, \quad B^\mu(1)_0. \tag{2.7}$$

The corresponding field strengths are given by

$$\begin{aligned} W_a^{\mu\nu} &= \partial^\mu W_a^\nu - \partial^\nu W_a^\mu - g\epsilon_{abc}W_b^\mu W_c^\nu, \\ B^{\mu\nu} &= \partial^\mu B^\nu - \partial^\nu B^\mu. \end{aligned} \tag{2.8}$$

The covariant derivative is

$$D^\mu = \partial^\mu + igW_a^\mu T_a + ig'YB^\mu. \tag{2.9}$$

$\mathcal{L}_{kin}$ includes the kinetic terms of all the fields:

$$\mathcal{L}_{\rm kin} = -\frac{1}{4}W_a^{\mu\nu}W_{a\mu\nu} - \frac{1}{4}B^{\mu\nu}B_{\mu\nu} - i\overline{L_{Li}}\not{D}L_{Li} - i\overline{E_{Ri}}\not{D}E_{Ri} - (D^\mu\phi)^\dagger(D_\mu\phi). \tag{2.10}$$

For the $SU(2)_L$ doublets $T_a = \frac{1}{2}\sigma_a$ (σ_a are the Pauli matrices), while for the $SU(2)_L$ singlets, $T_a = 0$. Explicitly,

$$\begin{aligned} D^\mu L_L &= \left(\partial^\mu + \frac{i}{2} g W_a^\mu \sigma_a - \frac{i}{2} g' B^\mu\right) L_L, \\ D^\mu E_R &= \left(\partial^\mu - i g' B^\mu\right) E_R, \\ D^\mu \phi &= \left(\partial^\mu + \frac{i}{2} g W_a^\mu \sigma_a + \frac{i}{2} g' B^\mu\right) \phi. \end{aligned} \tag{2.11}$$

For $SU(2)_L$ triplets, $(T_a)_{bc} = \epsilon_{abc}$, which has already been used in writing (2.8).

We remind the reader that in $\mathcal{L}_{\rm LSM}$ there are no mass terms for the gauge bosons, as that would violate the gauge symmetry.

Where is QED in all of this? We defined Q, the generator of $U(1)_{\rm EM}$, as follows:

$$Q = T_3 + Y. \tag{2.12}$$

Let us write explicitly the two components of $SU(2)_L$ doublets:

$$L_{L1} = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}. \tag{2.13}$$

Then

$$Q\nu_{eL} = 0, \qquad Qe_L^- = -e_L^-, \qquad Qe_R^- = -e_R^-, \qquad Q\phi^+ = \phi^+, \qquad Q\phi^0 = 0. \tag{2.14}$$

For now, ν_{eL} (ϕ^+) and e_L (ϕ^0) stand for, respectively, the $T_3 = +1/2$ and $T_3 = -1/2$ components of the lepton (scalar) doublet. If $SU(2)_L \times U(1)_Y$ were an exact symmetry of Nature, there would be no way of distinguishing particles of different electric charges in the same $SU(2)_L$ multiplet. We make this choice as it will give us the correct QED after SSB as we see next.

2.2.2 $\mathcal{L}_\psi = 0$

There are no mass terms the fermions in the LSM. We cannot write Dirac mass terms for the fermions because they are assigned to chiral representations of the gauge symmetry. We cannot write Majorana mass terms for the fermions because they all have $Y \neq 0$.

2.2.3 $\mathcal{L}_{\rm Yuk}$

The Yukawa part of the Lagrangian is given by

$$\mathcal{L}_{\rm Yuk} = Y_{ij}^e \overline{L_{Li}} E_{Rj}\, \phi + {\rm h.c.}, \tag{2.15}$$

where $i, j = 1, 2, 3$ are flavor indices. The Yukawa matrix Y^e is a general complex 3×3 matrix of dimensionless couplings. Without loss of generality, we can choose a basis where Y^e is diagonal and real (see the discussion in subsection 2.5.1):

$$Y^e = {\rm diag}(y_e, y_\mu, y_\tau). \tag{2.16}$$

2.2.4 $\mathcal{L}_\phi$ and spontaneous symmetry breaking

The Higgs potential, which leads to the spontaneous symmetry breaking, is given by

$$\mathcal{L}_\phi = -\mu^2 \phi^\dagger \phi - \lambda \left(\phi^\dagger \phi\right)^2 . \tag{2.17}$$

The quartic coupling λ is dimensionless and real, and has to be positive for the potential to be bounded from below. The quadratic coupling μ^2 has mass dimension 2 and is real. It can a-priori have either sign, but if the gauge symmetry is to be spontaneously broken, Eq. (2.4), then we must take $\mu^2 < 0$. Defining

$$v^2 = -\frac{\mu^2}{\lambda}, \tag{2.18}$$

we can rewrite Eq. (2.17) as follows (up to a constant term):

$$\mathcal{L}_\phi = -\lambda \left(\phi^\dagger \phi - \frac{v^2}{2}\right)^2 . \tag{2.19}$$

The scalar potential (2.19) implies that the scalar field acquires a VEV, $\langle\phi\rangle = \frac{v}{\sqrt{2}}$. This VEV breaks the $SU(2) \times U(1)$ symmetry down to a $U(1)$ subgroup. We choose the unbroken subgroup to be $U(1)_{\rm EM}$, generated by Q of Eq. (2.12).

Let us denote the four real components of the scalar doublet as follows:

$$\phi(x) = \exp\left[i\frac{\sigma_i}{2}\theta^i(x)\right] \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}. \tag{2.20}$$

The local $SU(2)_L$ symmetry of the Lagrangian allows one to rotate away any dependence on the three θ^i. They represent the three would-be Goldstone bosons that are eaten by the three gauge bosons that acquire masses as a result of the SSB.

The remaining scalar degree of freedom, $h(x)$, represents a real massive scalar degree of freedom, the Higgs boson. Its mass is given by

$$m_h = \sqrt{2\lambda} v. \tag{2.21}$$

Experiment gives (PDG 2014)

$$m_h = 125.9 \pm 0.4 \text{ GeV}. \tag{2.22}$$

Note that we had to give the VEV to the $T_3 = -1/2$ component of ϕ, because this is the electromagnetically neutral component ($Q = T_3 + Y = 0$), and we want $U(1)_{\rm EM}$ to remain unbroken.

The main two points of this section are thus the following:

1. We have a mechanism to spontaneously break $SU(2)_L \times U(1)_Y \to U(1)_{\rm EM}$.

2. The model predicts the existence of a single, electromagnetically neutral, real scalar field, the Higgs boson. We discuss the properties of the Higgs boson in Section 2.4.1.

2.3 The LSM Spectrum

2.3.1 Back to $\mathcal{L}_{\rm kin}(\phi)$: The vector boson spectrum

Since the symmetry that is related to three out of the four generators is spontaneously broken, three of the four vector bosons acquire masses, while one remains massless. To see how this happens, we write the terms in $(D_\mu\phi)^\dagger(D^\mu\phi)$ (see Eq. (2.11) for the explicit expression for $D_\mu\phi$) setting $\theta^i(x)=0$, that is, working in the unitary (physical) gauge. The terms that are proportional to v^2 are given by (we omit Lorentz indices):

$$\frac{1}{8}(0\ v)\begin{pmatrix} gW_3+g'B & g(W_1-iW_2) \\ g(W_1+iW_2) & -gW_3+g'B\end{pmatrix}^\dagger \begin{pmatrix} gW_3+g'B & g(W_1-iW_2) \\ g(W_1+iW_2) & -gW_3+g'B\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix}. \tag{2.23}$$

We define

$$\begin{aligned} W^\pm &= \frac{1}{\sqrt{2}}(W_1\mp iW_2), \\ Z^0 &= \frac{1}{\sqrt{g^2+g'^2}}(gW_3-g'B), \\ A &= \frac{1}{\sqrt{g^2+g'^2}}(g'W_3+gB). \end{aligned} \tag{2.24}$$

Note that the $W^\pm$ are charged under electromagnetism (hence the superscripts $\pm$), while A and Z^0 are not. In terms of the vector boson fields of Eq. (2.24), we rewrite the mass terms of Eq. (2.23) as follows:

$$\frac{1}{4}g^2v^2W^+W^- + \frac{1}{8}(g^2+g'^2)v^2Z^0Z^0. \tag{2.25}$$

We learn that the four states of Eq. (2.24) are the mass eigenstates, with masses

$$m_W^2=\frac{1}{4}g^2v^2, \qquad m_Z^2=\frac{1}{4}(g^2+g'^2)v^2, \qquad m_A^2=0. \tag{2.26}$$

Two points are worth emphasizing:

1. As anticipated, three vector boson acquire masses.

2. $m_A^2=0$ is not a prediction, it is a consistency check on our calculation.

We define an angle θ_W via

$$\tan\theta_W \equiv \frac{g'}{g}. \tag{2.27}$$

Then

$$Z^\mu = \cos\theta_W W_3^\mu - \sin\theta_W B^\mu, \qquad A^\mu = \sin\theta_W W_3^\mu + \cos\theta_W B^\mu. \tag{2.28}$$

We learn that θ_W represents a rotation angle from the "interaction" basis (where fields have well-defined transformation properties under the gauge symmetry), W_3 and B, into the mass basis for the vector bosons, Z and A.

While θ_W depends on the two gauge couplings, g and g', and can thus be extracted from various interaction rates, it further provides a relation between the vector boson masses:

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2\theta_W} = 1. \tag{2.29}$$

This relation is testable. Note that the $\rho = 1$ relation is a consequence of the SSB by scalar doublets. (See your homework for other possibilities.) It thus tests this specific ingredient of the SM.

The weak gauge boson masses are (PDG 2014)

$$m_W = 80.385 \pm 0.015 \text{ GeV}; \qquad m_Z = 91.1876 \pm 0.0021 \text{ GeV}. \tag{2.30}$$

The ratio is

$$\frac{m_W}{m_Z} = 0.8815 \pm 0.0002 \quad \Longrightarrow \quad \sin^2\theta_W = 1 - (m_W/m_Z)^2 = 0.2229 \pm 0.0004. \tag{2.31}$$

Below we describe the determination of $\sin^2\theta_W$ by various interaction rates. We will see that the $\rho = 1$ is indeed realized in Nature (within experimental errors, and up to calculable quantum corrections).

2.3.2 Back to $\mathcal{L}_{\text{Yuk}}$: The fermion spectrum

Next we see how the chiral fermions acquire their masses. The Yukawa part of the Lagrangian is given by Eq. (2.15). The SSB allows us to tell the upper and lower components of the doublet. In the basis defined in Eq. (2.16), we denote these components as follows:

$$\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \qquad \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \qquad \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}, \tag{2.32}$$

where e, μ, τ are ordered by the size of $y_{e,\mu,\tau}$ (from smallest to largest). Eq. (2.12) tells us that the neutrinos, ν_{eL}, $\nu_{\mu L}$ and $\nu_{\tau L}$, have charge zero, while the charged leptons, e_L, μ_L and τ_L, carry charge -1. Similarly, the right handed fields, e_R, μ_R and τ_R, carry charge -1.

With ϕ^0 acquiring a VEV, $\langle \phi^0 \rangle = v/\sqrt{2}$, (2.15) has a piece that corresponds to the charged lepton masses:

$$-\frac{y_e v}{\sqrt{2}} \overline{e_L} e_R - \frac{y_\mu v}{\sqrt{2}} \overline{\mu_L} \mu_R - \frac{y_\tau v}{\sqrt{2}} \overline{\tau_L} \tau_R + \text{h.c.}. \tag{2.33}$$

namely

$$m_e = \frac{y_e v}{\sqrt{2}}, \quad m_\mu = \frac{y_\mu v}{\sqrt{2}}, \quad m_\tau = \frac{y_\tau v}{\sqrt{2}}. \tag{2.34}$$

The crucial point in this discussion is that, while the leptons are in a chiral representation of the full gauge group $SU(2)_L \times U(1)_Y$, the charged leptons – e, μ, τ – are in a vectorial representation of the subgroup that is not spontaneously broken, that is $U(1)_{\text{EM}}$. This situation is the key to opening the possibility of acquiring masses as a result of the SSB, as realized in Eq. (2.33).

Table 2.1: The LSM particles

particle	spin	Q	mass (theo) $[v]$
$W^\pm$	1	± 1	$\frac{1}{2}g$
Z^0	1	0	$\frac{1}{2}\sqrt{g^2+g'^2}$
A^0	1	0	0
h	0	0	$\sqrt{2\lambda}$
e	1/2	-1	$y_e/\sqrt{2}$
μ	1/2	-1	$y_\mu/\sqrt{2}$
τ	1/2	-1	$y_\tau/\sqrt{2}$
ν_e	1/2	0	0
ν_μ	1/2	0	0
ν_τ	1/2	0	0

These three masses have been measured:

$$m_e = 0.510998928(11)\ \text{MeV}, \quad m_\mu = 105.6583715(35)\ \text{MeV}, \quad m_\tau = 1776.82(16)\ \text{MeV}. \tag{2.35}$$

Note that the neutrino are massless in this model. There are no right handed neutrinos, $N_i(1)_0$, in the SM so the neutrinos cannot acquire Dirac mass. A-priori, since the neutrinos have no charge under the remaining subgroup $U(1)_{\rm EM}$, the possibility of acquiring Majorana masses is not closed. Yet, lepton number is an accidental symmetry of the theory (see Section 2.5.2) and thus the neutrinos do not acquire Majorana masses from renormalizable terms.

In your homework you will find that the number of Higgs representations that can give the gauge boson their masses is large, but only very few also give masses to the fermions.

2.3.3 Summary

We presented the details of the spectrum of the leptonic standard model. These are summarized in Table 2.1. All masses are proportional to the VEV of the scalar field, v. For the three massive gauge bosons, and for the three charged leptons, this is expected: In the absence of spontaneous symmetry breaking, the former would be protected by the gauge symmetry and the latter by their chiral nature. For the Higgs boson, the situation is different, as a mass-squared term does not violate any symmetry.

Next we further discuss the interactions of the model.

2.4 The LSM interactions

2.4.1 The Higgs boson

Out of the four scalar degrees of freedom, three are the would-be Goldstone bosons eaten by the $W^\pm$ and Z^0, and one is a physical scalar h called *the Higgs boson*.

The kinetic, gauge-interaction, self-interaction and Yukawa interaction terms of h are given by

$$\begin{aligned}\mathcal{L}_H &= \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}m_h^2 h^2 - \frac{m_h^2}{2v}h^3 - \frac{m_h^2}{8v^2}h^4 \\ &+ m_W^2 W_\mu^- W^{\mu+}\left(\frac{2h}{v}+\frac{h^2}{v^2}\right) + \frac{1}{2}m_Z^2 Z_\mu Z^\mu \left(\frac{2h}{v}+\frac{h^2}{v^2}\right) \\ &- \frac{h}{v}\left(m_e\,\overline{e_L}\,e_R + m_\mu\,\overline{\mu_L}\,\mu_R + m_\tau\,\overline{\tau_L}\,\tau_R + \text{h.c.}\right). \end{aligned} \tag{2.36}$$

Note that all of the Higgs couplings can be written in terms of the masses of the particles to which it couples.

The Higgs mass is given by

$$m_h = \sqrt{2\lambda}v. \tag{2.37}$$

It determines its quartic self-coupling, $\frac{m_h^2}{2v^2} = \lambda$, which is unchanged from the quartic coupling in (2.17), and its trilinear self-coupling, $\frac{m_h^2}{2v} = \lambda v$, which arises as a consequence of the SSB.

The Higgs coupling to the weak interaction gauge bosons is proportional to their masses-squared. The dimensionless $hhVV$ couplings, $\frac{m_W^2}{v^2} = \frac{g^2}{4}$ and $\frac{m_Z^2}{2v^2} = \frac{g^2+g'^2}{8}$ are unchanged from Eq. (2.10). The hVV couplings, $\frac{2m_W^2}{v} = \frac{g^2 v}{2}$ and $\frac{m_Z^2}{v} = \frac{(g^2+g'^2)v}{4}$, arise as a consequence of the SSB.

There is neither an hAA nor $hhAA$ coupling. One can understand the absence of these couplings in two ways. First, the Higgs boson is electromagnetically neutral, so it should not couple to the electromagnetic force carrier. Second, the photon is massless, so it should not couple to the Higgs boson.

The Yukawa couplings of the Higgs bosons to the charged leptons are proportional to their masses: the heavier the lepton, the stronger the coupling. Note that these couplings, $m_\ell/v = y_\ell/\sqrt{2}$, are unchanged from Eq. (2.15).

2.4.2 QED: Electromagnetic interactions

This subsection is based in part on Ref. [2]. By construction, the local $U(1)_{\rm EM}$ symmetry survives the SSB. Our theory has thus one massless gauge boson that we identify with the photon. Let us make sure that it indeed couples like the photon. From Eq. (2.9) we learn that the couplings of the neutral gauge fields are of the form

$$g\not{W}_3 T_3 + g'\not{B}Y. \tag{2.38}$$

Eq. (2.28) gives

$$W_3^\mu = \cos\theta_W Z^\mu + \sin\theta_W A^\mu, \qquad B^\mu = -\sin\theta_W Z^\mu + \cos\theta_W A^\mu. \tag{2.39}$$

Inserting that into Eq. (2.38), we obtain

$$\not{A}(g\sin\theta_W\ T_3 + g'\cos\theta_W\ Y) + \not{Z}(g\cos\theta_W\ T_3 - g'\sin\theta_W\ Y). \tag{2.40}$$

The photon field couples to $eQ = e(T_3 + Y)$, so we must have

$$g = \frac{e}{\sin\theta_W}, \qquad g' = \frac{e}{\cos\theta_W}. \tag{2.41}$$

Thus, the electromagnetic interactions are described by the QED Lagrangian, which is the part of the SM Lagrangian that involves the A^0 photon field and the charged fermions:

$$\mathcal{L}_{\rm QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + eA_\mu\overline{\ell_i}\gamma^\mu\ell_i, \tag{2.42}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $\ell_{1,2,3} = e, \mu, \tau$ are the Dirac fermions with $Q = -1$ that are formed from the $T_3 = -1/2$ component of a left-handed lepton doublet and a right-handed lepton singlet.

This Lagrangian gives rise to the well-known Maxwell equations:

$$\partial_\mu F^{\mu\nu} = eJ^\nu \equiv -e\bar{\ell}\gamma^\nu\ell. \tag{2.43}$$

The most stringent QED test comes from high precision measurements of the e and μ anomalous magnetic moments, $a_\ell \equiv (g_\ell^\gamma - 2)/2$, with $\vec{\mu}_\ell \equiv g_\ell^\gamma(e/2m_\ell)\vec{S}_\ell$:

$$a_e = (1159652180.76 \pm 0.27) \times 10^{-12}, \quad a_\mu = (11659209 \pm 6) \times 10^{-10}. \tag{2.44}$$

To the level of experimental sensitivity, a_e arises entirely from virtual electrons and photons. These contributions are fully known to $\mathcal{O}(\alpha^4)$, and many $\mathcal{O}(\alpha^5)$ corrections have been computed. The impressive agreement between theory and experiment has promoted QED to the level of the best theory ever built to describe Nature. The theoretical error is dominated by the uncertainty in the input value of the QED coupling $\alpha \equiv e^2/(4\pi)$. Turning things around, a_e provides the most accurate determination of the fine structure constant,

$$\alpha^{-1} = 137.035999074 \pm 0.000000044. \tag{2.45}$$

The anomalous magnetic moment of the muon is sensitive to small corrections from heavier states; compared with a_e, they scale with the mass ratio m_μ^2/m_e^2. Electroweak effects from virtual $W^\pm$ and Z^0 bosons amount to a contribution of $(15.4\pm0.2)\times10^{-10}$, which is larger than the present experimental accuracy. Thus a_μ allows one to test the entire SM. The main theoretical uncertainty comes from strong interactions. We will not enter a detailed discussion, but only mention that presently there is a discrepancy between theory and experiment at a level of above 3σ.

Eq. (2.42) reveals some further important features of the model:

1. The photon couplings are *vector-like*: It couples to the left-handed and right-handed fields in the same way.

2. Thus, electromagnetic interactions are *parity conserving*.

3. *Diagonality.* The photon couples to e^+e^-, $\mu^+\mu^-$ and $\tau^+\tau^-$, but not to $e^\pm\mu^\mp$, $e^\pm\tau^\mp$ or $\mu^\pm\tau^\mp$ pairs. Thus, electromagnetic interactions do not change flavor. This is a result of the unbroken local $U(1)_{\rm EM}$ symmetry.

4. *Universality*: the couplings of the photon to the different generations are universal. This is a result of the $U(1)_{\rm EM}$ gauge invariance.

2.4.3 Neutral current weak interactions

The Z couplings to fermions, given in Eq. (2.40), can be written as follows:

$$\mathcal{L} = \frac{e}{\sin\theta_W\cos\theta_W}(T_3 - \sin^2\theta_W Q)\ \bar\psi \not{Z}\psi\,, \tag{2.46}$$

where T_3 and Q are specific to the fermion ψ. For example, for left handed electrons, $T_3 = -1/2$ and $Q = -1$. Explicitly, we find the following three types of Z couplings in a lepton generation:

$$\mathcal{L} = \frac{e}{\sin\theta_W\cos\theta_W}\left[-\left(\frac{1}{2} - \sin^2\theta_W\right)\ \overline{e_L}\not{Z}e_L + \sin^2\theta_W\ \overline{e_R}\not{Z}e_R + \frac{1}{2}\ \overline{\nu_L}\not{Z}\nu_L\right]. \tag{2.47}$$

Note that, unlike the photon, the Z couples to neutrinos. Z-exchange gives rise to *neutral current weak interactions*. Eq. (2.47) reveals some further important features of the model:

1. The Z-boson couplings are chiral: It couples to left-handed and right-handed fields with different strength.

2. Thus, the Z-interactions are *parity violating*.

3. *Diagonality.* Consequently, there are no flavor changing neutral currents (FCNCs). This is a result of an accidental $U(1)^3$ symmetry of the model.

4. *Universality*: the couplings of the Z-boson to the different generations are universal. This is a result of a special feature of the LSM: all fermions of given chirality and given charge come from the same $SU(2)\times U(1)$ representation.

The branching ratios of the Z-boson into charged lepton pairs,

$$\begin{aligned} {\rm BR}(Z\to e^+e^-) &= (3.363\pm 0.004)\%\,, \\ {\rm BR}(Z\to \mu^+\mu^-) &= (3.366\pm 0.007)\%\,, \\ {\rm BR}(Z\to \tau^+\tau^-) &= (3.367\pm 0.008)\%\,. \end{aligned} \tag{2.48}$$

beautifully confirms universality:

$$\begin{aligned}\Gamma(\mu^+\mu^-)/\Gamma(e^+e^-) &= 1.0009 \pm 0.0028,\\ \Gamma(\tau^+\tau^-)/\Gamma(e^+e^-) &= 1.0019 \pm 0.0032.\end{aligned}$$

Diagonality is also tested by the following experimental searches:

$$\begin{aligned}\mathrm{BR}(Z \to e^+\mu^-) &< 1.7 \times 10^{-6},\\ \mathrm{BR}(Z \to e^+\tau^-) &< 9.8 \times 10^{-6},\\ \mathrm{BR}(Z \to \mu^+\tau^-) &< 1.2 \times 10^{-5}.\end{aligned} \tag{2.49}$$

The branching ratio of Z decays into invisible final states which, in our model, is interpreted as the decay into final neutrinos, is measured to be

$$\mathrm{BR}(Z \to \nu\bar{\nu}) = (20.00 \pm 0.06)\%. \tag{2.50}$$

From Eq. (2.47) we obtain

$$\frac{\mathrm{BR}(Z \to \ell^+\ell^-)}{\mathrm{BR}(Z \to \nu_\ell\bar{\nu}_\ell)} = \frac{(1/2 - \sin^2\theta_W)^2 + \sin^4\theta_W}{1/4} = 1 - 4\sin^2\theta_W + 8\sin^4\theta_W. \tag{2.51}$$

We can thus extract $\sin^2\theta_W$ from the experimental data, $\sin^2\theta_W = 0.226$, consistent with Eq. (2.31).

2.4.4 Charged current weak interactions

We now study the interactions that change particle identity, namely the couplings of W_1^μ and W_2^μ. Inserting the explicit form of the T_a matrices (Pauli matrices for doublets, 0 for singlets) in $\bar{\psi}\not{D}\psi$, we obtain the following interaction terms:

$$\begin{aligned}-\frac{g}{2}\{&\overline{\nu_{eL}}(\not{W}_1 - i\not{W}_2)e_L^- + \overline{e_L^-}(\not{W}_1 + i\not{W}_2)\nu_{eL}\\ &+\overline{\nu_{\mu L}}(\not{W}_1 - i\not{W}_2)\mu_L^- + \overline{\mu_L^-}(\not{W}_1 + i\not{W}_2)\nu_{\mu L}\\ &+\overline{\nu_{\tau L}}(\not{W}_1 - i\not{W}_2)\tau_L^- + \overline{\tau_L^-}(\not{W}_1 + i\not{W}_2)\nu_{\tau L}\}.\end{aligned} \tag{2.52}$$

In terms of the charged gauge bosons, $W^{\pm\mu} = \frac{1}{\sqrt{2}}(W_1^\mu \mp iW_2^\mu)$, the interaction term for the electron and the electron-neutrino is

$$-\frac{g}{\sqrt{2}}\,\overline{\nu_{eL}}\,\not{W}^+ e_L^- + \mathrm{h.c.}, \tag{2.53}$$

and similarly for the muon and the tau. The interactions mediated by the $W^\pm$ vector-bosons are called *charged current interactions*.

Eq. (2.53) reveals some important features of the model:

1. Only *left-handed* particles take part in charged-current interactions. (We remind the reader that we use the term "left-handed" to denote a chirality eigenstate. These are identical to helicity eigenstates in the massless limit.)

2. *Parity violation*: a consequence of the previous feature is that the W-mediated interactions violate parity.

3. *Diagonality*: the charged current interactions couple each charged lepton to a single neutrino, and each neutrino to a single charged lepton. Note that a global $SU(2)$ symmetry would allow off-diagonal couplings; It is the local symmetry that leads to diagonality.

4. *Universality*: the couplings of the W-boson to $\tau\bar{\nu}_\tau$, to $\mu\bar{\nu}_\mu$ and to $e\bar{\nu}_e$ are equal. Again, a global symmetry would have allowed an independent coupling to each lepton pair.

All of these predictions have been experimentally tested. As an example of how well universality works, consider the decay rates of the W-bosons to the three lepton pairs:

$$\begin{aligned} \text{BR}(W^+ \to e^+\nu_e) &= (10.75 \pm 0.13) \times 10^{-2}, \\ \text{BR}(W^+ \to \mu^+\nu_\mu) &= (10.57 \pm 0.15) \times 10^{-2}, \\ \text{BR}(W^+ \to \tau^+\nu_\tau) &= (11.25 \pm 0.20) \times 10^{-2}. \end{aligned} \tag{2.54}$$

You must be impressed by the nice agreement!

The charged current interaction gives rise to all flavor changing weak decays. One example is the $\mu^- \to e^-\nu_\mu\bar{\nu}_e$ decay. One can use this decay rate as yet another independent way to determine $\sin^2\theta_W$ from an interaction rate. The W-propagator is well approximated via a four fermion coupling:

$$\frac{g^2}{m_W^2 - q^2} \approx \frac{g^2}{m_W^2} = \frac{4\pi\alpha}{\sin^2\theta_W m_W^2} \equiv 4\sqrt{2}G_F. \tag{2.55}$$

The measured muon lifetime,

$$\tau_\mu = (2.197034 \pm 0.000021) \times 10^{-6}\ \text{s}, \tag{2.56}$$

determines G_F via

$$\Gamma_\mu = \frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3} f(m_e^2/m_\mu^2)(1+\delta_{\text{RC}}), \quad f(x) = 1 - 8x + 8x^3 - x^4 - 12x^2\log x, \tag{2.57}$$

where δ_{RC} is a correction factor from radiative corrections, which is known to $\mathcal{O}(\alpha^2)$. One gets:

$$G_F = 1.16637(1) \times 10^{-5}\ \text{GeV}^{-2}. \tag{2.58}$$

Using α of Eq. (2.45), m_W of Eq. (2.30) and G_F of Eq. (2.58), we obtain

$$\sin^2\theta_W = 0.215, \tag{2.59}$$

in good agreement with Eq. (2.31). The difference between the two is accounted for by higher order radiative corrections.

Note that G_F determines also the VEV:

$$v = (\sqrt{2}G_F)^{-1/2} = 246 \text{ GeV}. \tag{2.60}$$

2.4.5 Gauge boson self interactions

The gauge boson self interactions that are presently most relevant to experiments are the WWV ($V = Z, A$) couplings which, in the Standard Model, have the following form:

$$\begin{aligned} \mathcal{L}_{WWV} &= ie\cot\theta_W \left[(W^{a\dagger}_{\mu\nu}W^{a\mu} - W^{a\dagger}_{\mu}W^{a\mu}_{\nu})Z^\mu + W^{a\dagger}_{\mu}W^{a}_{\nu}Z^{\mu\nu}\right] \\ &+ ie\left[(W^{a\dagger}_{\mu\nu}W^{a\mu} - W^{a\dagger}_{\mu}W^{a\mu}_{\nu})A^\mu + W^{a\dagger}_{\mu}W^{a}_{\nu}A^{\mu\nu}\right]. \end{aligned} \tag{2.61}$$

The most general form of the C and P conserving $W^+W^-V^0$ ($V = A, Z$) couplings, assuming electromagnetic gauge invariance, is given by

$$\begin{aligned} \frac{\mathcal{L}_{WWZ}}{g_{WWZ}} &= ig_1^Z\left(W^\dagger_{\mu\nu}W^\mu Z^\nu - W^\dagger_\mu Z_\nu W^{\mu\nu}\right) + i\kappa_Z W^\dagger_\mu W_\nu Z^{\mu\nu} + \frac{i\lambda_Z}{m_W^2}W^\dagger_{\lambda\mu}W^\mu_\nu Z^{\lambda\nu}, \\ \frac{\mathcal{L}_{WW\gamma}}{g_{WW\gamma}} &= i\kappa_\gamma W^\dagger_\mu W_\nu A^{\mu\nu} + \frac{i\lambda_\gamma}{m_W^2}W^\dagger_{\lambda\mu}W^\mu_\nu A^{\lambda\nu}. \end{aligned} \tag{2.62}$$

Here W^μ is the W^- field, and $W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$. The SM tree level values are

$$\kappa_\gamma = \kappa_Z = g_1^Z = 1, \quad \lambda_\gamma = \lambda_Z = 0. \tag{2.63}$$

The experimental values are

$$\begin{aligned} g_1^Z &= 0.98 \pm 0.02, \quad \kappa_Z = 0.92 \pm 0.07, \quad \lambda_Z = -0.09 \pm 0.07, \\ \kappa_\gamma &= 0.97 \pm 0.04, \quad \lambda_\gamma = -0.03 \pm 0.02. \end{aligned} \tag{2.64}$$

2.4.6 Summary

Leptons have four types of interactions. These interactions are summarized in Table 2.1.

2.5 Some general comments

2.5.1 The interaction basis and the mass basis

The interaction basis is the one where all fields have well-defined transformation properties under the symmetries of the Lagrangian. In particular, in this basis, the gauge interactions are universal.

If there are several fields with the same quantum numbers, then the interaction basis is not unique. The kinetic and gauge terms are invariant under a global unitary transformation among

Table 2.1: The LSM lepton interactions

interaction	force carrier	coupling	range
electromagneric	γ	eQ	long
NC weak	Z^0	$\frac{e(T_3-s_W^2Q)}{s_Wc_W}$	short
CC weak	$W^\pm$	g	short
Yukawa	h	y_ℓ	short

these fields. On the other hand, the Yukawa terms and the fermion mass terms are, in general, not invariant under a unitary transformation among fermion fields with the same quantum numbers, $f_i \to U^f_{ji} f_i$, while the Yukawa terms and scalar potential are, in general, not invariant under a unitary transformation among scalar fields with the same quantum numbers, $s_i \to U^s_{ji} s_i$. Thus, by performing such transformations, we are changing the interaction basis.

In the LSM, there are three copies of $(2)_{-1/2}$ fermions and three copies of $(1)_{-1}$ fermions. Transforming the first by a unitary transformation U_L, and the latter by an independent unitary transformation U_R, the Yukawa matrix Y^e is transformed into $U_L Y^e U_R^\dagger$. The matrix Y^e is a 3×3 complex matrix and thus has, in general, nine complex parameters. We can always find a bi-unitary transformation that would make Y^e real and diagonal, and thus depend on only three real parameters:

$$Y^e \;\to\; U_L Y^e U_R^\dagger = Y^e_{\rm diag} = {\rm diag}(y_e, y_\mu, y_\tau). \tag{2.65}$$

Often one chooses a basis where the number of Lagrangian parameters is minimal, as is the case with the diagonal basis of Eq. (2.65). One could work in any other interaction basis. However, when calculating physical observables, only the eigenvalues of $Y_e^\dagger Y_e$ would play a role. Using the diagonal basis just provides a shortcut to this result.

The mass basis is the one where all fields have well defined transformation properties under the symmetries that are not spontaneously broken and are mass eigenstates. The fields in this basis correspond to the particles that are eigenstates of free propagation in spacetime. The Lagrangian parameters in this basis correspond directly to physical observables.

For the LSM, the interaction eigenstates have well defined transformation properties under the $SU(2)_L \times U(1)_Y$ symmetry:

$$W_a(3)_0,\; B(1)_0,\; L_{L1,2,3}(2)_{-1/2},\; E_{R1,2,3}(1)_{-1},\; \phi(2)_{+1/2}. \tag{2.66}$$

The mass eigenstates have well defined electromagnetic charge and mass:

$$W^\pm,\; Z^0,\; A^0,\; e^-,\; \mu^-,\; \tau^-,\; \nu_e,\; \nu_\mu,\; \nu_\tau,\; h^0. \tag{2.67}$$

The number of degrees of freedom is the same in both bases. To verify this statement one has to take into account the following features:

1. W_a and B have only transverse components, while $W^\pm$ and Z^0 have also a longitudinal one.

2. L_L and E_R are Weyl fermions, while e, μ, τ are Dirac fermions.

3. ϕ is a complex scalar, while h is a real one.

The three electromagnetically neutral neutrino states are, at the renormalizable level, massless and, in particular, degenerate. Thus, there is freedom in choosing the basis for the neutrinos. We choose the basis where the $W^\pm$ couplings to the charged lepton mass eigenstates are diagonal.

One could choose a different mass basis, related o the one we chose by a unitary transformation of the three neutrino fields,

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \rightarrow \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = U^\dagger \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}. \tag{2.68}$$

Let us see how the decay rate of the W-boson into an electron and a neutrino is calculated in this basis. Since the experiment cannot distinguish between $\nu_1,\ \nu_2,\ \nu_3$, one has to sum over all three species:

$$\begin{aligned} \Gamma(W^+ \to e^+\nu) &= \sum_{i=1,2,3} \Gamma(W^+ \to e^+\nu_i) = \Gamma(W^+ \to e^+\nu_e)(|U_{e1}|^2 + |U_{e2}|^2 + |U_{e3}|^2) \\ &= \Gamma(W^+ \to e^+\nu_e). \end{aligned} \tag{2.69}$$

Thus, if the neutrinos are degenerate, the elements of the matrix U have no physical significance; They cannot appear in any physical observable. Our choice of basis $(\nu_e, \nu_\mu, \nu_\tau)$ provides a shortcut to this result.

Later we will see that non-renormalizable terms provide the neutrinos with (non-degenerate) masses, and then the mass basis becomes unique.

2.5.2 Accidental symmetries

If we set the Yukawa couplings to zero, $\mathcal{L}_{\rm Yuk} = 0$, the leptonic SM (LSM) gains a large accidental global symmetry:

$$G^{\rm global}_{\rm LSM}(Y^e = 0) = U(3)_L \times U(3)_E, \tag{2.70}$$

where $U(3)_L$ has (L_{L1}, L_{L2}, L_{L3}) transforming as an $SU(3)_L$ triplet, and all other fields singlets, while $U(3)_E$ has (E_{R1}, E_{R2}, E_{R3}) transforming as an $SU(3)_E$ triplet, and all other fields singlets.

The Yukawa couplings break this symmetry into the following subgroup:

$$G^{\rm global}_{\rm LSM} = U(1)_e \times U(1)_\mu \times U(1)_\tau, \tag{2.71}$$

where ℓ_L, ℓ_R and ν_ℓ carry charge $+1$ under $U(1)_\ell$. Total lepton number is a subgroup of $G^{\rm global}_{\rm LSM}$ and is thus conserved, which explains why the neutrinos do not acquire Majorana masses.

Thus, electron number, muon number, tau number, and total lepton number are accidental symmetries of the SM. This situation allows, for example, the muon decay mode $\mu^- \to e^- \overline{\nu_e} \nu_\mu$, but forbids $\mu^- \to e^- \gamma$ and $\mu^- \to e^- e^+ e^-$. Also scattering processes such as $e^+ e^- \to \mu^+ \mu^-$ are allowed, but $e^+ \mu^- \to \mu^+ e^-$ is forbidden.

These accidental symmetries are, however, all broken by nonrenormalizable terms of the form $(1/\Lambda) L_{Li} L_{Lj} \phi\phi$. If the scale Λ is high enough, these breaking effects are very small. It means that the 'forbidden' processes mentioned above are expected to occur, but at very slow rates.

Finally, let us point out that the breaking of the symmetry (2.70) into (2.71) is by the Yukawa couplings – y_e, y_μ, y_τ – which are small, of $\mathcal{O}(10^{-6}, 10^{-3}, 10^{-2})$, respectively. Thus, the full $[SU(3)]^2$ remains an approximate symmetry of the SM.

2.5.3 The LSM parameters

There are seven independent parameters in the leptonic $SU(2) \times U(1)$ model. They can be chosen to be

$$g,\ g',\ v,\ \lambda,\ y_e,\ y_\mu,\ y_\tau. \tag{2.72}$$

There are, however, other possible choices. Another example would be $m_W,\ m_Z,\ m_h,\ m_e,\ m_\mu,\ m_\tau$ and v. This example shows that by measuring the spectrum of the LSM, and a single weak interaction rate (*e.g.* the muon decay rate) to determine v, all other interaction rates are predicted. A good choice of parameters would be one where the experimental errors in their determination are very small. Such a set is the following:

$$\alpha, \quad G_F, \quad m_e, \quad m_\mu, \quad m_\tau, \quad m_Z, \quad m_h. \tag{2.73}$$

By now, all seven parameters have been measured, with m_h (or, equivalently, λ in the previous list) the latest addition. In the following we use the 7 parameters to test the model.

2.6 Low Energy Tests of the LSM

Nowadays, experiments produce the W and Z bosons and measure their properties directly. It is interesting to understand, however, how the SM was tested at the time before the energy in experiments became high enough for such direct production. It is not only the historical aspect that is interesting; It is also important to see how we can use low energy data to understand shorter distances.

2.6.1 CC in neutrino–electron scattering

Let us compare the charged current contributions to the two elastic scattering processes $\bar{\nu}_e e^- \to \bar{\nu}_\mu \mu^-$ and $\nu_\mu e^- \to \nu_e \mu^-$ scattering. Since these are flavor changing processes, the only contributions come from W exchange. We consider scattering with a center-of-mass energy in the range $m_\mu^2 \ll s \ll m_W^2$. In particular, we can consider the leptons massless.

We define θ to be the angle between the incoming (anti)neutrino and the outgoing muon. Then $\cos\theta = 1$ corresponds to backward scattering of the beam particle. For the $\bar{\nu}_e e^-$ scattering, $\overline{\nu_L}$ and ℓ_L have positive and negative helicities, respectively. Thus, in the center of mass frame, their spins are in the same direction. Therefore $(J_z)_i = +1$. When the scattering is backwards, the respective momenta of the antineutrinos and the charged leptons change to the opposite directions, and so do their helicities: $(J_z)_f = -1$. Therefore, backward $\bar{\nu}\ell$ scattering is forbidden by angular momentum conservation. In fact, the process $\bar{\nu}_e e \to \bar{\nu}_\mu \mu$ proceeds entirely in a $J = 1$ state with net helicity $+1$. That is, only one of the three states is allowed. In contrast, in $\nu_\mu e \to \nu_e \mu$, backward scattering has $(J_Z)_i = (J_Z)_f = 0$ and all helicity states are allowed. The full $SU(2) \times U(1)$ calculation yields, for $m_\mu^2 \ll s \ll m_W^2$:

$$\frac{d\sigma(\nu_\mu e^-)}{d\Omega} = \frac{G_F^2 s}{4\pi^2}; \qquad \frac{d\sigma(\bar{\nu}_e e^-)}{d\Omega} = \frac{G_F^2 s}{16\pi^2}(1-\cos\theta)^2. \tag{2.74}$$

$$\sigma(\nu_\mu e^-) = \frac{G_F^2 s}{\pi}; \qquad \sigma(\bar{\nu}_e e^-) = \frac{G_F^2 s}{3\pi}. \tag{2.75}$$

2.6.2 NC in neutrino–electron scattering

There are several observables that can be used to test neutral currents interactions. The first example is low energy $\nu_\mu e^- \to \nu_\mu e^-$ scattering. Since the W-boson couples diagonally, it does not couple to a $\nu_\mu e^-$ pair. Consequently, the elastic scattering $\nu_\mu e \to \nu_\mu e$ is mediated purely by the Z-boson.

We can use the ratio

$$R \equiv \frac{\sigma(\nu_\mu e \to \nu_\mu e)}{\sigma(\bar{\nu}_\mu e \to \bar{\nu}_\mu e)} \tag{2.76}$$

to fix $\sin\theta_W$:

$$\sigma_{\nu_\mu} = \frac{G_F^2 m_e E_\nu}{2\pi}\left[(g_L^e)^2 + \frac{1}{3}(g_R^e)^2\right], \qquad \sigma_{\bar{\nu}_\mu} = \frac{G_F^2 m_e E_\nu}{2\pi}\left[(g_R^e)^2 + \frac{1}{3}(g_L^e)^2\right], \tag{2.77}$$

where

$$g_L^e = -1/2 + \sin^2\theta_W; \qquad g_R^e = \sin^2\theta_W. \tag{2.78}$$

We also use the notations

$$g_V^e = g_L^e + g_R^e = -1/2 + 2\sin^2\theta_W; \qquad g_A^e = g_L^e - g_R^e = -1/2. \tag{2.79}$$

From PDG00 we find (p. 101) $g_A^e = -0.503 \pm 0.017$ and $g_V^e = -0.041 \pm 0.015$. This gives

$$\sin^2\theta_W = 0.230 \pm 0.008. \tag{2.80}$$

2.6.3 Forward-backward asymmetry

We consider $e^+e^- \to \mu^+\mu^-$ scattering. This process is mediated by both QED interactions and NC weak interactions. The former are vector-like contributions, and therefore conserve parity. The latter are parity violating. The interference between the photon-mediated contribution and the Z-mediated contribution leads to a forward-backward asymmetry, which is a manifestation of parity violation.

The forward-backward asymmetry is defined as follows:

$$A_{\rm FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B}, \quad \sigma_F = 2\pi \int_0^1 d\cos\theta\, \frac{d\sigma}{d\cos\theta}, \quad \sigma_B = 2\pi \int_{-1}^0 d\cos\theta\, \frac{d\sigma}{d\cos\theta}. \tag{2.81}$$

A detailed calculation gives, for $m_\mu^2 \ll s \ll m_Z^2$,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left[1 + \cos^2\theta - \frac{4g_A^2}{c_W^2 s_W^2}\frac{s}{m_Z^2}\cos\theta\right], \tag{2.82}$$

yielding

$$A_{\rm FB}(m_\mu^2 \ll s \ll m_Z^2) = -\frac{3g_A^2}{2c_W^2 s_W^2}\frac{s}{m_Z^2}. \tag{2.83}$$

Chapter 3

The full Standard Model

3.1 Defining the Standard Model

The Standard Model (SM) is defined as follows:

(i) The symmetry is a local

$$SU(3)_C \times SU(2)_L \times U(1)_Y. \tag{3.1}$$

(ii) The pattern of spontaneous symmetry breaking is as follows:

$$SU(3)_C \times SU(2)_L \times U(1)_Y \to SU(3)_C \times U(1)_{\rm EM} \quad (Q_{\rm EM} = T_3 + Y). \tag{3.2}$$

(iii) There are three fermion generations, each consisting of five different representations:

$$Q_{Li}(3,2)_{+1/6}, \quad U_{Ri}(3,1)_{+2/3}, \quad D_{Ri}(3,1)_{-2/3}, \quad L_{Li}(1,2)_{-1/2}, \quad E_{Ri}(1,1)_{-1}, \quad i = 1,2,3. \tag{3.3}$$

There is a single scalar multiplet:

$$\phi(1,2)_{+1/2}. \tag{3.4}$$

The fermions that transform as triplets of $SU(3)_C$ are called quarks, while those that transform as singlets of $SU(3)_C$ are called leptons.

3.2 The SM Lagrangian

The most general renormalizable Lagrangian with scalar and fermion fields can be decomposed into

$$\mathcal{L} = \mathcal{L}_{\rm kin} + \mathcal{L}_\psi + \mathcal{L}_{\rm Yuk} + \mathcal{L}_\phi. \tag{3.5}$$

Here $\mathcal{L}_{\rm kin}$ describes the free propagation in spacetime, as well as the gauge interactions, $\mathcal{L}_\psi$ gives the fermion mass terms, $\mathcal{L}_{\rm Yuk}$ describes the Yukawa interactions, and $\mathcal{L}_\phi$ gives the scalar potential.

It is now our task to find the specific form of the Lagrangian made of the fermion fields Q_{Li}, U_{Ri}, D_{Ri}, L_{Li} and E_{Ri} (3.3), and the scalar field ϕ (3.4), subject to the gauge symmetry (3.1) and leading to the SSB of Eq. (3.2).

3.2.1 $\mathcal{L}_{\rm kin}$

The gauge group is given in Eq. (3.1). It has twelve generators: eight L_a's that form the $SU(3)$ algebra, three T_b's that form the $SU(2)$ algebra, and a single Y that generates the $U(1)$ algebra:

$$[L_a, L_b] = i f_{abc} L_c, \qquad [T_a, T_b] = i\epsilon_{abc} T_c, \qquad [L_a, T_b] = [L_a, Y] = [T_b, Y] = 0. \tag{3.6}$$

Thus there are three independent coupling constants in $\mathcal{L}_{\rm kin}$: g_s related to the $SU(3)_C$ subgroup, g related to the $SU(2)_L$ subgroup, and g' related to the $U(1)_Y$ subgroup.

The local symmetry requires twelve gauge bosons, eight in the adjoint representation of $SU(3)_C$, three in the adjoint representation of $SU(2)_L$, and one related to the $U(1)_Y$ symmetry:

$$G^\mu_a(8,1)_0, \quad W^\mu_a(1,3)_0, \quad B^\mu(1,1)_0. \tag{3.7}$$

The corresponding field strengths are given by

$$\begin{aligned} G^{\mu\nu}_a &= \partial^\mu G^\nu_a - \partial^\nu G^\mu_a - g_s f_{abc} G^\mu_b G^\nu_c, \\ W^{\mu\nu}_a &= \partial^\mu W^\nu_a - \partial^\nu W^\mu_a - g\epsilon_{abc} W^\mu_b W^\nu_c, \\ B^{\mu\nu} &= \partial^\mu B^\nu - \partial^\nu B^\mu. \end{aligned} \tag{3.8}$$

The covariant derivative is

$$D^\mu = \partial^\mu + i g_s G^\mu_a L_a + i g W^\mu_b T_b + i g' Y B^\mu. \tag{3.9}$$

$\mathcal{L}_{\rm kin}$ includes the kinetic terms of all the fields:

$$\begin{aligned} \mathcal{L}_{\rm kin} = &- \frac{1}{4} G^{\mu\nu}_a G_{a\mu\nu} - \frac{1}{4} W^{\mu\nu}_b W_{b\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \\ &- i\overline{Q_{Li}} \not{D} Q_{Li} - i\overline{U_{Ri}} \not{D} U_{Ri} - i\overline{D_{Ri}} \not{D} D_{Ri} - i\overline{L_{Li}} \not{D} L_{Li} - i\overline{E_{Ri}} \not{D} E_{Ri} \\ &- (D^\mu \phi)^\dagger (D_\mu \phi). \end{aligned} \tag{3.10}$$

For the $SU(3)_C$ triplets $L_a = \frac{1}{2}\lambda_a$ (λ_a are the Gell-Mann matrices), while for the $SU(3)_C$ singlets, $L_a = 0$. For the $SU(2)_L$ doublets $T_b = \frac{1}{2}\sigma_b$ (σ_b are the Pauli matrices), while for the $SU(2)_L$ singlets, $T_b = 0$. Explicitly, the covariant derivatives acting on the various fermion fields are given by

$$\begin{aligned} D^\mu Q_L &= \left(\partial^\mu + \frac{i}{2} g_s G^\mu_a \lambda_a + \frac{i}{2} g W^\mu_b \sigma_b + \frac{i}{6} g' B^\mu\right) Q_L, \\ D^\mu U_R &= \left(\partial^\mu + \frac{i}{2} g_s G^\mu_a \lambda_a + \frac{2i}{3} g' B^\mu\right) U_R, \end{aligned}$$

$$
\begin{aligned}
D^\mu D_R &= \left(\partial^\mu + \frac{i}{2} g_s G_a^\mu \lambda_a - \frac{i}{3} g' B^\mu\right) U_R, \\
D^\mu L_L &= \left(\partial^\mu + \frac{i}{2} g W_b^\mu \sigma_b - \frac{i}{2} g' B^\mu\right) L_L, \\
D^\mu E_R &= \left(\partial^\mu - i g' B^\mu\right) E_R.
\end{aligned}
\tag{3.11}
$$

For $SU(3)_C$ adjoints, $(L_a)_{bc} = f_{abc}$ and for $SU(2)_L$ adjoints, $(T_a)_{bc} = \epsilon_{abc}$, which have already been used in writing (3.8).

We remind the reader that in $\mathcal{L}_{\rm SM}$ there are no mass terms for the gauge bosons, as that would violate the gauge symmetry.

What are the electromagnetic charges of the quark fields? We have $Q = T_3 + Y$. Let us write explicitly the two components of $SU(2)_L$ doublets:

$$
Q_{L1} = \begin{pmatrix} u_L \\ d_L \end{pmatrix}. \tag{3.12}
$$

Then

$$
Qu_L = +\frac{2}{3} u_L, \qquad Qd_L = -\frac{1}{3} d_L, \qquad Qu_R = +\frac{2}{3} u_R, \qquad Qd_R = -\frac{1}{3} d_R. \tag{3.13}
$$

For now, u_L and d_L stand for, respectively, the $T_3 = +1/2$ and $T_3 = -1/2$ components of the quark doublet. If $SU(2)_L \times U(1)_Y$ were an exact symmetry of Nature, there would be no way of distinguishing particles of different electric charges in the same $SU(2)_L$ multiplet. We make this choice as it will give us the correct QED after SSB as we see below.

3.2.2 $\mathcal{L}_\psi$

There are no mass terms for the fermions of the SM,

$$
\mathcal{L}_\psi = 0. \tag{3.14}
$$

We have seen already that this is the case for leptons. Note that a larger symmetry means stronger constraints, hence it is impossible that lepton masses would become allowed when the gauge symmetry is extended to include $SU(3)_C$. As concerns the quarks, we cannot write Dirac mass terms because they are assigned to chiral representations of the $SU(2)_L \times U(1)_Y$ gauge symmetry. We cannot write Majorana mass terms for the quarks because they all have $Y \neq 0$.

3.2.3 $\mathcal{L}_{\rm Yuk}$

The Yukawa part of the Lagrangian is given by

$$
\mathcal{L}_{\rm Yuk} = Y^u_{ij} \overline{Q_{Li}} U_{Rj} \, \widetilde{\phi} + Y^d_{ij} \overline{Q_{Li}} D_{Rj} \, \phi + Y^e_{ij} \overline{L_{Li}} E_{Rj} \, \phi + \text{h.c.}, \tag{3.15}
$$

where $i, j = 1, 2, 3$ are flavor indices, and $\widetilde{\phi} = i\sigma_2 \phi^*$. The Yukawa matrices Y^u, Y^d and Y^e are general complex 3×3 matrices of dimensionless couplings.

Without loss of generality, we can use a bi-unitary transformation,

$$Y^e \to \hat{Y}_e = U_{eL} Y^e U^\dagger_{eR}, \tag{3.16}$$

to change the basis to one where Y^e is diagonal and real:

$$\hat{Y}^e = \mathrm{diag}(y_e, y_\mu, y_\tau). \tag{3.17}$$

In the basis defined in Eq. (3.17), we denote the components of the lepton $SU(2)$-doublets, and the three lepton $SU(2)$-singlets, as follows:

$$\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \qquad \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \qquad \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}; \qquad e_R, \qquad \mu_R, \qquad \tau_R, \tag{3.18}$$

where e, μ, τ are ordered by the size of $y_{e,\mu,\tau}$ (from smallest to largest).

Similarly, without loss of generality, we can use a bi-unitary transformation,

$$Y^u \to \hat{Y}_u = V_{uL} Y^u V^\dagger_{uR}, \tag{3.19}$$

to change the basis to one where $\hat{Y}^u$ is diagonal and real:

$$\hat{Y}^u = \mathrm{diag}(y_u, y_c, y_t). \tag{3.20}$$

In the basis defined in Eq. (3.20), we denote the components of the quark $SU(2)$-doublets, and the quark up $SU(2)$-singlets, as follows:

$$\begin{pmatrix} u_L \\ d_{uL} \end{pmatrix}, \qquad \begin{pmatrix} c_L \\ d_{cL} \end{pmatrix}, \qquad \begin{pmatrix} t_L \\ d_{tL} \end{pmatrix}; \qquad u_R, \qquad c_R, \qquad t_R, \tag{3.21}$$

where u, c, t are ordered by the size of $y_{u,c,t}$ (from smallest to largest).

We can use yet another bi-unitary transformation,

$$Y^d \to \hat{Y}_d = V_{dL} Y^d V^\dagger_{dR}, \tag{3.22}$$

to change the basis to one where $\hat{Y}^d$ is diagonal and real:

$$\hat{Y}^d = \mathrm{diag}(y_d, y_s, y_b). \tag{3.23}$$

In the basis defined in Eq. (3.23), we denote the components of the quark $SU(2)$-doublets, and the quark down $SU(2)$-singlets, as follows:

$$\begin{pmatrix} u_{dL} \\ d_L \end{pmatrix}, \qquad \begin{pmatrix} u_{sL} \\ s_L \end{pmatrix}, \qquad \begin{pmatrix} u_{bL} \\ b_L \end{pmatrix}; \qquad d_R, \qquad s_R, \qquad b_R, \tag{3.24}$$

where d, s, b are ordered by the size of $y_{d,s,b}$ (from smallest to largest).

Note that if $V_{uL} \neq V_{dL}$, as is the general case, then the interaction basis defined by (3.20) is different from the interaction basis defined by (3.23). In the former, Y^d can be written as a unitary matrix times a diagonal one,

$$Y^u = \hat{Y}^u, \quad Y^d = V\hat{Y}^d. \tag{3.25}$$

In the latter, Y^u can be written as a unitary matrix times a diagonal one,

$$Y^d = \hat{Y}^d, \quad Y^u = V^\dagger \hat{Y}^u. \tag{3.26}$$

In either case, the matrix V is given by

$$V = V_{uL} V_{dL}^\dagger, \tag{3.27}$$

where V_{uL} and V_{dL} are defined in Eqs. (3.19) and (3.22), respectively. Note that V_{uL}, V_{uR}, V_{dL} and V_{dR} depend on the basis from which we start the diagonalization. The combination $V = V_{uL}V_{dL}^\dagger$, however, does not. This is a hint that V is physical. Indeed, below we see that it plays a crucial role in the charged current interactions.

3.2.4 $\mathcal{L}_\phi$

The scalar field is a singlet of the $SU(3)_C$ group. Thus, the form of $\mathcal{L}_\phi$ is the same as in the LSM,

$$\mathcal{L}_\phi = -\mu^2 \phi^\dagger \phi - \lambda \left(\phi^\dagger \phi\right)^2 . \tag{3.28}$$

Choosing $\mu^2 < 0$ and $\lambda > 0$ leads, as in the LSM, to spontaneous symmetry breaking, with $|\langle\phi\rangle| = v/\sqrt{2}$, with $v^2 = -\mu^2/\lambda$. Since ϕ is $SU(3)_C$ singlet, the $SU(3)_C$ subgroup remains unbroken, and the pattern of spontaneous symmetry breaking is as required by Eq. (3.2).

3.2.5 Summary

The renormalizable part of the Standard Model Lagrangian is given by

$$\begin{aligned}
\mathcal{L}_{\rm SM} = & - \frac{1}{4}G_a^{\mu\nu}G_{a\mu\nu} - \frac{1}{4}W_b^{\mu\nu}W_{b\mu\nu} - \frac{1}{4}B^{\mu\nu}B_{\mu\nu} - (D^\mu\phi)^\dagger(D_\mu\phi) \\
& - i\overline{Q_{Li}}\not{D}Q_{Li} - i\overline{U_{Ri}}\not{D}U_{Ri} - i\overline{D_{Ri}}\not{D}D_{Ri} - i\overline{L_{Li}}\not{D}L_{Li} - i\overline{E_{Ri}}\not{D}E_{Ri} \\
& + \left(Y_{ij}^u\overline{Q_{Li}}U_{Rj}\,\tilde{\phi} + Y_{ij}^d\overline{Q_{Li}}D_{Rj}\,\phi + Y_{ij}^e\overline{L_{Li}}E_{Rj}\,\phi + \text{h.c.}\right) \\
& - \mu^2\phi^\dagger\phi - \lambda\left(\phi^\dagger\phi\right)^2 ,
\end{aligned} \tag{3.29}$$

where $i, j = 1, 2, 3$.

3.3 The SM Spectrum

3.3.1 Bosons

Given the spontaneous breaking of the $SU(2)_L \times U(1)_Y$ symmetry to the $U(1)_{\rm EM}$ subgroup, the spectrum of the electroweak gauge bosons remains the same as in the LSM: three massive vector bosons, $W^\pm$ and Z^0 that carry the weak interactions, and a massless photon, A^0, which mediates the electromagnetic interactions. Furthermore, since the breaking is induced by an $SU(2)_L$ Higgs doublet, the $\rho \equiv m_W^2/(m_Z^2 \cos^2\theta_W) = 1$ relation holds.

The new ingredient is the existence of eight gluons that mediate the strong interactions. Since the $SU(3)_C$ gauge symmetry remains unbroken, the gluons are massless.

As concerns scalars, the three would-be Goldstone bosons become the longitudinal components of the three massive vector bosons. The fourth scalar degree of freedom is the Higgs boson h, a real massive scalar field,

3.3.2 Fermions

Since the SM allows no bare mass terms for the fermions, their masses can only arise from the Yukawa part of the Lagrangian, which is given in Eq. (3.15). Indeed, with $\langle\phi^0\rangle = v/\sqrt{2}$, Eq. (3.15) has a piece that corresponds to charged lepton masses:

$$m_e = \frac{y_e v}{\sqrt{2}}, \quad m_\mu = \frac{y_\mu v}{\sqrt{2}}, \quad m_\tau = \frac{y_\tau v}{\sqrt{2}}, \tag{3.30}$$

a piece that corresponds to up-type quark masses,

$$m_u = \frac{y_u v}{\sqrt{2}}, \quad m_c = \frac{y_c v}{\sqrt{2}}, \quad m_t = \frac{y_t v}{\sqrt{2}}, \tag{3.31}$$

and a piece that corresponds to down-type quark masses,

$$m_d = \frac{y_d v}{\sqrt{2}}, \quad m_s = \frac{y_s v}{\sqrt{2}}, \quad m_b = \frac{y_b v}{\sqrt{2}}. \tag{3.32}$$

We conclude that all charged fermions acquire Dirac masses as a result of the spontaneous symmetry breaking. The key to this feature is that, while the charged fermions are in chiral representations of the full gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$, they are in vector-like representations of the $SU(3)_C \times U(1)_{\rm EM}$ group:

- The LH and RH charged lepton fields, e, μ and τ, are in the $(1)_{-1}$ representation.
- The LH and RH up-type quark fields, u, c and t, are in the $(3)_{+2/3}$ representation.
- The LH and RH down-type quark fields, d, s and b, are in the $(3)_{-1/3}$ representation.

Table 3.1: The SM particles

particle	spin	color	Q	mass $[v]$
$W^\pm$	1	(1)	± 1	$\frac{1}{2}g$
Z^0	1	(1)	0	$\frac{1}{2}\sqrt{g^2+g'^2}$
A^0	1	(1)	0	0
g	1	(8)	0	0
h	0	(1)	0	$\sqrt{2\lambda}$
e, μ, τ	1/2	(1)	-1	$y_{e,\mu,\tau}/\sqrt{2}$
ν_e, ν_μ, ν_τ	1/2	(1)	0	0
u, c, t	1/2	(3)	$+2/3$	$y_{u,c,t}/\sqrt{2}$
d, s, b	1/2	(3)	$-1/3$	$y_{d,s,b}/\sqrt{2}$

On the other hand, the neutrinos remain massless:

$$m_{\nu_e} = m_{\nu_\mu} = m_{\nu_\tau} = 0. \tag{3.33}$$

This is the case in spite of the fact that they are in the $(1)_0$ representation of $SU(3)_C \times U(1)_{\rm EM}$, which allows for Majorana masses. Such masses require a VEV carried by a scalar field in the $(1,3)_{+1}$ representation of the $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry, but there is no such field in the SM.

The experimental values of the charged fermion masses are

$$\begin{aligned}
&m_e = 0.510998910(13)\ \text{MeV}, \quad m_\mu = 105.658367(4)\ \text{MeV}, \quad m_\tau = 1776.82(16)\ \text{MeV}, \\
&m_u = 1.5 - 3.1\ \text{MeV}, \quad m_c = 1.29^{+0.05}_{-0.11}\ \text{GeV}, \quad m_t = 172.9 \pm 0.12\ \text{GeV}, \\
&m_d = 4.1 - 5.7\ \text{MeV}, \quad m_s = 100^{+30}_{-20}\ \text{MeV}, \quad m_b = 4.9^{+0.18}_{-0.06}\ \text{GeV},
\end{aligned} \tag{3.34}$$

where the quark masses are given at a scale $\mu = 2$ GeV.

3.3.3 Summary

We presented the details of the spectrum of the standard model. These are summarized in Table 3.1. All masses are proportional to the VEV of the scalar field, v. For the three massive gauge bosons, and for the fermions, this is expected: In the absence of spontaneous symmetry breaking, the former would be protected by the gauge symmetry and the latter by their chiral nature. For the Higgs boson, the situation is different, as a mass-squared term does not violate any symmetry.

3.4 The SM Interactions

In this section, we discuss the interactions of the fermion and scalar mass eigenstates of the Standard Model. Before we start, let us mention that a reliable calculation of the experimental consequences is possible in the perturbative regime. If a coupling grows large, one loses the ability to make accurate predictions. In fact, when the coupling grows large, we expect that bound states made from the particles that carry the corresponding charge will form, providing a new set of particles and a new set of effective interactions among them.

As concerns the three gauge couplings of the SM, their renormalization group equations are the following:

$$\begin{aligned} \frac{dg_1}{dt} &= +\frac{41}{6}g_1^3, \\ \frac{dg_2}{dt} &= -\frac{19}{6}g_2^3, \\ \frac{dg_3}{dt} &= -7g_3^3, \end{aligned} \tag{3.35}$$

where $t = \log(E/m_Z)/(16\pi^2)$. Given the measured values of these couplings at m_Z, they remain perturbative up to the Planck scale. However, g_3 becomes strong at energies lower than a few hundreds MeV. We discuss the resulting spectrum below.

Among the Yukawa coupling, the top-Yukawa is the largest. Its renormalization group equation is given by

$$\frac{dy_t}{dt} = +y_t\left(9y_t^2 - 8g_3^2 - \frac{9}{4}g_2^2 - \frac{17}{12}g_1^2\right). \tag{3.36}$$

With the value of $y_t(\approx 1)$ which corresponds to $m_t \approx 173$ GeV, it remains perturbative up to the Planck scale.

3.4.1 QED: Electromagnetic Interactions

The photon couples to the electromagnetic charge,

$$\mathcal{L}_{\text{QED, fermions}} = eQ_i\overline{\psi_i}\not{A}\psi_i. \tag{3.37}$$

We have already seen that the neutrinos are electromagnetically neutral, and therefore have no tree level coupling to the photon, while the charged leptons have $Q = -1$. What is the electromagnetic charge of the quark fields? Using $Q = T_3 + Y$, we obtain

$$Qu_L = +\frac{2}{3}u_L, \quad Qd_L = -\frac{1}{3}d_L, \quad Qu_R = +\frac{2}{3}u_R, \quad Qd_R = -\frac{1}{3}d_R. \tag{3.38}$$

The photon interactions with leptons and quarks are vector-like, parity-conserving, diagonal and universal.

3.4.2 QCD: Strong Interactions

The gluons couple to all colored particles. Among the fermions, all quarks are in triplets of $SU(3)_C$, and therefore have the same strong coupling, while all leptons are singlets of $SU(3)_C$, and therefore do not couple to gluons:

$$\mathcal{L}_{\text{QCD, fermions}} = -\frac{1}{2} g_S \bar{q} \lambda_a \not{G}_a q \quad (q = u, c, t, d, s, b), \tag{3.39}$$

where λ_a are the Gell-Mann matrices. The gluon interactions with quarks are vector-like, parity-conserving, diagonal and universal.

The strong coupling constant is unique among the SM coupling constants in its dependence on distance (or, equivalently, on energy): The more distant the quarks or antiquarks are from each other, the larger the strong coupling constant becomes. This leads to *confinement* among quarks.

Hadrons

We do not observe free quarks in Nature. Instead, we observe bound states of quarks, which we call *hadrons*. They come in three types: *Mesons*, which have quark-antiquark constituents, $M = q\bar{q}$; *Baryons*, which have three quark constituents, $B = qqq$; *Antibaryons*, which have three antiquark constituents, $\bar{B} = \bar{q}\bar{q}\bar{q}$. For example, the lightest mesons are the pions: $\pi^+ = u\bar{d}$, $\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$ and $\pi^- = d\bar{u}$. The lightest baryons are the proton, $p = uud$, and the neutron, $n = udd$.

To explain why mesons, baryons and antibaryons exist in Nature, and other quark combinations do not, one has to postulate that all asymptotic states are singlets of $SU(3)_C$. This is the *confinement* hypothesis: Quarks are $S(3)_C$ triplets and therefore they must be confined within color-singlet bound states.

The only stable hadron is the proton. This can be understood by the $U(1)_B$ symmetry. The proton is the lightest particle that carries baryon number. Most of the hadrons decay very fast, for example, the ρ meson has a lifetime $\sim 4 \times 10^{-24}$ s, and its width is of the order of its mass, $\Gamma/m \sim 0.2$. Other examples of fast decaying hadrons include K^*, D^* and B^*. These particles have not been observed by looking at their trajectories, but only by detecting their decay products. These mesons appear as resonances in a production process of some particles. Therefore, the fast decaying hadrons are often called resonances, rather than particles. Other hadrons have much longer lifetimes, for example, $\pi^\pm$ ($\tau_{\pi^\pm} = 2.6 \times 10^{-8}$ s), $K^\pm$ ($\tau_{K^\pm} = 1.2 \times 10^{-12}$ s), and $B^\pm$ ($\tau_{B^\pm} = 1.6 \times 10^{-12}$ s).

These properties are explained by the fact that the hadrons are built from quarks. The fast decaying hadrons decay via strong or electromagnetic interactions. The lowest mass flavored (*i.e.* non-singlets of the $[U(3)]^5$ symmetry) states cannot decay the strong or electromagnetic interactions, because these interactions conserve the $[U(3)]^5$ symmetry. They can only decay via the weak interaction. Thus, the lifetimes of these states are much longer, and their decays are a probe of W-mediated interactions and, in particular, of the CKM parameters. Of particular importance in this study are mesons:

$$\begin{aligned}
K-\text{mesons}: &\qquad K^+(\bar{s}u), K^0(\bar{s}d), \overline{K^0}(s\bar{d}), K^-(s\bar{u}), \\
D-\text{mesons}: &\qquad D^+(c\bar{d}), D^0(c\bar{u}), \overline{D^0}(\bar{c}u), D^-(\bar{c}d), \\
D_s-\text{mesons}: &\qquad D_s^+(c\bar{s}), D_s^-(\bar{c}s), \\
B-\text{mesons}: &\qquad B^+(\bar{b}u), B^0(\bar{b}d), \overline{B^0}(b\bar{d}), B^-(b\bar{u}), \\
B_s-\text{mesons}: &\qquad B_s(\bar{b}s), \overline{B_s}(b\bar{s}).
\end{aligned} \tag{3.40}$$

3.4.3 Neutral current weak interactions

The Z couplings to fermions can be written as follows:

$$\mathcal{L}_{Z,\text{fermions}} = \frac{e}{\sin\theta_W \cos\theta_W}(T_{3i} - \sin^2\theta_W Q_i)\, \overline{\psi_i} \not{Z} \psi_i\,. \tag{3.41}$$

Using the T_3 and Y assignments of the various fermion fields, we find the following types of Z couplings in each generation:

$$\begin{aligned}
\mathcal{L} \;=\; & \frac{e}{s_W c_W}\left[-\left(\frac{1}{2} - s_W^2\right)\, \overline{e_L}\not{Z}e_L + s_W^2\, \overline{e_R}\not{Z}e_R + \frac{1}{2}\, \overline{\nu_L}\not{Z}\nu_L \right. \\
& \left. + \left(\frac{1}{2} - \frac{2}{3}s_W^2\right)\, \overline{u_L}\not{Z}u_L - \frac{2}{3}s_W^2\, \overline{u_R}\not{Z}u_R - \left(\frac{1}{2} - \frac{1}{3}s_W^2\right)\, \overline{d_L}\not{Z}d_L + \frac{1}{3}s_W^2\, \overline{d_R}\not{Z}d_R\right].
\end{aligned} \tag{3.42}$$

The Z couplings are chiral, parity-violating, diagonal and universal.

Thus, omitting common factors (particularly, a factor of $\frac{e^2}{4s_W^2 c_W^2}$) and phase-space factors, we obtain the following predictions for the Z decays into a one-generation fermion-pair of each type:

$$\begin{aligned}
\Gamma(Z \to \nu\bar{\nu}) &\propto 1, \\
\Gamma(Z \to \ell\bar{\ell}) &\propto 1 - 4s_W^2 + 8s_W^4, \\
\Gamma(Z \to u\bar{u}) &\propto 3\left(1 - \frac{8}{3}s_W^2 + \frac{32}{9}s_W^4\right), \\
\Gamma(Z \to d\bar{d}) &\propto 3\left(1 - \frac{4}{3}s_W^2 + \frac{8}{9}s_W^4\right).
\end{aligned} \tag{3.43}$$

Putting $s_W^2 = 0.225$, we obtain

$$\Gamma_\nu : \Gamma_\ell : \Gamma_u : \Gamma_d = 1 : 0.505 : 1.74 : 2.24. \tag{3.44}$$

Experiments measure the following average branching ratio into a single generation of each fermion species:

$$\begin{aligned}
\text{BR}(Z \to \nu\bar{\nu}) &= (6.67 \pm 0.02)\%, \\
\text{BR}(Z \to \ell\bar{\ell}) &= (3.37 \pm 0.01)\%, \\
\text{BR}(Z \to u\bar{u}) &= (11.6 \pm 0.6)\%, \\
\text{BR}(Z \to d\bar{d}) &= (15.6 \pm 0.4)\%,
\end{aligned} \tag{3.45}$$

which gives

$$\Gamma_\nu : \Gamma_\ell : \Gamma_u : \Gamma_d = 1 : 0.505 : 1.74 : 2.34. \tag{3.46}$$

3.4.4 Charged current weak interactions

We now study the couplings of the charged vector bosons, $W^\pm$, to fermion pairs. For the lepton mass eigenstates, things are simple, because there exists an interaction basis that is also a mass basis. Thus, the W interactions must be universal also in the mass basis:

$$-\frac{g}{\sqrt{2}}\left(\overline{\nu_{eL}}\, \not{W}^+ e_L^- + \overline{\nu_{\mu L}}\, \not{W}^+ \mu_L^- + \overline{\nu_{\tau L}}\, \not{W}^+ \tau_L^- + \text{h.c.}\right). \tag{3.47}$$

As concerns quarks, things are more complicated, since there is no interaction basis that is also a mass basis. In the interaction basis where the down quarks are mass eigenstates (3.24), the W interactions have the following form:

$$-\frac{g}{\sqrt{2}}\left(\overline{u_{dL}}\, \not{W}^+ d_L + \overline{u_{sL}}\, \not{W}^+ s_L + \overline{u_{bL}}\, \not{W}^+ b_L + \text{h.c.}\right). \tag{3.48}$$

The Yukawa matrices in this basis have the form (3.26), and in particular, for the up sector, we have

$$\mathcal{L}^u_{\text{Yuk}} = (\overline{u_{dL}}\ \overline{u_{sL}}\ \overline{u_{bL}}) V^\dagger \hat{Y}^u \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix}, \tag{3.49}$$

which tells us straightforwardly how to transform to the mass basis:

$$\begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix} = V \begin{pmatrix} u_{dL} \\ u_{sL} \\ u_{bL} \end{pmatrix}. \tag{3.50}$$

Using Eq. (3.50), we obtain the form of the W interactions (3.48) in the mass basis:

$$-\frac{g}{\sqrt{2}}\,(\overline{u_L}\ \overline{c_L}\ \overline{t_L})\ V\ \not{W}^+ \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} + \text{h.c.}. \tag{3.51}$$

You can easily convince yourself that we would have obtained the same form starting from any arbitrary interaction basis. We remind you that $V = V_{uL} V_{dL}^\dagger$ is basis independent.

Eq. (3.51) reveals some important features of the model:

1. Only left-handed particles take part in charged-current interactions. Consequently, parity is violated by these interactions.

2. The W couplings to the quark mass eigenstates are neither universal nor diagonal. The universality of gauge interactions is hidden in the unitarity of the matrix V.

The matrix V is called the CKM matrix.

Omitting common factors (particularly, a factor of $\frac{g^2}{4}$) and phase-space factors, we obtain the following predictions for the W decays:

$$
\begin{aligned}
\Gamma(W^+ \to \ell^+ \nu_\ell) &\propto 1, \\
\Gamma(W^+ \to u_i \overline{d_j}) &\propto 3|V_{ij}|^2 \quad (i=1,2;\ j=1,2,3).
\end{aligned} \tag{3.52}
$$

The top quark is not included because it is heavier than the W boson. Taking this fact into account, and the CKM unitarity relations

$$
|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = |V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 = 1, \tag{3.53}
$$

we obtain

$$
\Gamma(W \to \text{hadrons}) \approx 2\Gamma(W \to \text{leptons}). \tag{3.54}
$$

Experimentally,

$$
\begin{aligned}
&\text{BR}_{\text{leptons}} = (32.40 \pm 0.27)\%, \\
&\text{BR}_{\text{hadrons}} = (67.60 \pm 0.27)\%, \\
\Longrightarrow\ &\Gamma(W \to \text{hadrons})/\Gamma(W \to \text{leptons}) = 2.09 \pm 0.01,
\end{aligned} \tag{3.55}
$$

in beautiful agreement with the SM prediction. The (hidden) universality within the quark sector is tested by the prediction

$$
\Gamma(W \to uX) = \Gamma(W \to cX) = \frac{1}{2}\Gamma(W \to \text{hadrons}). \tag{3.56}
$$

Experimentally,

$$
\Gamma(W \to cX)/\Gamma(W \to \text{hadrons}) = 0.49 \pm 0.04. \tag{3.57}
$$

We discuss more aspects of the phenomenology related to the CKM matrix in Section 5.

3.4.5 Interactions of the Higgs boson

The Higgs boson has self-interactions, weak interactions, and Yukawa interactions:

$$
\begin{aligned}
\mathcal{L}_h &= \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}m_h^2 h^2 - \frac{m_h^2}{2v}h^3 - \frac{m_h^2}{8v^2}h^4 \\
&+ m_W^2 W_\mu^- W^{\mu +}\left(\frac{2h}{v} + \frac{h^2}{v^2}\right) + \frac{1}{2}m_Z^2 Z_\mu Z^\mu \left(\frac{2h}{v} + \frac{h^2}{v^2}\right) \\
&- \frac{h}{v}\Big(m_e\, \overline{e_L}\, e_R + m_\mu\, \overline{\mu_L}\, \mu_R + m_\tau\, \overline{\tau_L}\, \tau_R \\
&\quad + m_u\, \overline{u_L}\, u_R + m_c\, \overline{c_L}\, c_R + m_t\, \overline{t_L}\, t_R + m_d\, \overline{d_L}\, d_R + m_s\, \overline{s_L}\, s_R + m_b\, \overline{b_L}\, b_R + \text{h.c.}\Big).
\end{aligned} \tag{3.58}
$$

To see that the Higgs boson couples diagonally to the quark mass eigenstates, let us start from an arbitrary interaction basis:

$$\begin{aligned} h\overline{D_L}Y^d D_R &= h\overline{D_L}(V_{dL}^\dagger V_{dL})Y^d(V_{dR}^\dagger V_{dR})D_R \\ &= h(\overline{D_L}V_{dL}^\dagger)(V_{dL}Y^d V_{dR}^\dagger)(V_{dR}D_R) \\ &= h(\overline{d_L}\ \overline{s_L}\ \overline{b_L})\hat{Y}^d(d_R\ s_R\ b_R)^T. \end{aligned} \tag{3.59}$$

We conclude that the Higgs couplings to the fermion mass eigenstates are diagonal, but not universal. Instead, they are proportional to the fermion masses: the heavier the fermion, the stronger the coupling.

Thus, the Higgs boson decay is dominated by the heaviest particle which can be pair-produced in the decay. For $m_h \sim 125$ GeV, this is the bottom quark. Indeed, the SM predicts the following branching ratios for the leading decay modes:

$$\mathrm{BR}_{\bar{b}b} : \mathrm{BR}_{WW^*} : \mathrm{BR}_{gg} : \mathrm{BR}_{\tau^+\tau^-} : \mathrm{BR}_{ZZ^*} : \mathrm{BR}_{c\bar{c}} = 0.58 : 0.21 : 0.09 : 0.06 : 0.03 : 0.03. \tag{3.60}$$

The following comments are in order with regard to Eq. (3.60):

1. From the six branching ratios, three (b, τ, c) stand for two-body tree-level decays. Thus, at tree level, the respective branching ratios obey $\mathrm{BR}_{\bar{b}b} : \mathrm{BR}_{\tau^+\tau^-} : \mathrm{BR}_{c\bar{c}} = 3m_b^2 : m_\tau^2 : 3m_c^2$. QCD radiative corrections somewhat suppress the two modes with the quark final states (b, c) compared to one with the lepton final state (τ).

2. The WW^* and ZZ^* modes stand for the three-body tree-level decays, where one of the vector bosons is on-shell and the other off-shell.

3. The Higgs boson does not have a tree-level coupling to gluons since it carries no color (and the gluons have no mass). The decay into final gluons proceeds via loop diagrams. The dominant contribution comes from the top-quark loop.

4. Similarly, the Higgs decays into final two photons via loop diagrams with small ($\mathrm{BR}_{\gamma\gamma} \sim 0.002$), but observable, rate. The dominant contributions come from the W and the top-quark loops which interfere destructively.

Experimentally, the decays into final ZZ^*, WW^* and $\gamma\gamma$ have been established.

3.4.6 Summary

Within the SM, quarks have five types of interactions. These interactions are summarized in Table 3.1.

Table 3.1: The SM quark interactions

interaction	force carrier	coupling	range
electromagneric	γ	eQ	long
Strong	g	g_s	long
NC weak	Z^0	$\frac{e(T_3-s_W^2Q)}{s_Wc_W}$	short
CC weak	$W^\pm$	gV	short
Yukawa	h	y_q	short

3.5 Accidental symmetries

If we set the Yukawa couplings to zero, $\mathcal{L}_{\rm Yuk}=0$, the SM gains a large accidental global symmetry:

$$G_{\rm SM}^{\rm global}(Y^{u,d,e}=0)=U(3)_Q\times U(3)_U\times U(3)_D\times U(3)_L\times U(3)_E, \tag{3.61}$$

where $U(3)_Q$ has (Q_1,Q_2,Q_3) transforming as an $SU(3)_Q$ triplet, and all other fields singlets, $U(3)_U$ has (U_1,U_2,U_3) transforming as an $SU(3)_U$ triplet, and all other fields singlets, $U(3)_D$ has (D_1,D_2,D_3) transforming as an $SU(3)_D$ triplet, and all other fields singlets, $U(3)_L$ has (L_1,L_2,L_3) transforming as an $SU(3)_L$ triplet, and all other fields singlets, and $U(3)_E$ has (E_1,E_2,E_3) transforming as an $SU(3)_E$ triplet, and all other fields singlets.

The Yukawa couplings break this symmetry into the following subgroup:

$$G_{\rm SM}^{\rm global}=U(1)_B\times U(1)_e\times U(1)_\mu\times U(1)_\tau. \tag{3.62}$$

Under $U(1)_B$, all quarks (antiquarks) carry charge $+1/3$ $(-1/3)$, while all other fields are neutral. It explains why proton decay has not been observed. Possible proton decay modes, such as $p\to\pi^0e^+$ or $p\to K^+\nu$, are not forbidden by the $SU(3)_C\times U(1)_{\rm EM}$ symmetry. However, they violate $U(1)_B$, and therefore do not occur within the SM.[1] The lesson here is quire general: The lightest particle that carries a conserved charge is stable. The accidental $U(1)_B$ symmetry also explains why neutron-antineutron oscillations have not been observed.

The accidental symmetries of the renormalizable part of the SM Lagrangian also explain the vanishing of neutrino masses. Indeed, the explanation provided in Section 3.3.2 [see the discussion below Eq. (3.33)], namely the fact that there are no scalars transforming in the $(1,3)_{+1}$ representation, proves only the absence of neutrino masses at tree level. However, a Majorana mass term violates the accidental $B-L$ symmetry by two units. Thus, the symmetry prevents mass terms not only at tree level but also to all orders in perturbation theory. Moreover, since the

[1]The $U(1)_B$ symmetry is anomalous. Thus, baryon number violating processes might occur non-perturbatively. However, the non-perturbative effects obey $\Delta B=\Delta L=3n$, with n =integer, and thus do not lead to proton decay.

symmetry is non-anomalous (unlike B or L separately), Majorana mass terms do not arise even at the non-perturbative level. We conclude that the renormalizable SM gives the *exact* prediction:

$$m_\nu = 0. \tag{3.63}$$

3.5.1 parametr counting

The rule given by (1.90) can be applied to the standard model. Consider the quark sector of the model. The kinetic term has a global symmetry

$$G_f = U(3)_Q \times U(3)_U \times U(3)_D. \tag{3.64}$$

An $U(3)$ has 9 generators (3 real and 6 imaginary), so the total number of generators of G_f is 27. The Yukawa interactions defined in (**??**), Y^F $(F = u, d)$, are 3×3 complex matrices, which contain a total of 36 parameters (18 real parameters and 18 phases) in a general basis. These parameters also break G_f down to the baryon number

$$U(3)_Q \times U(3)_U \times U(3)_D \to U(1)_B. \tag{3.65}$$

While $U(3)^3$ has 27 generators, $U(1)_B$ has only one and thus $N_{\rm broken} = 26$. This broken symmetry allows us to rotate away a large number of the parameters by moving to a more convenient basis. Using (**??**), the number of physical parameters should be given by

$$N_{\rm phys} = 36 - 26 = 10. \tag{3.66}$$

These parameters can be split into real parameters and phases. The three unitary matrices generating the symmetry of the kinetic and gauge terms have a total of 9 real parameters and 18 phases. The symmetry is broken down to a symmetry with only one phase generator. Thus,

$$N^{(r)}_{\rm phys} = 18 - 9 = 9, \qquad N^{(i)}_{\rm phys} = 18 - 17 = 1. \tag{3.67}$$

We interpret this result by saying that of the 9 real parameters, 6 are the fermion masses and three are the CKM matrix mixing angles. The one phase is the CP-violating phase of the CKM mixing matrix.

In your homework you will count the number of parameters for different models.

3.6 The CKM parameters

The Cabibbo-Kobayashi-Maskawa (CKM) matrix determines the strength of the couplings of the W boson to quark-antiquark pairs,

$$\mathcal{L}_{Wq\bar{q}} = -\frac{g}{\sqrt{2}}\, \overline{u_{Li}}\, \gamma^\mu\, V_{ij}\, d_{Lj}\, W^+_\mu \;+\; \text{h.c.}. \tag{3.68}$$

Here $(u_1, u_2, u_3) = (u, c, t)$ and $(d_1, d_2, d_3) = (d, s, b)$:

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{3.69}$$

Since the W-bosons couple to pairs that belong to different generations, V is often referred to as the mixing matrix for quarks.

The present status of our knowledge of the absolute values of the various entries in the CKM matrix can be summarized as follows:

$$|V| = \begin{pmatrix} 0.97383 \pm 0.00024 & 0.2272 \pm 0.0010 & (3.96 \pm 0.09) \times 10^{-3} \\ 0.2271 \pm 0.0010 & 0.97296 \pm 0.00024 & (4.221^{+0.010}_{-0.080}) \times 10^{-2} \\ (8.14^{+0.32}_{-0.64}) \times 10^{-3} & (4.161^{+0.012}_{-0.078}) \times 10^{-2} & 0.999100^{+0.000034}_{-0.000004} \end{pmatrix}. \tag{3.70}$$

We emphasize that there is an inherent difficulty in determining the CKM parameters. While the SM Lagrangian has the quarks as its degrees of freedom, in Nature they appear only within hadrons. Thus, for example, the V_{cb} matrix element affects the rates of $b \to c\ell\nu$ decays, but what can be measured are hadronic processes such as $\bar{B} \to D\ell\nu$ decay. How can we relate the two processes? Our best chances of doing so in a reliable way arise when we can use approximate symmetries of QCD. An example of how isospin symmetry relates hadron decays to the $u \to d\ell\nu$ decays is given in Appendix 3.A. Approximate symmetries such as isospin, $SU(3)$-flavor and heavy quark symmetry are useful for semileptonic or leptonic decays, where the relevant operators involve only two quarks.

While a general 3×3 unitary matrix depends on three mixing angles and six phases, the freedom to redefine the phases of the quark mass eigenstates can be used to remove five of the phases, leaving a single physical phase, the Kobayashi-Maskawa phase, that is responsible for all CP violation in meson decays in the Standard Model. The freedom of redefining phases can be understood by examining the Lagrangian mass terms for quarks,

$$\mathcal{L}_{m_q} = m_u \overline{u_L} u_R + m_c \overline{c_L} c_R + m_t \overline{t_L} t_R + m_d \overline{d_L} d_R + m_s \overline{s_L} s_R + m_b \overline{b_L} b_R + \text{h.c.}. \tag{3.71}$$

$\mathcal{L}_{m_q}$ and, obviously, all kinetic and $SU(3)_C \times U(1)_{\rm EM}$ terms are invariant under the following phase transformations:

$$\begin{aligned} &u_{L,R} \to e^{i\phi_u} u_{L,R}, \quad c_{L,R} \to e^{i\phi_c} c_{L,R}, \quad t_{L,R} \to e^{i\phi_t} t_{L,R}, \\ &d_{L,R} \to e^{i\phi_d} d_{L,R}, \quad s_{L,R} \to e^{i\phi_s} s_{L,R}, \quad b_{L,R} \to e^{i\phi_b} b_{L,R}. \end{aligned} \tag{3.72}$$

Thus, by performing (3.72), we are changing from one mass basis to another. The $Zq\bar{q}$ couplings, Eq. (3.42), are also invariant under (3.72),but the $Wq\bar{q}$, Eq. (3.68), are not: $V_{ij} \to e^{i(\phi_{u_i} - \phi_{d_j})} V_{ij}$. A-priori, we may think that we have all six phases of (3.72) at our disposal to affect the parameters of V. However, V is invariant under the transformation where all six phases are equal, $\phi_u = \phi_c =$

$\phi_t = \phi_d = \phi_s = \phi_b$. This is just a manifestation of the fact that $U(1)_B$ is an accidental symmetry of the Standard Model.We can thus remove only five phases from V, leaving us with three real mixing angles and a single phase.

The fact that one can parameterize V with three real and one imaginary physical parameters can be made manifest by choosing an explicit parametrization. Given the hierarchy in the values of the various entries as reflected in Eq. (3.70), the Wolfenstein parametrization, with three real parameters, λ, A and ρ, and one imaginary parameter, $i\eta$, is particularly useful:

$$V = \begin{pmatrix} 1-\frac{1}{2}\lambda^2-\frac{1}{8}\lambda^4 & \lambda & A\lambda^3(\rho-i\eta) \\ -\lambda+\frac{1}{2}A^2\lambda^5[1-2(\rho+i\eta)] & 1-\frac{1}{2}\lambda^2-\frac{1}{8}\lambda^4(1+4A^2) & A\lambda^2 \\ A\lambda^3[1-(1-\frac{1}{2}\lambda^2)(\rho+i\eta)] & -A\lambda^2+\frac{1}{2}A\lambda^4[1-2(\rho+i\eta)] & 1-\frac{1}{2}A^2\lambda^4 \end{pmatrix}. \tag{3.73}$$

Here $\lambda(\approx 0.23)$ plays the role of an expansion parameter. Terms of $\mathcal{O}(\lambda^6)$ and higher were neglected.

The unitarity of the CKM matrix, $(VV^\dagger)_{ij} = (V^\dagger V)_{ij} = \delta_{ij}$, leads to twelve distinct complex relations among the matrix elements. The six relations with $i \neq j$ can be represented geometrically as triangles in the complex plane. One of these,

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0, \tag{3.74}$$

is particularly useful. It has terms of equal order, $\mathcal{O}(A\lambda^3)$, and so has a corresponding triangle whose interior angles are all $\mathcal{O}(1)$ physical quantities that can be independently measured:

$$\alpha \equiv \arg\left[-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right], \quad \beta \equiv \arg\left[-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right], \quad \gamma \equiv \arg\left[-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right]. \tag{3.75}$$

Appendix

3.A Isospin symmetry

The interpretation of measured rates of weak decays of hadrons in terms of parameters of the Standard Model Lagrangian are complicated by strong interaction effects which are not subject to perturbative expansion. However, the use of approximate symmetries of the strong interactions in analyzing various semileptonic processes allows one, in some cases, to overcome these difficulties and obtain a quantitatively clean interpretation. We emphasize that such interpretation may be possible for semi-leptonic decays, where the matrix elements of two-quark operators between hadronic states are required, but impossible for non-leptonic decays, where the matrix elements of four-quark operators are required.

In the limit

$$m_u = m_d, \tag{3.76}$$

the QCD Lagrangian for quarks,

$$\mathcal{L}_{\rm QCD} = g_s \bar{q}\gamma^\mu G_\mu q + m_q \bar{q} q, \tag{3.77}$$

has a global $SU(2)$ symmetry called isospin, under which the up and down quarks transform as a doublet:

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}. \tag{3.78}$$

Isospin is broken by the electromagnetic (and weak) interactions, and by the $u-d$ mass difference. The respective dimensionless breaking parameters can be taken as α and $(m_d - m_u)/\Lambda_{\rm QCD}$ and are thus of order a percent. Indeed, isospin is observed to be an excellent approximate symmetry of the strong interactions. As far as spectroscopy is concerned, it explains for example the quasi-degeneracy of $n-p$ ($m_p = 938.27$ MeV, $m_n = 939.57$ MeV, $\Delta m/m \sim 0.001$), of $\pi^+ - \pi^0$ ($m_{\pi^+} = 139.57$ MeV, $m_{\pi^0} = 134.98$ MeV, $\Delta m/m \sim 0.03$), of $K^+ - K^0$ ($m_{K^+} = 493.68$ MeV, $m_{K^0} = 497.61$ MeV, $\Delta m/m \sim 0.008$), and many other isospin multiplets.

The charged current weak interactions involve quark bilinears of the form

$$\bar{u}\gamma^\mu(1-\gamma_5)d, \tag{3.79}$$

which can be written as a combination of a vector and axial isospin currents,

$$j_a^\mu = \frac{1}{2}\bar{\psi}\gamma^\mu\sigma_a\psi, \tag{3.80}$$

$$j_{5a}^\mu = \frac{1}{2}\bar{\psi}\gamma^\mu\gamma_4\sigma_a\psi. \tag{3.81}$$

The (approximately) conserved isospin charge is

$$Q_a = \int d^3x\ j_a^0(x). \tag{3.82}$$

We can use symmetry considerations to determine the matrix elements of Q_a, and therefore of the conserved current j_+^μ of Eq. (3.79)..

Since we do not have a similar tool to find the matrix element of the axial current, we better focus on processes where only the vector current contributes. The relevant processes are of the class $M_1 \to M_2\ell\nu$, where M_1 and M_2 are in the same isospin multiplet, and are both spin-0 and of the same parity (namely, both scalars or both pseudoscalars). For such processes, the decay amplitude has the form

$$A = \frac{G_F}{\sqrt{2}}|V_{ud}|^2\langle M_2|j^\mu - j_5^\mu|M_1\rangle l_\mu, \tag{3.83}$$

where l_μ is the lepton current matrix element. Since parity is conserved by the strong interactions, we have $\langle M_2|j_5^\mu|M_1\rangle = 0$, and

$$A = \frac{G_F}{\sqrt{2}}|V_{ud}|^2\langle M_2(p_2)|j^\mu|M_1(p_1)\rangle l_\mu. \tag{3.84}$$

Define

$$q^\mu = p_1^\mu - p_2^\mu. \tag{3.85}$$

Then, Lorentz invariance implies

$$\langle M_2(p_2)|j^\mu(0)|M_1(p_1)\rangle = C(q^2)(p_1^\mu + p_2^\mu) + D(q^2)q^\mu. \tag{3.86}$$

In the isospin limit, the current j^μ is conserved. Thus, q_μ contracted with (3.86) must vanish. We thus obtain

$$q^2D(q^2) = 0 \implies D(q^2) = 0 \text{ for } q^2 \neq 0. \tag{3.87}$$

We are left with $C(q^2)$. Here we note that the matrix element of the charge Q_a between M_1 and M_2 is completely determined. Since M_1 and M_2 are in the same $SU(2)_I$ multiplet, we can write

$$\begin{aligned} |M_1, p_1\rangle &= |j, m_1, p_1\rangle, \\ |M_2, p_2\rangle &= |j, m_2, p_2\rangle, \end{aligned} \tag{3.88}$$

where j and m_i are the isospin and Q_3 values. In general,

$$Q_a|j, m, p\rangle = |j, m', p\rangle(T_a)_{m'm}. \tag{3.89}$$

The two-quark operator (3.79) corresponds to the raising operator,

$$Q_+ = Q_1 + iQ_2 = \int d^3x j^0(x) = \int d^3x u^\dagger(x)d(x), \tag{3.90}$$

which satisfies

$$Q_+|j,m,p\rangle = \sqrt{(j-m)(j+m+1)}|j,m+1,p\rangle, \tag{3.91}$$

so that m_2 in (3.88) is m_1+1, and thus

$$\langle M_2,p_2|Q_+|M_1,p_1\rangle = \sqrt{(j-m)(j+m+1)}(2\pi)^3 2p^0\delta^{(3)}(\vec{p}_1-\vec{p}_2), \tag{3.92}$$

for states with standard normalization.

Now, we calculate (3.92) in a different way. From translational invariance,

$$\langle M_2,p_2|j^0(x)|M_1,p_1\rangle = e^{ix\cdot(p_1-p_2)}\langle M_2,p_2|j^0(0)|M_1,p_1\rangle. \tag{3.93}$$

From Eqs. (3.86) and (3.87),

$$\langle M_2,p_2|Q_+|M_1,p_1\rangle = C(0)(2\pi)^3 2p^0\delta^{(3)}(\vec{p}_1-\vec{p}_2). \tag{3.94}$$

As expected, the time dependence of Q_+ goes away in the symmetry limit. This can be traced to the fact that $p_1^0 = p_2^0$ when $\vec{p}_2 = \vec{p}_1$. Comparing (3.92) and (3.94), we obtain

$$C(0) = \sqrt{(j-m)(j+m+1)}. \tag{3.95}$$

We thus know the value of $C(0)$ in the symmetry limit. This should be an excellent approximation to $C(0)$, since it is violated by effects of order $\Delta m/\Lambda_{\rm QCD} = \mathcal{O}(0.01)$. As concerns the q^2 dependence of $C(q^2)$, we can go beyond the estimate that this dependence is determined by $\Lambda_{\rm QCD}$ while $q^2 < (\Delta m)^2$ by modeling it or, even better, measuring it.

Among the processes to which we can apply such an analysis, we have the decays $\pi^+ \to \pi^0 e^+\nu$, ^{34}Cl $\to$ ^{34}S $e^+\nu$, ^{14}O $\to$ ^{14}N $e^+\nu$, and ^{26}Al $\to$ ^{26}Mg $e^+\nu$.

Chapter 4

The SM beyond tree level

4.1 Introduction

The SM is not a full theory of Nature. It is only a low energy effective theory, valid below some scale $\Lambda(\gg m_Z)$. Then, the SM Lagrangian should be extended to include all non-renormalizable terms, suppressed by powers of Λ:

$$\mathcal{L} = \mathcal{L}_{\rm SM} + \frac{1}{\Lambda} O_{d=5} + \frac{1}{\Lambda^2} O_{d=6} + \cdots, \tag{4.1}$$

where $O_{d=n}$ represents operators that are products of SM fields, transforming as singlets under the SM gauge group, of overall dimension n in the fields. As explained above, for physics at an energy scale E well below Λ, the effects of operators of dimension $n > 4$ is suppressed by $(E/\Lambda)^{n-4}$. Thus, the higher the dimension of an operator, the smaller its effect at low energies.

In previous sections, we studied the gauge sector of the SM at tree level and with only renormalizable terms. We can classify the effects of including loop corrections and nonrenormalizable terms into three broad categories:

1. *Forbidden processes:* Various processes are forbidden by the accidental symmetries of the Standard Model. Nonrenormalizable terms (but not loop corrections!) can break these accidental symmetries and allow the forbidden processes to occur. Examples include neutrino masses and proton decays.

2. *Rare processes:* Various processes are not allowed at tree level. These effects can often be related to accidental symmetries that hold within a particular sector of, but not in the entire SM. Here both loop corrections and nonrenormalizable terms can contribute. Examples include FCNC processes.

3. *Tree level processes:* Often tree level processes in a particular sector depend on a very small subset of the SM parameters. This situation leads to relations among different processes

within this sector. These relations can be violated by both loop processes and nonrenormalizable terms. Here, precision measurements (and precision theory calculations) are needed to observe these small effects. Examples include electroweak precision measurements.

As concerns the last two types of effects, where loop corrections and nonrenormalizable terms may both contribute, their use in phenomenology can be divided to two eras. Before all the SM particles have been directly discovered and all the SM parameters measured, one could assume the validity of the renormalizable SM and predict the properties of yet unobserved particles. Indeed, both the top mass and the Higgs boson mass were predicted in this way. Once all the particles have been observed and the parameters measured, the loop corrections can be quantitatively determined, and effects of nonrenormalizable terms can be unambiguously probed. Thus, at present, all three classes of processes serve to search for new physics.

4.2 Electroweak Precision Measurements (EWPM)

At tree level, all (flavor diagonal) electroweak processes depend on only three of the renormalizable SM parameters. This is the starting point of using precision measurements of electroweak processes to determine the effects of loop corrections and to probe nonrenormalizable terms.

At the language of the Lagrangian of Eq. (???), the three parameters are g, g' and v. It is convenient for our purposes to work with combinations of these parameters that are best measured: α, m_Z and G_F.

The number of relevant observables is much larger than three. Thus, at tree level, a large number of relations among these observables are predicted. These predictions are, however, violated by SM loop effects, and possibly by nonrenormalizable operators that are generated by BSM physics.

The program of EWPM takes advantage of the fact that in any quantum theory, if one sector is subject to a symmetry that is broken by other sectors, at the quantum level such a symmetry is broken in all sectors. While the electroweak gauge sector of the SM has three parameters, the full SM has eighteen. Thus, once we go to the quantum level, the fifteen other parameters modify the tree level relations. It is these effects that are probed with the EWPM program. Eleven of the fifteen parameters (eight of the fermion masses and the three CKM mixing angles) are small, and thus they have negligible effects on deviations from the tree level relations. The four large parameters are the Kobayashi-Maskawa (KM) phase, the strong coupling constant, the Higgs self-coupling and the top Yukawa coupling. The KM phase has negligible effects on flavor diagonal processes. As concerns the strong coupling constant, its universality and the fact that the electroweak vector bosons do not couple directly to gluons combine to make its effect on the relevant parameters very small. Thus, in practice, there are only two SM parameters that have significant one loop effects on the EWPM: m_t/v and m_H/v. In the past, when these masses had not yet been directly measured, the EWPM were used to predict their values. Now, that the top quark and the Higgs boson have been discovered and their masses are known from direct measurements, the EWPM are used to probe NR operators, that is, Beyond the SM (BSM) physics.

As concerns the experimental aspects of the EWPM program, we note that the relevant processes can be divided to two classes: low-energy and high-energy. The "low-energy" observables involve processes with a characteristic energy scale well below m_W and m_Z, so that the intermediate W-boson or Z-boson are far off-shell. The "high-energy" observables are measured in processes where the W-boson or the Z-boson are on-shell. The low-energy EWPM include measurements of G_F and α, as well as data from neutrino scattering, deep inelastic scattering (DIS), atomic parity violation (APV) and low energy e^+e^- scattering. The high-energy EWPM include measurements of the masses, the total widths and partial decay widths of the W and Z bosons. In the appendix we discuss some examples of these observables in detail and derive their dependence on the SM parameters.

4.2.1 The weak angle

As our first example we consider three definitions of the weak angle. Each definition involves a different set of observables. At tree level all three definitions are identical,

$$\tan\theta_{\rm tree} \equiv g'/g, \tag{4.2}$$

but at one loop they are not.

1. Definition in terms of α, G_F and m_Z:

$$\sin^2\theta_0 \equiv \frac{4\pi\alpha(m_Z)}{\sqrt{2}G_F m_Z^2}. \tag{4.3}$$

   Quantitatively, θ_0 is defined in terms of the best measured observables and thus has the smallest experimental uncertainties.

2. Definition in terms of m_W and m_Z:

$$\sin^2\theta_W \equiv 1 - \frac{m_W^2}{m_Z^2}. \tag{4.4}$$

   This definition is based on the tree level relation $\rho = 1$ discussed earlier.

3. Definition in terms of g_V and g_A:

$$\sin^2\theta_*^i \equiv \frac{g_A^i - g_V^i}{2Q_i}. \tag{4.5}$$

   These parameters refer to the Z couplings to fermions:

$$\mathcal{L}_{Z\psi\overline{\psi}} \propto \overline{\psi}_i\gamma^\mu\left(g_V^i - g_A^i\gamma_5\right)\psi_i Z_\mu, \tag{4.6}$$

where i is a flavor index that is not summed. In principle we have here nine different definitions of θ_*, one for each charged fermion. Each of them can be measured using parity violating observables, such as forward-backward asymmetries, that are sensitive to g_A/g_V. We discuss these observables in more detail in the appendix.

To proceed, we make the following working assumption:

1. The only significant effects are in the electroweak gauge boson propagators. These effects are called *oblique corrections.*

2. All θ_*^i are equal, that is, the relevant one loop effects are flavor universal.

Note that the first assumption implies the second. Yet, one can relax the strong first assumption and maintain the milder second assumption, thus including a larger class of BSM models.

We denote the oblique corrections to the electroweak gauge boson propagators by $\Pi_{AB}(q^2)$. The propagators P_{AB} are defined as follows:

$$P_{AB}(q^2) = \frac{-i}{q^2 - m_A^2}\left[\delta_{AB} + \frac{-i\Pi_{AB}(q^2)}{q^2 - m_B^2}\right]. \tag{4.7}$$

Taking charge conservation into account, we learn that there are four Π_{AB}'s that do not vanish: Π_{WW}, Π_{ZZ}, $\Pi_{\gamma\gamma}$ and $\Pi_{\gamma Z}(= \Pi_{Z\gamma})$. At $q^2 = 0$, Π can be identified as correction to the masses of the gauge bosons. Thus, gauge invariance guarantees that

$$\Pi_{\gamma\gamma}(q^2 = 0) = \Pi_{\gamma Z}(q^2 = 0) = 0. \tag{4.8}$$

How do the various corrections to the propagators affect the three differently-defined θ's?

1. Corrections to θ_0:

$$\Delta \sin^2\theta_0 = \frac{\tan^2\theta}{4}\left[\Pi'_{\gamma\gamma}(0) + \frac{\Pi_{WW}(0)}{m_W^2} - \frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2}\right]. \tag{4.9}$$

   The three terms correspond to corrections to α, G_F and m_Z, respectively.

2. Corrections to θ_W:

$$\Delta \sin^2\theta_W = \left(\Pi_{WW} - \frac{m_W^2}{m_Z^2}\Pi_{ZZ}\right) \tag{4.10}$$

   The two terms correspond to corrections to the tree level heavy gauge boson masses,

$$\Delta m_W^2 = \Pi_{WW}(m_W^2), \qquad \Delta m_Z^2 = \Pi_{ZZ}(m_Z^2), \tag{4.11}$$

3. Corrections to θ_*:

$$\Delta \sin^2\theta_* = -\sin\theta\cos\theta\frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2} \tag{4.12}$$

   The corrections arise from the mixing between the off-shell photon and the Z-boson.

We learn that the loop effects are different for the three definitions. Once we extract the values of $\sin\theta$ from each of the three sets of observables, we can probe these effects.

To proceed, we expand the Π_{AB}'s in q^2, and make one more working assumption:

3. Terms of order $(q^2)^2$ and higher can be neglected. We thus keep only the first two terms in the q^2-expansion:

$$\Pi_{AB}(q^2) = \Pi_{AB}(0) + q^2\Pi'_{AB}(0), \qquad \Pi'(q^2) \equiv \frac{d\Pi(q^2)}{dq^2}. \tag{4.13}$$

The latter assumption, in combination with Eq. (4.8), replace the four functions $\Pi_{AB}(q^2)$ with six quantities: $\Pi'_{\gamma\gamma}(0)$, $\Pi'_{\gamma Z}(0)$, $\Pi'_{WW}(0)$, $\Pi'_{ZZ}(0)$, $\Pi_{WW}(0)$ and $\Pi_{ZZ}(0)$.

4.2.2 Within the SM

Our analysis so far can be applied to all models where only oblique corrections are significant and where the q^2 expansion holds. We now proceed to the specific case of the SM. For the SM, we can safely make the following approximation:

4. Two loop corrections can be neglected. In the fermion sector, only the one loop top contributions are significant.

Thus, for the SM, there are four one-loop diagrams that are important for our purposes. They will be added to this text (but for now you can just find them in 21.13 of Peskin's book).

The next step is the actual calculation of these diagrams. We do not reproduce the calculation here, but we make one comment regarding the finiteness of the results. Naively, each diagram is quadratically divergent. Ward identities prevent, however, the quadratic divergences, leaving only logarithmic ones. The logarithmic divergences appear in each of the $\Delta\sin^2\theta$, but they cancel in the differences between any two observable quantities. The final results are the following:

$$\sin^2\theta_0 - \sin^2\theta_* = \frac{3\alpha}{16\pi\cos^2 2\theta}\frac{m_t^2}{m_Z^2}, \tag{4.14}$$

$$\sin^2\theta_W - \sin^2\theta_* = \frac{-3\alpha}{16\pi\sin^2\theta}\frac{m_t^2}{m_Z^2}. \tag{4.15}$$

The factor of $\alpha/16\pi$ is typical of electroweak one-loop effects. The factor of 3 is the color factor of the top quark in the loop. The factor of m_t^2/m_Z^2 deserves a more detailed discussion, which we now turn to.

Naively, quadratic dependence on the top mass is puzzling since it seems to violate the so-called *decoupling theorem*. The theorem states that the effect of heavy states on low energy observables must go to zero as their mass goes to infinity. The intuition behind this theorem is straightforward. The heavier a state is, the smaller its effects (when off-shell) become. This can be understood based on the uncertainty principle, or on second order perturbation theory, or simply by considering the form of propagators in QFT. Why doesn't this theorem apply to the top contribution to EWPM?

The solution to the puzzle lies in the fact that the SM quarks acquire their masses from the Higgs mechanism. Consequently, their Yukawa couplings are proportional to their masses. The heavier the top, the stronger its Yukawa coupling becomes. Indeed, the top-related loop corrections to EWPM depend on the top couplings to the longitudinal W and Z, which are its Yukawa couplings. The quadratic dependence on the top mass reflects the proportionality of the loop corrections to the top Yukawa coupling, and not to its mass. (In fact, the $m_t \to \infty$ cannot be taken, because perturbation theory does not hold anymore.)

The one loop corrections to EWPM are sensitive also to the Higgs mass. It turns out the m_H-dependance is logarithmic. This result is known as the *screening theorem*. We do not discuss it in detail here.

4.2.3 Beyond the SM

Within the SM, the EWPM program is sensitive at tree level to three input parameters and at the loop level to a few more. Since we have more observables than relevant SM parameters, EWPM can be used to test the SM. So far, no significant deviation from the SM was found. Furthermore, all the relevant SM parameters are now directly measured. Thus, the data can be use to constrain BSM physics.

As concerns new physics, one can either consider a specific model, or add nonrenormalizable terms to the SM. We here focus on extensions of the SM that fulfill the following three conditions:

1. The scale of the new physics is much higher than the electroweak breaking scale, $\Lambda \gg m_W$. (This condition holds, by definition, to new physics whose effects can be represented by nonrenormalizable terms.)

2. The effects of the new physics generate only oblique corrections.

3. There are no new bosons that mediate the relevant electroweak processes at tree level. In particular, the electroweak symmetry remains $SU(2)_L \times U(1)_Y$.

In the above case we can parametrize all the NP by three new parameters. Working to leading order we know we have 6 parameters, $\Pi_{ZZ}(0)$, $\Pi_{WW}(0)$ and $\Pi'(0)$ for the other four combinations. Thus, we can relate any six measurements to the six theoretical parameters. In the SM at tree level there are three parameters, so we end up with three new parameters.

With these three conditions, there is a convenient intermediate step in the procedure of translating the EWPM to constraints on the new physics parameters. This intermediate step involves three appropriately defined parameters (the so-called S, T and U parameters) that are affected by new physics:

$$\alpha T = \frac{\Pi_{WW}}{M_W^2} - \frac{\Pi_{ZZ}}{M_Z^2}, \tag{4.16}$$

$$\frac{\alpha S}{4\sin^2 2\theta} = \Pi'_{ZZ} - \frac{2\cos^2 2\theta}{\sin^2 2\theta}\Pi'_{Z\gamma} - \Pi'_{\gamma\gamma}, \tag{4.17}$$

$$\frac{\alpha U}{4\sin^2\theta} = \Pi'_{WW} - \cos^2\theta\Pi'_{ZZ} - \sin 2\theta\Pi'_{\gamma Z} - \sin^2\theta\Pi'_{\gamma\gamma}. \tag{4.18}$$

A few remarks are in order:

1. S, T and U are pure numbers. They are scaled by α just to make them of $O(1)$ in the SM.

2. These new parameters receive one loop contributions in the SM as well as potential BSM ones. One can subtract the SM values and redefine them such that within the SM they vanish. With this new definition, a non-zero value would be a sign of new physics.

3. The U parameter rarely provides a significant constraint. As we explain below, the reason is that U arises from a dimension-8 operator, while S and T arise from dimension-6 operators.

4. An important role in this discussion is played by the so-called *custodial symmetry.* which is discussed in the next subsection. Here we only mention that T is related to custodial symmetry breaking while S is not.

As a simple example, consider a four generation extension of the SM. We denote the new quarks by t' and b'. Assuming no flavor mixing between the fourth generation and the lighter three known, we obtain: We learn that T is related to the mass splitting between t' and b', while S, in a way, "counts" the number of new $SU(2)_L$ doublets.

4.2.4 Custodial symmetry

In the SM, the Higgs potential has an accidental symmetry. This symmetry has important implications on the EWPM. Consider the SM Higgs potential:

$$V = -\mu^2|\phi|^2 + \lambda|\phi|^4. \tag{4.19}$$

Since ϕ is a complex, $SU(2)_L$-doublet scalar field, it has four degrees of freedom:

$$\phi = \begin{pmatrix} \phi_3 + i\phi_4 \\ \phi_1 + i\phi_2 \end{pmatrix}. \tag{4.20}$$

The scalar potential, when written in terms of these four components, depends only on the combination $\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2$, and thus has manifestly an $SO(4)$ symmetry. At the algebra level, $SO(4) \sim SU(2) \times SU(2)$. Out of the six generators, four are also generators of the gauge group

$SU(2)_L \times U(1)_Y$. The two extra generators are then related to an accidental symmetry of the scalar sector of the SM.

The VEV of the Higgs field breaks three of the generators, leaving (within the pure Higgs sector) an unbroken $SU(2)$ symmetry. This symmetry is called the custodial symmetry. Under this symmetry, the (W_1, W_2, W_3) DOFs transform as a triplet. Consequently, the mass terms induced by the spontaneous symmetry breaking are equal for these three DOFS.

The most general mass matrix in the (W_1, W_2, W_3, B) basis, that is consistent with $U(1)_{\rm EM}$ gauge invariance, is given by

$$\begin{pmatrix} m_W^2 & & & \\ & m_W^2 & & \\ & & m_Z^2 c_W^2 & m_Z^2 c_W s_W \\ & & m_Z^2 c_W s_W & m_Z^2 s_W^2 \end{pmatrix}. \tag{4.21}$$

The custodial symmetry requires that the top three diagonal terms are equal and thus that $m_Z^2 c_W^2 = m_W^2$, namely the $\rho = 1$ relation.

The custodial symmetry holds at tree level for models with any number of scalar doublets and singlets. It is however not a symmetry of the full SM. Thus, loop effects violate the predictions that follow from the custodial symmetry. The most significant violation within the SM comes from $m_t \neq m_b$. This is the reason that the leading correction to the $\rho = 1$ relation is proportional to $m_t^2 - m_b^2$.

1. *It can be generelized, such that the massive gauge bosons masses are related by some CG coefficients.*

2. *In general, when we have several different representations the mass eigenstase are not related by CG coefficients.*

3. *Due to the W^0 and B mixing, the Z and photon are the mass eigenstate. This make the situation more complictaed and result in the $\rho = 1$ relation.*

4.2.5 Nonrenormalizable terms

The most general way of including the effects of heavy new physics is to add nonrenormalizable terms to the SM. The most severe constraints from EWPM apply to flavor conserving dimension-six terms. There are 28 such operators. Seven of these do not affect EWPM, so we need to consider 21 operators. Such a general analysis, taking into account correlations among operators, was done in [3]. It turns out that the most important operators are those that correspond to the T and S parameters:

$$T \sim \frac{(H^\dagger D_\mu H)^2}{\Lambda^2}, \qquad S \sim \frac{H^\dagger W^{\mu\nu} B_{\mu\nu} H}{\Lambda^2}. \tag{4.22}$$

The reason is that indeed the assumption that the most important effect are in the oblique corrections. The bound the data give on the scale of the operators is of order 10 TeV. You can read the bounds at [3].

Chapter 5

Flavor physics

5.1 Introduction

The effects of non-renormalizable terms might be observed in rare processes, where the contribution from the renormalizable SM is highly suppressed. The prime example of such processes are flavor changing neutral current processes. In this section, we explain what these processes are, describe the phenomenological constraints on deviations from the SM predictions, and extract lower bounds on the scale that suppresses dimension-six terms that contribute to these processes.

The term "flavor" is used, in the jargon of particle physics, to describe several copies of the same gauge representation. Within the Standard Model, each of the four different types of fermionic particles comes in three flavors:

- Up-type quarks in the $(3)_{+2/3}$ representation: u, c, t;
- Down-type quarks in the $(3)_{-1/3}$ representation: d, s, b;
- Charged leptons in the $(1)_{-1}$ representation: e, μ, τ;
- Neutrinos in the $(1)_0$ representation: ν_1, ν_2, ν_3.

The term "*flavor physics*" refers to interactions that distinguish between flavors. Within the SM, these are the W-mediated weak interactions and the Yukawa interactions. The term "*flavor parameters*" refers to parameters that carry flavor indices. Within the SM, there are 13 flavor parameters: the 9 charged fermion masses and the four CKM parameters. (As explained in Section 4.1, if one augments the SM with Majorana mass terms for neutrinos, one should add to the list 9 parameters: 3 neutrino masses, and 6 parameters of the leptonic mixing matrix.) The term "*flavor universal*" refers to interactions with couplings (or to parameters) that are proportional to a unit matrix in flavor space. Within the SM, the strong, electromagnetic, and Z-mediated weak interactions are flavor-universal. The term "*flavor diagonal*" refers to interactions with couplings

Table 5.1: Measurements related to neutral meson mixing

Sector	CP-conserving	CP-violating
sd	$\Delta m_K/m_K = 7.0 \times 10^{-15}$	$\epsilon_K = 2.3 \times 10^{-3}$
cu	$\Delta m_D/m_D = 8.7 \times 10^{-15}$	$A_\Gamma/y_{\rm CP} \lesssim 0.2$
bd	$\Delta m_B/m_B = 6.3 \times 10^{-14}$	$S_{\psi K} = +0.67 \pm 0.02$
bs	$\Delta m_{B_s}/m_{B_s} = 2.1 \times 10^{-12}$	$S_{\psi\phi} = -0.04 \pm 0.09$

(or to parameters) that are diagonal, but not necessarily universal, in flavor space. Within the SM, the Yukawa interactions are flavor-diagonal.

A central role in testing the CKM sector of the SM is played by flavor changing processes. The term "*flavor-changing*" refers to processes where the initial and final flavor-numbers (that is, the number of particles of a certain flavor minus the number of anti-particles of the same flavor) are different. In "*flavor changing charged current*" processes, both up-type and down-type flavors, and/or both charged lepton and neutrino flavors are involved. Examples are $K^- \to \mu^-\bar\nu_\mu$ which corresponds, at the quark level, to $s\bar u \to \mu^-\bar\nu_\mu$ transition, and $\bar B \to \psi K_S$ ($b \to c\bar c s$ transition). Within the Standard Model, these processes are mediated by the W-bosons and occur at tree level. In "*flavor changing neutral current*" (FCNC) processes, either up-type or down-type flavors but not both, and/or either charged lepton or neutrino flavors but not both, are involved. Examples of FCNC decays include $\mu \to e\gamma$, $K_L \to \mu^+\mu^-$ ($s\bar d \to \mu^+\mu^-$ transition), and $B \to \phi K_S$ ($b \to s\bar s s$ transition). Within the Standard Model, these processes do not occur at tree level, and are strongly suppressed.

5.1.1 Flavor changing neutral current (FCNC) processes

A very useful class of FCNC is that of neutral meson mixing. Nature provides us with four pairs of neutral mesons: $K^0 - \overline{K}^0$, $B^0 - \overline{B}^0$, $B_s^0 - \overline{B}_s^0$, and $D^0 - \overline{D}^0$. Mixing in this context refers to a transition such as $K^0 \to \overline{K}^0$ ($\bar s d \to \bar d s$).[1] The experimental results for CP conserving and CP violating observables related to neutral meson mixing (mass splittings and CP asymmetries in tree level decays, respectively) are given in Table 5.1.

Our aim in this subsection is to explain the suppression factors that affect FCNC within the SM.

(a) **Loop suppression.** The W-boson cannot mediate FCNC processes at tree level, since it couples to up-down pairs, or to neutrino-charged lepton pairs. Obviously, only neutral bosons

[1] These transitions involve four-quark operators. When calculating the matrix elements of these operators between meson-antimeson states, approximate symmetries of QCD are of no help. Instead, one uses lattice calculations to relate, for example, the $B^0 \to \overline{B}^0$ transition to the corresponding quark process, $\bar b d \to \bar d b$.

can mediate FCNC at tree level. The SM has four different types of neutral bosons: the gluons, the photon, the Z-boson and the Higgs-boson. As concerns the massless gauge bosons, the gluons and the photon, their couplings are flavor-universal and, in particular, flavor-diagonal. This is guaranteed by gauge invariance. The universality of the kinetic terms in the canonical basis requires universality of the gauge couplings related to the unbroken symmetries. Hence neither the gluons nor the photon can mediate flavor changing processes at tree level. The situation concerning the Z-boson and the Higgs-boson is more complicated. In fact, the diagonality of their tree-level couplings is a consequence of special features of the SM, and can be violated with new physics.

The Z-boson, similarly to the W-boson, does not correspond to an unbroken gauge symmetry (as manifest in the fact that it is massive). Hence, there is no fundamental symmetry principle that forbids flavor changing couplings. Yet, as mentioned in Section 3.4.3, in the SM this does not happen. The key point is the following. For each sector of mass eigenstates, characterized by spin, $SU(3)_C$ representation and $U(1)_{\rm EM}$ charge, there are two possibilities:

1. All mass eigenstates in this sector originate from interaction eigenstates in the same $SU(2)_L \times U(1)_Y$ representation.

2. The mass eigenstates in this sector mix interaction eigenstates of different $SU(2)_L \times U(1)_Y$ representations (but, of course, with the same $T_3 + Y$).

Let us examine the Z couplings in the interaction basis in the subspace of all states that mix within a given sector of mass eigenstates:

1. In the first class, the Z couplings in this subspace are universal, namely they are proportional to the unit matrix (times $T_3 - Q \sin^2\theta_W$ of the relevant interaction eigenstates). The rotation to the mass basis maintains the universality: $V_{fM} \times \mathbf{1} \times V_{fM}^\dagger = \mathbf{1}$ ($f = u, d, e$; $M = L, R$).

2. In the second class, the Z couplings are only "block-universal". In each sub-block i of m_i interaction eigenstates that have the same $(T_3)_i$, they are proportional to the $m_i \times m_i$ unit matrix, but the overall factor of $(T_3)_i - Q\sin^2\theta_W$ is different between the sub-blocks. In this case, the rotation to the mass basis, $V_{fM} \times {\rm diag}\{[(T_3)_1 - Qs_W^2]\mathbf{1}_{m_1}, [(T_3)_2 - Qs_W^2]\mathbf{1}_{m_2}, \ldots\} \times V_{fM}^\dagger$, does not maintain the universality, nor even the diagonality.

The special feature of the SM fermions is that they belong to the first class: All fermion mass eigenstates in a given $SU(3)_C \times U(1)_{\rm EM}$ representation come from the same $SU(3)_C \times SU(2)_L \times U(1)_Y$ representation.[2] For example, all the left-handed up quark mass eigenstates, which are in the $(3)_{+2/3}$ representation, come from interaction eigenstates in the $(3,2)_{+1/6}$ representation. This is the reason that the SM predicts universal Z couplings to fermions. If, for example, Nature had

[2] This is not true for the SM bosons. The vector boson mass eigenstates in the $(1)_0$ representation come from interaction eigenstates in the $(1,3)_0$ and $(1,1)_0$ representations (W_3 and B, respectively).

left-handed quarks in the $(3,1)_{+2/3}$ representation, then the Z couplings in the left-handed up sector would be non-universal and the Z could mediate FCNC. In your homework, you will work out an explicit example.

The Yukawa couplings of the Higgs boson are not universal. In fact, in the interaction basis, they are given by completely general 3×3 matrices. Yet, as explained in Section 3.4.5, in the fermion mass basis they are diagonal. The reason is that the fermion mass matrix is proportional to the corresponding Yukawa matrix. Consequently, the mass matrix and the Yukawa matrix are simultaneously diagonalized. The special features of the SM in this regard are the following:

1. All the SM fermions are chiral, and therefore there are no bare mass terms.

2. The scalar sector has a single Higgs doublet.

In contrast, either of the following possible extensions would lead to flavor changing Higgs couplings:

1. There are quarks or leptons in vector-like representations, and thus there are bare mass terms.

2. There is more than one $SU(2)_L$-doublet scalar.

It is interesting to note, however, that not all multi Higgs doublet models lead to flavor changing Higgs couplings. If all the fermions of a given sector couple to one and the same doublet, then the Higgs couplings in that sector would still be diagonal. For example, in a model with two Higgs doublets, ϕ_1 and ϕ_2, and Yukawa terms of the form

$$\mathcal{L}_{\rm Yuk} = Y^u_{ij}\overline{Q_{Li}}U_{Rj}\,\phi_2 + Y^d_{ij}\overline{Q_{Li}}D_{Rj}\,\phi_1 + Y^e_{ij}\overline{L_{Li}}E_{Rj}\,\phi_1 + {\rm h.c.}, \tag{5.1}$$

the Higgs couplings are flavor diagonal. In the physics jargon, we say that such models have *natural flavor conservation* (NFC).

We conclude that within the SM, all FCNC processes are loop suppressed. However, in extensions of the SM, FCNC can appear at the tree level, mediated by the Z boson or by the Higgs boson or by new massive bosons.

(b) **CKM suppression.** Obviously, all flavor changing processes are proportional to off-diagonal entries in the CKM matrix. A quick look at the absolute values of the off-diagonal entries of the CKM matrix, Eq. (3.70), reveals that they are small. A rough estimate of the CKM suppression can be acquired by counting powers of λ in the Wolfenstein parametrization, Eq. (3.73): $|V_{us}|$ and $|V_{cd}|$ are suppressed by λ, $|V_{cb}|$ and $|V_{ts}|$ by λ^2, $|V_{ub}|$ and $|V_{td}|$ by λ^3.

For example, the amplitude for $b \to s\gamma$ decay comes from penguin diagrams, dominated by the intermediate top quark, and suppressed by $|V_{tb}V_{ts}| \sim \lambda^2$. As another example, the $B^- - \overline{B}^0$ mixing amplitude comes from box diagrams, dominated by intermediate top quarks, and suppressed by $|V_{tb}V_{td}|^2 \sim \lambda^6$.

(c) **GIM suppression.** If all quarks in a given sector were degenerate, then there would be no flavor changing W-couplings. A consequence of this fact is that FCNC in the down (up) sector are proportional to mass-squared differences between the quarks of the up (down) sector. For FCNC processes that involve only quarks of the first two generations, this leads to a strong suppression factor related to the light quark masses, and known as Glashow-Iliopoulos-Maiani (GIM) suppression.

Let us take as an example Δm_K, the mass splitting between the two neutral K-mesons. (A more detailed discussion of neutral meson mixing can be found in Appendix 5.A.) We have $\Delta m_K = 2|M_{K\bar{K}}|$, where $M_{K\bar{K}}$ corresponds to the $\overline{K}^0 \to K^0$ transition and comes from box diagrams. The top contribution is CKM-suppressed compared to the contributions from intermediate up and charm, so we consider only the latter:

$$M_{K\bar{K}} \simeq \sum_{i,j=u,c} \frac{G_F^2}{16\pi^2} \langle K^0 | (\bar{d}_L \gamma^\mu s_L)^2 | \overline{K}^0 \rangle (V_{is} V_{id}^* V_{js} V_{jd}^*) \times F(x_i, x_j), \tag{5.2}$$

where $x_i = m_i^2/m_W^2$. If we had $m_u = m_c$, the amplitude would be proportional to $(V_{us}V_{ud}^* + V_{cs}V_{cd}^*)^2$, which vanishes in the two generation limit. We conclude that $\Delta m_K \propto (m_c^2 - m_u^2)/m_W^2$, which is the GIM suppression factor.

For the $B^0 - \overline{B}^0$ and $B_s - \overline{B}_s$ mixing amplitudes, the top-mediated contribution is not CKM suppressed compared to the lighter generations. The mass ratio m_t^2/m_W^2 enhances, rather than suppresses, the top contribution. Consequently, the $M_{B\bar{B}}$ amplitude is dominated by the top contribution:

$$M_{B\bar{B}} \simeq \frac{G_F^2 m_t^2}{16\pi^2} \langle B^0 | (\bar{d}_L \gamma^\mu b_L)^2 | \overline{B}^0 \rangle (V_{tb} V_{td}^*)^2 \times F\left(\frac{m_W^2}{m_t^2}\right). \tag{5.3}$$

Before we turn to discuss the present situation, we should mention that historically, FCNC have served important role in predicting the existence of SM particles before they were directly discovered, and in predicting their masses:

- The smallness of $\frac{\Gamma(K_L \to \mu^+\mu^-)}{\Gamma(K^+ \to \mu^+\nu)}$ led to predicting a fourth (the charm) quark;
- The size of Δm_K led to a successful prediction of the charm mass;
- The measurement of ε_K led to predicting the third generation;
- The size of Δm_B led to a successful prediction of the top mass.

5.1.2 Testing the CKM sector: The $\rho - \eta$ plane

According to the SM, all quark flavor changing processes depend on only four independent CKM parameters, that can be chosen to be those of the Wolfenstein parametrization (3.73): λ, A, ρ and η. The number of flavor changing processes that can be measured is much larger, thus providing a stringent test of the CKM picture of flavor physics.

The values of λ and A are known rather accurately from, respectively, $K \to \pi\ell\nu$ and $B \to X_c\ell\nu$ decays:

$$\lambda = 0.2254 \pm 0.0007, \quad A = 0.811^{+0.022}_{-0.012}. \tag{5.4}$$

Then, one can express all the relevant observables as a function of the two remaining parameters, ρ and η, and check whether there is a range in the $\rho - \eta$ plane that is consistent with all measurements. The list of observables includes the following:

- The rates of inclusive and exclusive charmless semileptonic B decays depend on $|V_{ub}|^2 \propto \rho^2 + \eta^2$;
- The CP asymmetry in $B \to \psi K_S$, $S_{\psi K_S} = \sin 2\beta = \frac{2\eta(1-\rho)}{(1-\rho)^2+\eta^2}$;
- The rates of various $B \to DK$ decays depend on the phase γ, where $e^{i\gamma} = \frac{\rho+i\eta}{\sqrt{\rho^2+\eta^2}}$;
- The rates of various $B \to \pi\pi, \rho\pi, \rho\rho$ decays depend on the phase $\alpha = \pi - \beta - \gamma$;
- The ratio between the mass splittings in the neutral B and B_s systems is sensitive to $|V_{td}/V_{ts}|^2 = \lambda^2[(1-\rho)^2 + \eta^2]$;
- The CP violation in $K \to \pi\pi$ decays, ϵ_K, depends in a complicated way on ρ and η.

The resulting constraints are shown in Fig. 5.1.

The consistency of the various constraints is impressive. In particular, the following ranges for ρ and η can account for all the measurements:

$$\rho = +0.131^{+0.026}_{-0.013}, \quad \eta = +0.345 \pm 0.014. \tag{5.5}$$

Given the consistency of the measurements with the renormalizable SM, and the fact that all the SM parameters are known, one can use the upper bounds on possible deviations from the SM predictions to set upper bounds on the size of non-renormalizable terms.

5.1.3 New flavor physics: The $h_d - \sigma_d$ plane

We now aim to go beyond testing the self-consistency of the CKM picture of flavor physics and CP violation. Based on experimental information, we can actually *prove* that the KM phase is different from zero and that, moreover, it is the dominant source of all observed CP violation. We can also *quantify* how much room is left for new physics in this regard. In proving that the KM mechanism is at work, we assume that charged-current tree-level processes are dominated by the W-mediated SM diagrams. This is a very plausible assumption. It is difficult to construct a model where new physics competes with the SM in flavor changing charged current processes, and does not violate the constraints from flavor changing neutral current processes.

Thus we can use all tree level processes and fit them to ρ and η, as we did before. The list of such processes includes the following:

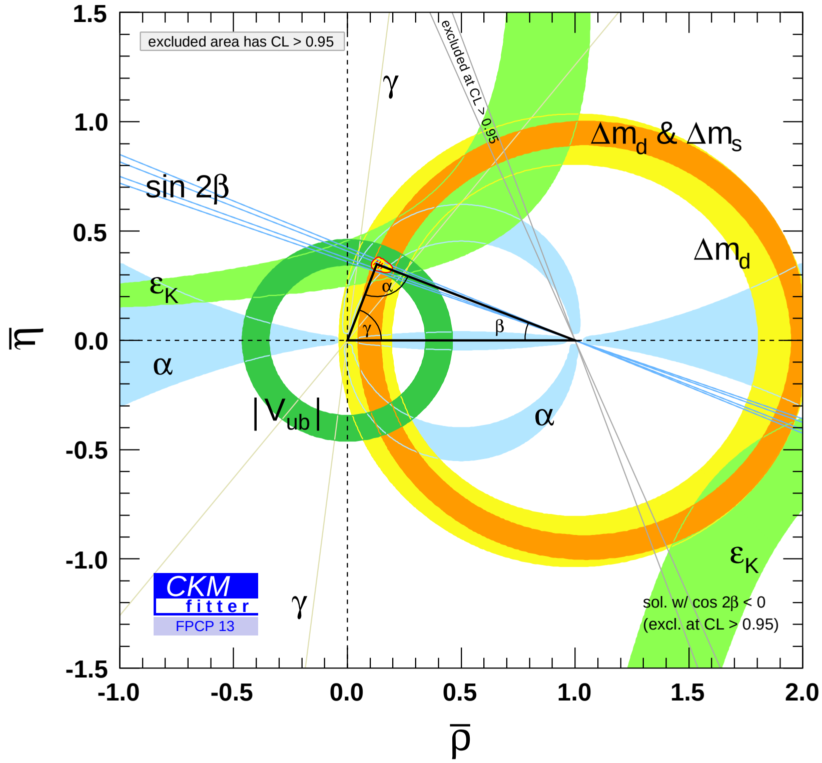


Figure 5.1: Allowed region in the ρ, η plane. Superimposed are the individual constraints from charmless semileptonic B decays ($|V_{ub}|$), mass differences in the B^0 (Δm_d) and B_s (Δm_s) neutral meson systems, and CP violation in $K \to \pi\pi$ (ε_K), $B \to \psi K$ ($\sin 2\beta$), $B \to \pi\pi, \rho\pi, \rho\rho$ (α), and $B \to DK$ (γ).

1. Charmless semileptonic B-decays, $b \to u\ell\nu$.

2. $B \to DK$ decays, which go through the quark transitions $b \to c\bar{u}s$ and $b \to u\bar{c}s$.

3. $B \to \rho\rho$ decays (and, similarly, $B \to \pi\pi$ and $B \to \rho\pi$ decays) go through the quark transition $b \to u\bar{u}d$. With an isospin analysis, one can determine the relative phase between the tree decay amplitude and the mixing amplitude. By incorporating the measurement of $S_{\psi K_S}$, one can subtract the phase from the mixing amplitude, finally providing a measurement of the angle γ.

In addition, we can use loop processes, but then we must allow for new physics contributions, in addition to the (ρ, η)-dependent SM contributions. Of course, if each such measurement adds a separate mode-dependent parameter, then we do not gain anything by using this information. However, there is a number of observables where the only relevant loop process is $B^0 - \overline{B}^0$ mixing. Within the SM, the $B^0 - \overline{B}^0$ mixing amplitude, M_{12}, is a function of ρ and η. We can parameterize the most general modification of the SM prediction of M_{12} in terms of two parameters, r_d^2 signifying the change in magnitude, and $2\theta_d$ signifying the change in phase:

$$M_{12} = r_d^2 \; e^{2i\theta_d} \; M_{12}^{\rm SM}(\rho, \eta). \tag{5.6}$$

The list of relevant observables includes $S_{\psi K_S}$, Δm_B and $\mathcal{A}_{\rm SL}$, the CP asymmetry in semileptonic B decays:

$$\begin{aligned} S_{\psi K_S} &= \sin(2\beta + 2\theta_d), \\ \Delta m_B &= r_d^2 (\Delta m_B)^{\rm SM}, \\ \mathcal{A}_{\rm SL} &= \left(\frac{\Delta\Gamma_B}{\Delta m_B}\right)^{\rm SM} \frac{\sin 2\theta_d}{r_d^2} + (\mathcal{A}_{\rm SL})^{\rm SM} \frac{\cos 2\theta_d}{r_d^2}. \end{aligned} \tag{5.7}$$

An alternative way to present the data is to use the h_d, σ_d parametrization,

$$r_d^2 e^{2i\theta_d} = 1 + h_d e^{2i\sigma_d}. \tag{5.8}$$

While the r_d, θ_d parameters give the relation between the full mixing amplitude and the SM one, and are convenient to apply to the measurements, the h_d, σ_d parameters give the relation between the new physics and SM contributions, and are more convenient in testing theoretical models:

$$M_{12}^{\rm NP} = h_d e^{2i\sigma_d} M_{12}^{\rm SM}. \tag{5.9}$$

Thus, we fit six observables to four parameters. The results of such fit, projected on the $\rho - \eta$ plane, can be seen in Fig. 5.2(a). It is clear that $\eta \neq 0$ is well established, proving that **the Kobayashi-Maskawa mechanism of CP violation is at work.**

In order to test model independently whether the SM dominates the observed CP violation, and to put an upper bound on the new physics contribution to $B^0 - \overline{B}^0$ mixing, we need to project the results of the fit on the $h_d - \sigma_d$ plane. If we find that $h_d \ll 1$, then the SM dominance in the $B^0 - \overline{B}^0$ mixing amplitude will be established. If $h_d \gtrsim 1$ for $\sigma_d \sim 0, \pi$ but $h_d \ll 1$ otherwise, then the SM dominance in the observed CP violation will be established. The constraints in the $h_d - \sigma_d$ plane are shown in Fig. 5.2(b). We can make the following two statements:

1. A new physics contribution to $B^0 - \overline{B}^0$ mixing amplitude that carries a phase that is significantly different from the KM phase is constrained to lie below the 10-20% level.

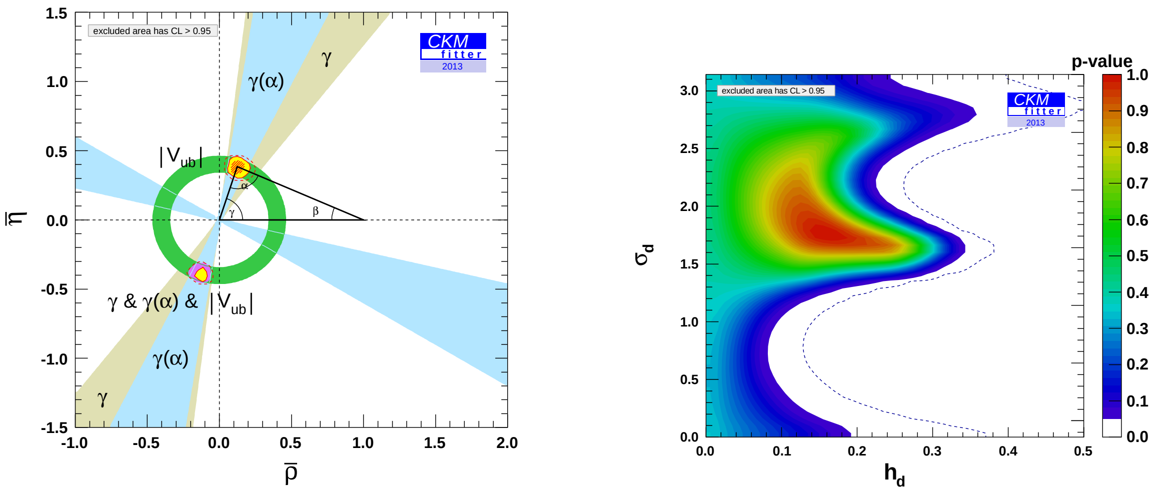
2. A new physics contribution to the $B^0 - \overline{B}^0$ mixing amplitude which is aligned with the KM phase lie below the the 30-40% level.

One can reformulate these statements as follows:

1. The KM mechanism dominates CP violation in $B^0 - \overline{B}^0$ mixing.

2. The CKM mechanism is a major player in $B^0 - \overline{B}^0$ mixing.

Figure 5.2: Constraints in the (a) $\rho-\eta$ plane, and (b) $h_d-\sigma_d$ plane, assuming that NP contributions to tree level processes are negligible.



5.1.4 Non-renormalizable terms

Given that the SM is only an effective low energy theory, one should consider the effects of non-renormalizable terms. As concerns quark flavor physics, consider, for example, the following dimension-six set of operators:

$$\mathcal{L}_{\rm NP}^{\Delta F=2} = \sum_{i\neq j} \frac{z_{ij}}{\Lambda^2}(\overline{Q_{Li}}\gamma_\mu Q_{Lj})^2, \tag{5.10}$$

where the z_{ij} are dimensionless couplings. The consistency of the experimental results with the SM predictions for neutral meson mixing, allows us to impose the condition $|M_{P\bar{P}}^{\rm NP}| < |M_{P\bar{P}}^{\rm SM}|$ for $P = K, B, B_s$, which implies that

$$\Lambda > \frac{4.4 \text{ TeV}}{|V_{ti}^* V_{tj}|/|z_{ij}|^{1/2}} \sim \begin{cases} 1.3 \times 10^4 \text{ TeV} \times |z_{sd}|^{1/2} \\ 5.1 \times 10^2 \text{ TeV} \times |z_{bd}|^{1/2} \\ 1.1 \times 10^1 \text{ TeV} \times |z_{bs}|^{1/2} \end{cases} \tag{5.11}$$

A more detailed list of the bounds derived from the $\Delta F = 2$ observables in Table 5.1 is given in Table 5.2. The bounds refer to two representative sets of dimension-six operators: (i) left-left operators, that are also present in the SM, and (ii) operators with different chirality, where the bounds are strongest because of larger hadronic matrix elements.

The first lesson that we draw from these bounds on Λ is that new physics can contribute to FCNC at a level comparable to the SM contributions even if it takes place at a scale that is six orders of magnitude above the electroweak scale. A second lesson is that if the new physics has a generic flavor structure, that is $z_{ij} = \mathcal{O}(1)$, then its scale must be above $10^4 - 10^5$ TeV (or, if

Table 5.2: Lower bounds on the scale of new physics Λ, in units of TeV, for $|z_{ij}| = 1$, and upper bounds on z_{ij}, assuming $\Lambda = 1$ TeV.

Operator	Λ [TeV] CPC	Λ [TeV] CPV	$\lvert z_{ij}\rvert$	$\mathcal{I}m(z_{ij})$	Observables
$(\bar{s}_L\gamma^\mu d_L)^2$	9.8×10^2	1.6×10^4	9.0×10^{-7}	3.4×10^{-9}	Δm_K; ϵ_K
$(\bar{s}_R d_L)(\bar{s}_L d_R)$	1.8×10^4	3.2×10^5	6.9×10^{-9}	2.6×10^{-11}	Δm_K; ϵ_K
$(\bar{c}_L\gamma^\mu u_L)^2$	1.2×10^3	2.9×10^3	5.6×10^{-7}	1.0×10^{-7}	Δm_D; A_Γ
$(\bar{c}_R u_L)(\bar{c}_L u_R)$	6.2×10^3	1.5×10^4	5.7×10^{-8}	1.1×10^{-8}	Δm_D; A_Γ
$(\bar{b}_L\gamma^\mu d_L)^2$	5.1×10^2	9.3×10^2	3.3×10^{-6}	1.0×10^{-6}	Δm_B; $S_{\psi K}$
$(\bar{b}_R d_L)(\bar{b}_L d_R)$	1.9×10^3	3.6×10^3	5.6×10^{-7}	1.7×10^{-7}	Δm_B; $S_{\psi K}$
$(\bar{b}_L\gamma^\mu s_L)^2$	1.1×10^2	1.1×10^2	7.6×10^{-5}	7.6×10^{-5}	Δm_{B_s}
$(\bar{b}_R s_L)(\bar{b}_L s_R)$	3.7×10^2	3.7×10^2	1.3×10^{-5}	1.3×10^{-5}	Δm_{B_s}

the leading contributions involve electroweak loops, above $10^3 - 10^4$ TeV). *If indeed $\Lambda \gg TeV$, it means that we have misinterpreted the hints from the fine-tuning problem and the dark matter puzzle.*

A different lesson can be drawn from the bounds on z_{ij}. *It could be that the scale of new physics is of order TeV, but its flavor structure is far from generic.* Specifically, if new particles at the TeV scale couple to the SM fermions, then there are two ways in which their contributions to FCNC processes, such as neutral meson mixing, can be suppressed: degeneracy and alignment. Either of these principles, or a combination of both, signifies non-generic structure.

5.2 CP violation

There are two main reasons for the interest in CP violation:

- CP asymmetries provide some of the theoretically cleanest probes of flavor physics. The reason for that is that CP is a good symmetry of the strong interactions. Consequently, for some hadronic decays, QCD-related uncertainties cancel out in the CP asymmetries.

- There is a cosmological puzzle related to CP violation. The baryon asymmetry of the Universe is a CP violating observable, and it is many orders of magnitude larger than the SM prediction. Hence, there must exist new sources of CP violation beyond the single phase of the CKM matrix.

In this section we explain why CP violation is related to complex parameters of the Lagrangian. Based on this fact, we prove that CP violation in a two generation SM is impossible, while CP violation in a three generation SM requires a long list of conditions on its flavor parameters in order. The formalism and SM calculation of CP asymmetries is presented in Appendix 5.B. The cosmological puzzle is presented in Section 7.1.

5.2.1 CP violation and complex couplings

The CP transformation combines charge conjugation C with parity P. Under C, particles and antiparticles are interchanged by conjugating all internal quantum numbers, *e.g.*, $Q \to -Q$. Under P, the handedness of space is reversed, $\vec{x} \to -\vec{x}$. Thus, for example, a left-handed electron e_L^- is transformed under CP into a right-handed positron, e_R^+.

At the Lagrangian level, CP is a good symmetry if there is a basis where all couplings are real. Let us provide a simple explanation of this statement. Consider fields Φ_i. We can define the CP transformation of the fields as

$$\Phi_i \to \Phi_i^\dagger. \tag{5.12}$$

Take, for example, terms in the Lagrangian that consist of three fields. (These could be Yukawa terms, if two of the Φ_i's are fermions and one is a scalar, or terms in the scalar potential, if all three are scalars, *etc.*) The hermiticity of the Lagrangian dictates that the following two terms should be included:

$$Y_{ijk}\Phi_i\Phi_j\Phi_k + Y_{ijk}^*\Phi_i^\dagger\Phi_j^\dagger\Phi_k^\dagger. \tag{5.13}$$

Under the CP transformation, the field content of the two terms is exchanged, but the couplings remain the same. Thus, CP is a good symmetry if $Y_{ijk} = Y_{ijk}^*$, *i.e.*, the coupling is real.

In practice, things are more subtle, since one can define the CP transformation as $\Phi_i \to e^{i\theta_i}\Phi_i^\dagger$, with θ_i a convention dependent phase. Then, there can be complex couplings, yet CP would be a good symmetry. Therefore, the correct statement is that CP is violated if, using all freedom to redefine the phases of the fields, one cannot find any basis where all couplings are real.

Let us examine the situation in the mass basis of the SM. The couplings of the gluons, the photon and the Z-boson are all real, as are the two parameters of the scalar potential. As concerns the fermion mass terms and the weak gauge interactions, the relevant CP transformation laws are

$$\bar{\psi}_i\psi_j \to \bar{\psi}_j\psi_i, \quad \bar{\psi}_i\gamma^\mu W_\mu^+(1-\gamma_5)\psi_j \to \bar{\psi}_j\gamma^\mu W_\mu^-(1-\gamma_5)\psi_i. \tag{5.14}$$

Thus the mass terms and CC weak interaction terms are CP invariant if all the masses and couplings are real. We can always choose the masses to be real. Then, let us focus on the couplings of $W^\pm$ to quarks:

$$-\frac{g}{\sqrt{2}}\left(V_{ij}\bar{u}_i\gamma^\mu W_\mu^+(1-\gamma_5)d_j + V_{ij}^*\bar{d}_j\gamma^\mu W_\mu^-(1-\gamma_5)u_i\right). \tag{5.15}$$

The CP operation exchanges the two terms, except that V_{ij} and V_{ij}^* are not interchanged. Thus CP is a good symmetry only if there is a mass basis and choice of phase convention where all couplings and masses are real.

5.2.2 SM2: CP conserving

Consider a two generation Standard Model, SM2. This model is similar to the one defined in Section 3.1, which in this section will be referred to as SM3, except that there are two, rather than three fermion generations. Many features if SM2 are similar to SM3, but there is one important difference: CP is a good symmetry of SM2, but not of SM3. To see how this difference comes about, let us examine the accidental symmetries of SM2. We follow here the line of analysis of SM3 in Section 3.5.

If we set the Yukawa couplings to zero, $\mathcal{L}_{\rm Yuk}=0$, SM2 gains an accidental global symmetry:

$$G^{\rm global}_{\rm SM2}(Y^{u,d,e}=0)=U(2)_Q\times U(2)_U\times U(2)_D\times U(2)_L\times U(2)_E, \tag{5.16}$$

where the two generations of each gauge representation are a doublet of the corresponding $U(2)$. The Yukawa couplings break this symmetry into the subgroup

$$G^{\rm global}_{\rm SM2}=U(1)_B\times U(1)_e\times U(1)_\mu. \tag{5.17}$$

A-priori, the Yukawa terms depend on three 2×2 complex matrices, namely 12_R+12_I parameters. The global symmetry breaking, $[U(2)]^5\to[U(1)]^3$, implies that we can remove $5\times(1_R+3_I)-3_I=5_R+12_I$ parameters. Thus the number of physical flavor parameters is 7 real parameters and no imaginary parameter. The real parameters can be identified as two charged lepton masses, four quark masses, and the single real mixing angle, $\sin\theta_c=|V_{us}|$.

The important conclusion for our purposes is that all imaginary couplings can be removed from SM2, and CP is an accidental symmetry of the model.

5.2.3 SM3: Not necessarily CP violating

A-priori, CP is not necessarily violated in SM3. If two quarks of the same charge had equal masses, one mixing angle and the phase could be removed from V. This can be written as a condition on the quark mass differences. CP violation requires

$$(m_t^2-m_c^2)(m_t^2-m_u^2)(m_c^2-m_u^2)(m_b^2-m_s^2)(m_b^2-m_d^2)(m_s^2-m_d^2)\neq0. \tag{5.18}$$

Likewise, if the value of any of the three mixing angles were 0 or $\pi/2$, then the phase can be removed. Finally, CP would not be violated if the value of the single phase were 0 or π. These last eight conditions are elegantly incorporated into one, parametrization-independent condition. To find this condition, note that the unitarity of the CKM matrix, $VV^\dagger=1$, requires that for any choice of $i,j,k,l=1,2,3$,

$$\mathcal{I}m[V_{ij}V_{kl}V^*_{il}V^*_{kj}]=J\sum_{m,n=1}^{3}\epsilon_{ikm}\epsilon_{jln}. \tag{5.19}$$

Then the conditions on the mixing parameters are summarized by

$$J\neq0. \tag{5.20}$$

The quantity J is of much interest in the study of CP violation from the CKM matrix. The maximum value that J could assume in principle is $1/(6\sqrt{3}) \approx 0.1$, but it is found to be $\sim 4 \times 10^{-5}$.

The fourteen conditions incorporated in Eqs. (5.18) and (5.20) can all be written as a single requirement on the quark mass matrices in the interaction basis:

$$X_{CP} \equiv \mathcal{I}m\left\{\det\left[M_d M_d^\dagger, M_u M_u^\dagger\right]\right\} \neq 0 \;\Leftrightarrow\; \text{CP violation}. \tag{5.21}$$

This is a convention independent condition.

Appendix

5.A Neutral meson mixing

Neutral meson mixing is an FCNC process. Within the SM, it provides indirect measurements of CKM parameters. Beyond the SM, it probes very high energy scales. In this Appendix, we present the formalism that is used to investigate these processes, explain how the time evolution of the neutral meson system depends on the tiny mass splitting between the two quasi-degenerate mass eigenstates, and present the SM expression for this mass splitting.

5.A.1 Flavor oscillations

There are four neutral meson-pairs where mixing can occur: $K^0-\overline{K}^0$, $D^0-\overline{D}^0$, $B^0-\overline{B}^0$, and $B_s^0-\overline{B}_s^0$. [3]

Consider a neutral meson P ($P=K,D,B$ or B_s). Initially ($t=0$) it is a superposition of P^0 and $\overline{P}^0$:

$$|\psi_P(0)\rangle = a(0)|P^0\rangle + b(0)|\overline{P}^0\rangle \; . \tag{5.22}$$

It evolves in time, and acquires components that correspond to all possible decay final states $\{f_1, f_2, \ldots\}$:

$$|\psi_P(0)\rangle = a(t)|P^0\rangle + b(t)|\overline{P}^0\rangle + c_1(t)|f_1\rangle + c_2(t)|f_2\rangle + \cdots \; . \tag{5.23}$$

Our interest lies in obtaining only $a(t)$ and $b(t)$. For this aim, one can use a simplified formalism, where the full Hamiltonian is replaced with a 2×2 effective Hamiltonian $\mathcal{H}$ that is not Hermitian. The non-Hermiticity is related to the possibility of decays, which makes the $\{P^0,\overline{P}^0\}$ system an open one. The complex matrix $\mathcal{H}$ can be written in terms of Hermitian matrices M and Γ as

$$\mathcal{H} = M - \frac{i}{2}\Gamma \; . \tag{5.24}$$

The matrices M and Γ are associated with $(P^0,\overline{P}^0) \leftrightarrow (P^0,\overline{P}^0)$ transitions via off-shell (dispersive) and on-shell (absorptive) intermediate states, respectively. Diagonal elements of M and Γ are associated with the flavor-conserving transitions $P^0\to P^0$ and $\overline{P}^0\to\overline{P}^0$. The CPT symmetry

[3] You may be wondering why there are only four such systems. If you do not wonder and do not know the answer, then you should wonder. You will answer this question in your homework.

implies that $M_{11} = M_{22}$ and $\Gamma_{11} = \Gamma_{22}$. The off-diagonal elements are associated with the flavor changing transitions $P^0 \leftrightarrow \overline{P}^0$.

Before we proceed, let us clarify a semantic issue. The effective Hamiltonian $\mathcal{H}$ and, similarly, its Hermitian part M, is a combination of operators. What we need for our purposes is its matrix element between specific meson states. With some abuse of language, we denote by M_{ij} both the operator and its matrix element. Model independently, the diagonal matrix elements fulfill $M_{11} = M_{22} = m$ and $\Gamma_{11} = \Gamma_{22} = \Gamma$. The off-diagonal elements are those of interest to us. When we refer to a specific meson system, we will use $M_{P\bar{P}}$ for the matrix element $\langle P^0 | M_{12} | \overline{P}^0 \rangle$.

In all cases ($P = K, D, B, B_s$), $\mathcal{H}$ is not a diagonal matrix. Thus, the states that have well defined masses and decay widths are not P^0 and $\overline{P}^0$, but rather the eigenvectors of $\mathcal{H}$. We denote the light and heavy eigenstates by P_L and P_H with masses $m_H > m_L$. (Another possible choice, which is standard for K mesons, is to define the mass eigenstates according to their lifetimes. We denote the short-lived and long-lived eigenstates by K_S and K_L with decay widths $\Gamma_S > \Gamma_L$. The K_L meson is experimentally found to be the heavier state.) The eigenstates of $\mathcal{H}$ are given by

$$|P_{L,H}\rangle = p|P^0\rangle \pm q|\overline{P}^0\rangle, \tag{5.25}$$

where

$$\left(\frac{q}{p}\right)^2 = \frac{M_{12}^* - (i/2)\Gamma_{12}^*}{M_{12} - (i/2)\Gamma_{12}}\,, \tag{5.26}$$

and with the normalization $|p|^2 + |q|^2 = 1$. Since $\mathcal{H}$ is not Hermitian, the eigenstates need not be orthogonal to each other.

The masses and decay-widths are given by the real and imaginary parts of the eigenvalues, respectively. The average mass and the average width are given by

$$m \equiv \frac{m_H + m_L}{2}, \qquad \Gamma \equiv \frac{\Gamma_H + \Gamma_L}{2}. \tag{5.27}$$

The mass difference Δm and the width difference $\Delta\Gamma$ are defined as follows:

$$\Delta m \equiv m_H - m_L, \qquad \Delta\Gamma \equiv \Gamma_H - \Gamma_L. \tag{5.28}$$

Here Δm is positive by definition, while the sign of $\Delta\Gamma$ is to be determined experimentally. (Alternatively, one can use the states defined by their lifetimes to have $\Delta\Gamma \equiv \Gamma_S - \Gamma_L$ positive by definition.) It is useful to define dimensionless ratios x and y:

$$x \equiv \frac{\Delta m}{\Gamma}, \qquad y \equiv \frac{\Delta\Gamma}{2\Gamma}. \tag{5.29}$$

We also define

$$\theta = \arg(M_{12}\Gamma_{12}^*). \tag{5.30}$$

Solving the eigenvalue equation gives

$$(\Delta m)^2 - \frac{1}{4}(\Delta\Gamma)^2 = 4|M_{12}|^2 - |\Gamma_{12}|^2, \qquad \Delta m \Delta\Gamma = 4\mathcal{R}e(M_{12}\Gamma_{12}^*). \tag{5.31}$$

We move on to study the time evolution of a neutral meson. For simplicity, we assume CP conservation. In Section 5.B we study CP violation, and there we relax this assumption. Many important points can, however, be understood in the simplified case where CP is conserved. If CP is a good symmetry of $\mathcal{H}$ then Γ_{12}/M_{12} is real, leading to

$$|q/p| = 1\ . \tag{5.32}$$

It follows that the mass eigenstates are also CP eigenstates, and are orthogonal to each other, $\langle P_H|P_L\rangle = |p|^2 - |q|^2 = 0$. The phase of q/p is convention dependent, and not a physical observable. As concerns the mass and decay widths, Eq. (5.31) simplifies to

$$\Delta m = 2|M_{12}|, \qquad |\Delta\Gamma| = 2|\Gamma_{12}|. \tag{5.33}$$

Let us denote the time-evolved state of an initial state $|P\rangle$ by $|P(t)\rangle$. For mass eigenstates, the time evolution is simple, $|P_{L,H}(t)\rangle = e^{-iE_{L,H}t}|P_{L,H}\rangle$. But the time evolution of $|P^0(t)\rangle$ and $|\overline{P}^0(t)\rangle$ is more complicated:

$$\begin{aligned} |P^0(t)\rangle &= \cos\left(\frac{\Delta E\, t}{2}\right)|P^0\rangle + i\sin\left(\frac{\Delta E\, t}{2}\right)|\overline{P}^0\rangle\,, \\ |\overline{P}^0(t)\rangle &= \cos\left(\frac{\Delta E\, t}{2}\right)|\overline{P}^0\rangle + i\sin\left(\frac{\Delta E\, t}{2}\right)|P^0\rangle\,. \end{aligned} \tag{5.34}$$

Since flavor is not conserved, the probability $\mathcal{P}$ to measure a specific flavor, that is P^0 or $\overline{P}^0$, oscillates in time:

$$\begin{aligned} \mathcal{P}(P^0 \to P^0)[t] &= \left|\langle P^0(t)|P^0\rangle\right|^2 = \frac{1+\cos(\Delta E t)}{2}, \\ \mathcal{P}(P^0 \to \overline{P}^0)[t] &= \left|\langle P^0(t)|\overline{P}^0\rangle\right|^2 = \frac{1-\cos(\Delta E t)}{2}. \end{aligned} \tag{5.35}$$

Thus, neutral meson mixing, $M_{12} \neq 0$, leads to flavor oscillations.

In the meson rest frame, $\Delta E = \Delta m$ and $t = \tau$, the proper time. Thus, Δm sets the frequency of the flavor oscillations. This is a very interesting result:

- On the theoretical side, Δm is related to FCNC transitions: the quark transitions that correspond to $K^0 - \overline{K}^0$, $D^0 - \overline{D}^0$, $B^0 - \overline{B}^0$, and $B^0_s - \overline{B}^0_s$ mixing are, respectively, $\bar{s}d \to s\bar{d}$, $\bar{u}c \to u\bar{c}$, $\bar{b}d \to b\bar{d}$, and $\bar{b}s \to b\bar{s}$. Thus, Δm for each of the four systems gives an indirect measurement of CKM parameters and can probe new physics.

- On the experimental side, we learn that by measuring the oscillation frequency we can determine the mass splitting between the two mass eigenstates. One way this can be done is by measuring the flavor of the meson both at production and decay. It is not trivial to measure the flavor at both ends, and we do not explain here how it is done, but you are encouraged to think and learn about it.

5.A.2 Time scales

There are various time scales involved in meson mixing, and understanding the hierarchy (or lack of hierarchy) between them leads to insights and simplifications.

The first important time scale is the oscillation period. As can be seen from Eq. (5.35), the oscillation time scale is given by Δm.[4]

To understand which other time scales are relevant, we need to introduce the notion of "flavor tagging." The flavor eigenstates P^0 and $\overline{P}^0$ have a well defined flavor content. For example, B^0 ($\overline{B}^0$) is a $\bar{b}d$ ($b\bar{d}$) bound state. The term 'flavor tagging' is used, in the physicists jargon, to the experimental determination of whether a neutral P meson is in a P^0 or $\overline{P}^0$ state. Flavor tagging is provided to us by Nature, when the meson decays into a flavor-specific final state, namely a state that can come from either P^0 or $\overline{P}^0$ state, but not from both.[5] Semi-leptonic decays are very good flavor tags. Take, for example, semileptonic b (anti)quark decays:

$$b \to c\mu^-\bar{\nu}, \qquad \bar{b} \to \bar{c}\mu^+\nu. \tag{5.36}$$

Thus, the charge of the lepton tells us the flavor: μ^+ comes from a B^0 (or B^+) decay, while μ^- comes from a $\overline{B}^0$ (or B^-) decay. Of course, before the meson decays it could be in a superposition of B^0 and a $\overline{B}^0$. The decay acts as a quantum measurement. In the case of semileptonic decay, it acts as a measurement of flavor *vs.* anti-flavor.

Thus, a second relevant time scale is that of flavor tagging. Since the flavor is tagged when the meson decays, the relevant time scale is determined by the decay width, Γ. We can then use the dimensionless quantity x [defined in Eq. (5.29)] to understand the possible hierarchies between these two time scales:

1. $x \ll 1$ ("slow oscillations"): The meson decays before it has time to oscillate, and thus flavor is conserved to good approximation. Putting $\cos(\Delta m t) \approx 1$ in Eq. (5.35), we obtain $\mathcal{P}(P^0 \to P^0) \approx 1$ and $\mathcal{P}(P^0 \to \overline{P}^0) \to 0$. A measurement of Δm is challenging, but experiments can provide a useful upper bound even before the required precision for an actual measurement is achieved. This case is relevant for the D system.

2. $x \gg 1$ ("fast oscillation"): The meson oscillates many times before decaying, and thus the oscillating term practically averages out to zero. Putting $\cos(\Delta m t) \approx 0$ in Eq. (5.35), we obtain $\mathcal{P}(P \to P) \approx \mathcal{P}(P \to \overline{P}) \approx 1/2$. A measurement of Δm is challenging, but experiments can provide a useful lower bound even before the required precision for an actual measurement is achieved. This case is relevant for the B_s system.

[4]The time scale is, of course, $1/\Delta m$. Physicists know, however, how to match dimensions. We thus interchange between time and energy freely, counting on the reader to understand what we mean.

[5]Final states that are common to the decays of both P and $\overline{P}$ are also very useful in flavor physics and, in particular, to the study of CP violation. They will be discussed in Section 5.B.

3. $x \sim 1$: The oscillation and decay times are roughly the same. The meson has time to oscillate and the oscillations do not average out. This is the case where it is experimentally easiest to measure Δm. This case is relevant to both the K and the B systems. We emphasize that the physics processes that determine Γ and Δm are unrelated, so there is no reason to expect $x \sim 1$. Yet, amazingly, Nature has been kind enough to choose flavor parameters such that $x \sim 1$ in two out of the four neutral meson systems.

Thus, flavor oscillations give us sensitivity to mass differences of the order of the width, which are much smaller than the mass itself. In fact, we have been able to measure mass differences that are 14 orders of magnitude smaller than the corresponding masses. It is due to the quantum mechanical nature of the oscillation that such high precision can be achieved.

In some cases there is one more time scale: $\Delta\Gamma$. In such cases, we have one more relevant dimensionless parameter $y \equiv \Delta\Gamma/(2\Gamma)$. Note that y is bounded, $|y| \leq 1$. (This is in contrast to x which has no upper bound.) Thus, we can talk about several cases depending on the values of y and x.

1. $|y| \ll 1$ and $y \ll x$. In this case the width difference is irrelevant. This is the case for the B^0 system.

2. $y \sim x$. In this case the width different is as important as the oscillation. This is the case in the D system where $y \ll 1$ and for the K system with $y \sim 1$.

3. $|y| \sim 1$ and $y \ll x$. In this case the oscillation averages out and the width difference can be observed simply as a difference in the lifetimes of the two mass eigenstates. This case is relevant to the B_s system, where $y \sim 0.1$.

There are few other limits (like $y \gg x$) that are not realized in the four meson systems. Yet, they might be realized in some other systems yet to be discovered.

To conclude this subsection, we present in Table 5.A.1 the experimental data on meson mixing. Note that in all cases (including the K meson system) we define x and y as in Eqs. (5.28) and (5.29). Note that for the B^0 system, there is only an upper bound on y.

5.A.3 The SM calculation of M_{12}

We now explain how the theoretical calculation of the mixing parameters is done. Our focus is on Δm. We present the SM calculation, but the tools that we develop can be used in a large class of models.

For the sake of concreteness, we discuss in this section the neutral B meson system. The operator M_{12} is given, within the SM, by $C_{\rm SM}(\bar{d}_L\gamma_\mu b_L)(\bar{d}_L\gamma^\mu b_L)$, where $C_{\rm SM}$ is the Wilson coefficient. The matrix element is given by

$$M_{B\bar{B}} = \frac{C_{\rm SM}}{2m_B}\langle B^0|(\bar{d}_L\gamma_\mu b_L)(\bar{d}_L\gamma^\mu b_L)|\overline{B}^0\rangle. \tag{5.37}$$

Table 5.A.1: Neutral meson mixing parameters

P	m [GeV]	Γ [GeV]	x	y
K^0	0.498	3.68×10^{-15}	0.945	-0.997
D^0	1.86	1.60×10^{-10}	0.0048 ± 0.0017	$+0.014 \pm 0.002$
B^0	5.28	4.33×10^{-13}	0.775 ± 0.006	-0.0075 ± 0.0090
B_s	5.37	4.34×10^{-13}	26.82 ± 0.23	-0.061 ± 0.008

The mass splitting is given by

$$\Delta m_B = 2|M_{B\bar{B}}|, \tag{5.38}$$

so that, within the SM, we have

$$\Delta m_B = -\frac{1}{3} m_B B_B f_B^2 C_{\rm SM}, \tag{5.39}$$

where we parameterized the hadronic matrix element as $\langle B^0|(\bar{d}_L\gamma_\mu b_L)(\bar{d}_L\gamma^\mu b_L)|\overline{B}^0\rangle = -\frac{1}{3}m_B^2 B_B f_B^2$ (lattice calculations give $\sqrt{B}f_B \approx 0.22$ GeV).

Our task is then to calculate $C_{\rm SM}$. Since the operator in Eq. (5.37) is an FCNC operator, within the SM it cannot be generated at tree level. The one loop diagrams that generate it are called "box diagrams". They are displayed in Fig. 5.A.1. The calculation of the box diagrams gives, to a good approximation,

$$M_{B\bar{B}} = \frac{G_F^2}{12\pi^2} m_B m_W^2 (B_B f_B^2) S_0(x_t) (V_{tb}V_{td}^*)^2, \tag{5.40}$$

where $x_t = m_t^2/m_W^2$. A few comments are in order:

1. The box diagrams have two W-boson propagators, which yield the G_F^2 factor.

2. The box diagrams have two up-type quark (i and j) propagators, yielding six different combinations: $ij = uu, cc, tt, uc, ut, ct$. Each such diagram depends on a different combination of CKM elements and quark masses, $(V_{ib}V_{id}^*)(V_{jb}V_{jd}^*)F(m_i^2/m_W^2, m_j^2/m_W^2)$.

3. The unitarity of the CKM matrix implies that any (m_i, m_j)-independent terms vanish.

4. The three CKM combinations $V_{id}^*V_{ib}$ are comparable in size. (They are all cubic in the Wolfenstein parameter λ.)

5. The six kinematic functions $F(m_i^2/m_W^2, m_j^2/m_W^2)$ are very different in size. In particular, $S_0(x_t) = F(x_t, x_t)$ (where $x_t = m_t^2/m_W^2$) is the largest.

$\bar{d}$ $\bar{u}_j$ $\bar{b}$

b u_i d

Figure 5.A.1: A box diagram that generate an operators that can lead to $B \leftrightarrow \overline{B}$ transition.

6. The conclusion of the last two statements is that the dominant contribution comes from the box diagram with two top-quark propagators. In the physicists' jargon we say that M_{12} is dominated by the top-quark.

The function $S_0(x_t)$ is quadratically sensitive to m_t. Similar to the EWPM, this non-decoupling effect is related to the fact that the larger the top mass, the stronger its Yukawa coupling. When Δm_B was first measured, the top quark has not yet been discovered, and one could use Eq. (5.40) to predict (correctly!) the top mass. At present, when the top mass is known (yielding $S_0(x_t) \approx 2.36$), Eq. (5.40) serves to constrain the CKM combination $|V_{tb}V_{td}^*|$.

Let us comment on the calculation of M_{12} in the other meson systems:

1. As concerns Δm_K, due to the CKM structure, it is dominated by the charm quark in the loop. Consequently, Δm_K is GIM suppressed by a factor of m_c^2/m_W^2. The lightness of the charm quark implies also considerably larger theoretical uncertainties in the calculation compared to Δm_B.

2. As concerns Δm_D, due to the CKM structure, the contributions involving the bottom quark are suppressed. The calculation of the box diagrams with intermediate down and strange quarks is not a good approximation to Δm_D.

3. As concerns Δm_{B_s}, the calculation goes along very similar lines to that of Δm_B. In the ratio $\Delta m_B/\Delta m_{B_s}$, much of the uncertainty in the calculation of the hadronic matrix elements cancels out, providing an excellent measurement of $|V_{td}/V_{ts}|$.

Finally, let us mention the calculation of Γ_{12}. An estimate of it can be made by calculating the on-shell part of the box diagram. Yet, since the intermediate quarks are light and on-shell, QCD effects are important, and the theoretical uncertainties in the calculation of Γ_{12} are large.

5.A.4 Homework

Question 5.A.1: The four mesons

It is now time to come back to the question of why there are only four meson pairs that are relevant to flavor oscillations. Explain why the following systems are irrelevant to flavor oscillations:

1. $B^+ - B^-$
2. $K - K^*$
3. $T - \overline{T}$ (a T is a meson made out of a t and a $\bar{u}$ quarks.)
4. $K^* - \overline{K}^*$ oscillation

Hint: The last three cases all have to do with time scales. In principle there are oscillations in these systems, but they are irrelevant.

Question 5.A.2: Kaons

Here we study some properties of the kaon system. We did not talk about it at all. You have to go back and recall (or learn) how kaons decay, and combine that with what we discussed in the lecture.

1. Explain why $y_K \approx 1$.
2. In a hypothetical world where we could change the mass of the kaon without changing any other masses, how would the value of y_K change if we made m_K smaller or larger.

Question 5.A.3: Mixing beyond the SM

Consider a model without a top quark, in which the first two generations are as in the SM, while the left–handed bottom (b_L) and the right–handed bottom (b_R) are $SU(2)$ singlets.

1. Draw a tree-level diagram that contributes to $B - \bar{B}$ mixing in this model.
2. Is there a tree-level diagram that contributes to $K - \bar{K}$ mixing?
3. Is there a tree-level diagram that contributes to $D - \bar{D}$ mixing?

5.B CP violation

To date, CP violation has been observed in about thirty different decay modes. It has not been observed in baryon decays, nor in the leptonic sector, nor in flavor diagonal processes, such as electric dipole moments. We thus present in this Appendix the formalism and the SM calculation of CP asymmetries in meson decays.

The experimental observation of CP violation is challenging for several reasons:

1. In order that there will be a CP asymmetry in a decay process, the presence of so-called "strong phases", which are CP conserving phases arising from intermediate on-shell particles, is needed. These phases might be small (or vanish) and suppress the CP asymmetry (or make it vanish).

2. CPT implies that the total width of a particle and its anti-particle are the same. Thus, any CP violation in one channel must be compensated by CP violation with an opposite sign in other channels. Consequently, CP violation is suppressed in inclusive measurements.

3. Within the SM, CP violation arises only when all three generations are involved. With the smallness of the CKM mixing angles, this means that either the CP asymmetries are small, or they appear in modes with small branching ratios.

CP violation in meson decays is an interference effect. In neutral meson decays the phenomenology of CP violation is particularly rich thanks to the fact that meson mixing, as described in Appendix 5.A, can contribute to the CP violating interference effects. One distinguishes three types of CP violation in meson decays, depending on which amplitudes interfere:

1. In decay: The interference is between two decay amplitudes.

2. In mixing: The interference is between the absorptive and dispersive mixing amplitudes.

3. In interference of decays with and without mixing: The interference is between the direct decay amplitude and a first-mix-then-decay amplitude.

The formalism and the SM calculation of the neutral meson mixing amplitude was presented in Appendix 5.A. Before we proceed to discuss in more detail each of these three types of CP violation, we present our notations, and some physics ingredients, concerning the decay amplitudes. We do so for the specific case of B-meson decays, but our discussion applies to all meson decays.

We denote the amplitude of $B \to f$ decay by A_f, and the amplitude of the CP conjugate process, $\overline{B} \to \overline{f}$, by $\overline{A}_{\overline{f}}$. There are two types of phases that may appear in these decay amplitudes. First, complex parameters in any Lagrangian term that contributes to A_f appear in a complex conjugate form in $\overline{A}_{\overline{f}}$. In other words, CP violating phases change sign between A_f and $\overline{A}_{\overline{f}}$. In the SM, these phases appear only in the couplings of the $W^\pm$-bosons, hence the CP violating phases are called "weak phases." Second, phases can appear in decay amplitudes even when the Lagrangian is real. They arise from contributions of intermediate on-shell states. These CP conserving phases appear with the same sign in A_f and $\overline{A}_{\overline{f}}$. In meson decays, such rescattering is usually driven by strong interactions, hence the CP conserving phases are called "strong phases."

It is useful to factorize each contribution a_i to A_f into three parts: the magnitude $|a_i|$, the weak phase ϕ_i, and the strong phase δ_i. If there are two such contributions, $A_f = a_1 + a_2$, we write

$$\begin{aligned} A_f &= |a_1| e^{i(\delta_1+\phi_1)} + |a_2| e^{i(\delta_2+\phi_2)}, \\ \overline{A}_{\overline{f}} &= |a_1| e^{i(\delta_1-\phi_1)} + |a_2| e^{i(\delta_2-\phi_2)}. \end{aligned} \tag{5.41}$$

It is further useful to define

$$\phi_f \equiv \phi_2 - \phi_1, \quad \delta_f \equiv \delta_2 - \delta_1, \quad r_f \equiv |a_2/a_1|. \tag{5.42}$$

Similarly, for neutral meson decays, it is useful to write

$$M_{12} = |M_{12}|e^{i\phi_M}, \qquad \Gamma_{12} = |\Gamma_{12}|e^{i\phi_\Gamma}\ . \tag{5.43}$$

Each of the phases appearing in Eqs. (5.41) and (5.43) is convention dependent, but combinations such as $\delta_1 - \delta_2$, $\phi_1 - \phi_2$, and $\phi_M - \phi_\Gamma$ are physical.

To discuss the modifications to the time evolution, which was presented in Eq. (5.34) for the CP conserving case, it is convenient to define another complex parameter,

$$\lambda_f \equiv (q/p)(\overline{A}_f/A_f). \tag{5.44}$$

The time evolution of a B^0 and $\overline{B}^0$ mesons is given by

$$\begin{aligned} |B^0(t)\rangle &= g_+(t)\,|B^0\rangle - (q/p)\; g_-(t)|\overline{B}^0\rangle, \\ |\overline{B}^0(t)\rangle &= g_+(t)\,|\overline{B}^0\rangle - (p/q)\; g_-(t)|B^0\rangle\ , \end{aligned} \tag{5.45}$$

where

$$g_\pm(t) \equiv \frac{1}{2}\left(e^{-im_H t - \frac{1}{2}\Gamma_H t} \pm e^{-im_L t - \frac{1}{2}\Gamma_L t}\right). \tag{5.46}$$

We define $\tau \equiv \Gamma t$. The time-dependent decay rate $\Gamma(B^0 \to f)[t]$ ($\Gamma(\overline{B}^0 \to f)[t]$) gives the probability for an initially pure B^0 ($\overline{B}^0$) meson to decay at time t to a final state f:

$$\begin{aligned} \Gamma(B^0 \to f)[t] = |A_f|^2 e^{-\tau} \quad & \Big\{(\cosh y\tau + \cos x\tau) + |\lambda_f|^2(\cosh y\tau - \cos x\tau) \\ & -2\mathrm{Re}\left[\lambda_f(\sinh y\tau + i\sin x\tau)\right]\Big\}, \\ \Gamma(\bar{B} \to f)[t] = |\overline{A}_f|^2 e^{-\tau} \quad & \Big\{(\cosh y\tau + \cos x\tau) + |\lambda_f|^{-2}(\cosh y\tau - \cos x\tau) \\ & -2\mathrm{Re}\left[\lambda_f^{-1}(\sinh y\tau + i\sin x\tau)\right]\Big\}. \end{aligned} \tag{5.47}$$

5.B.1 CP violation in decay

CP violation in decay corresponds to

$$|\overline{A}_{\overline{f}}/A_f| \neq 1. \tag{5.48}$$

In charged meson decays, this is the only possible contribution to the CP asymmetry:

$$\mathcal{A}_{f^\pm} \equiv \frac{\Gamma(B^- \to f^-) - \Gamma(B^+ \to f^+)}{\Gamma(B^- \to f^-) + \Gamma(B^+ \to f^+)} = \frac{|\overline{A}_{f^-}/A_{f^+}|^2 - 1}{|\overline{A}_{f^-}/A_{f^+}|^2 + 1}. \tag{5.49}$$

Using Eq. (5.41), we obtain for $r \ll 1$

$$\mathcal{A}_{f^\pm} = 2r_f \sin\phi_f \sin\delta_f. \tag{5.50}$$

This result shows explicitly that we need two decay amplitudes, that is, $r_f \neq 0$, with different weak phases, $\phi_f \neq 0, \pi$ and different strong phases $\delta_f \neq 0, \pi$.

A few comments are in order:

1. In order to have a large CP asymmetry, we need each of the three factors in (5.50) to be large.

2. A similar expression holds for the contribution of CP violation in decay in neutral meson decays. In this case there are, however, additional contributions.

3. Another complication with regard to neutral meson decays is that it is not always possible to tell the flavor of the decaying meson, that is, if it is B^0 or $\overline{B}^0$. This can be a problem or a virtue.

4. In general the strong phase is not calculable since it is related to QCD. This is not a problem if the aim is just to demonstrate CP violation, but it is if we want to extract the weak parameter ϕ_f. In some cases, however, the phase can be independently measured, eliminating this particular source of theoretical uncertainty.

$D \to K^+K^-$

We give here an example of the SM contribution to CP violation in decay in the $D \to K^+K^-$ mode. This decay proceeds via the quark transition $c \to s\bar{s}u$. Within the SM, there are contributions from both tree (t) and penguin (p^q, where $q = d, s, b$ is the quark in the loop) diagrams. Factoring out the CKM dependence, we have

$$A_{K^+K^-} = (V_{cs}^* V_{us}) t_{KK} + \sum_{q=d,s,b} (V_{cq}^* V_{uq}) p_{KK}^q. \tag{5.51}$$

Using CKM unitarity, $A_{K^+K^-}$ can be written in terms of just two CKM combinations:

$$A_{K^+K^-} = (V_{cs}^* V_{us}) T_{KK} + (V_{cb}^* V_{ub}) P_{KK}^b, \tag{5.52}$$

where $T_{KK} = t_{KK} + p_{KK}^s - p_{KK}^d$ and $P_{KK}^b = p_{KK}^b - p_{KK}^d$. CP violating phases appear only in the CKM elements, so that

$$\frac{\overline{A}_{K^+K^-}}{A_{K^+K^-}} = \frac{(V_{cs}^* V_{us}) T_{KK} + (V_{cb}^* V_{ub}) P_{KK}^b}{(V_{cs} V_{us}^*) T_{KK} + (V_{cb} V_{ub}^*) P_{KK}^b}. \tag{5.53}$$

Due to CKM suppression and loop suppression, we expect the P_{KK}^b-related contribution to be much smaller than the T_{KK}-related contribution, and thus the contribution from CP violation in decay to the CP asymmetry is given by

$$\mathcal{A}_{K^+K^-}^d \approx -2\mathcal{I}m\left(\frac{P_{KK}^b}{T_{KK}}\right) \frac{|V_{cb}^* V_{ub}|}{|V_{cs}^* V_{us}|} \sin\gamma, \tag{5.54}$$

where γ is defined in Eq. (3.75). The super-index d on $\mathcal{A}_{K^+K^-}^d$ denotes that we include here only the contribution from CP violation in decay.

The CKM parameters are known, and generate a suppression factor of $\mathcal{O}(10^{-3})$. The factor of $\mathcal{I}m(P_{KK}^b/T_{KK})$ depends on the relative size of the penguin and tree contributions, as well as the relative strong phase. Both ingredients arise from QCD dynamics at the scale of m_D. At present, there is no rigorous way to calculate this factor. Thus, one cannot use a measurement of $\mathcal{A}_{K^+K^-}$ to extract, for example, the value of the CP violating phase γ.

5.B.2 CP violation in mixing

CP violation in mixing corresponds to

$$|q/p| \neq 1 \; . \tag{5.55}$$

In decays into flavor specific final states ($\overline{A}_f = 0$ and, consequently, $\lambda_f = 0$), and, in particular, semileptonic neutral meson decays, this is the only source of CP violation:[6]

$$\mathcal{A}_{\rm SL}(t) \equiv \frac{\Gamma[\overline{B}^0(t) \to \ell^+ X] - \Gamma[B^0(t) \to \ell^- X]}{\Gamma[\overline{B}^0(t) \to \ell^+ X] + \Gamma[B^0(t) \to \ell^- X]} = \frac{1-|q/p|^4}{1+|q/p|^4}. \tag{5.56}$$

Using Eq. (5.26), we obtain for $|\Gamma_{12}/M_{12}| \ll 1$,

$$\mathcal{A}_{\rm SL} = -\,|\Gamma_{12}/M_{12}|\sin(\phi_M - \phi_\Gamma). \tag{5.57}$$

A few comments are in order:

1. Eq. (5.56) implies that this asymmetry of time-dependent decay rates is actually time independent.

2. The calculation of $|\Gamma_{12}/M_{12}|$ is difficult, since it depends on low-energy QCD effects. Hence, it would be difficult in general to extract the value of the CP violating phase $\phi_M - \phi_\Gamma$ from a measurement of $\mathcal{A}_{\rm SL}$.

$K \to \ell\nu\pi$

We give here an example of the SM contribution to CP violation in $K^0-\overline{K}^0$ mixing. It is measured via the semileptonic asymmetry which is defined as follows:

$$\delta_L \equiv \frac{\Gamma(K_L \to \ell^+\nu_\ell\pi^-) - \Gamma(K_L \to \ell^-\nu_\ell\pi^+)}{\Gamma(K_L \to \ell^+\nu_\ell\pi^-) + \Gamma(K_L \to \ell^-\nu_\ell\pi^+)} = \frac{1-|q/p|^2}{1+|q/p|^2}. \tag{5.58}$$

This asymmetry is somewhat different from the one defined in Eq. (5.56), in that the decaying meson is the neutral mass eigenstate, rather than the flavor eigenstate. Hence also the different dependence on $|q/p|$. The experimental value is $\delta_L = (3.32 \pm 0.06) \times 10^{-3}$.

Here one can overcome the difficulty of calculating $|\Gamma_{12}|$ by taking into account the experimental result that $\Delta\Gamma_K/\Delta m_K \approx -2$, and that, given that the CP violating effects are experimentally determined to be small, $\Delta\Gamma_K/\Delta m_K \simeq |\Gamma_{K\bar{K}}/M_{K\bar{K}}|$. Then one obtains

$$\mathcal{R}e(\epsilon_K) = \frac{1}{4}(1-|q/p|^2) \simeq \frac{\mathcal{I}m(M_{K\bar{K}})}{2\Delta m_K}, \tag{5.59}$$

[6]This statement holds within the SM where, to lowest order in G_F, $|A_{\ell^+X}| = |\overline{A}_{\ell^-X}|$ and $A_{\ell^-X} = \overline{A}_{\ell^+X} = 0$.

where we use connect the commonly used CP violating parameter ϵ_K to our notations. Thus, to find $\mathcal{R}e(\epsilon_K)$ we need to obtain the SM contribution to $M_{K\bar{K}}$. Similarly to the neutral B system, this contribution comes from box diagrams with intermediate up-type quarks, leading to

$$M_{K\bar{K}} = \frac{G_F^2 m_W^2}{12\pi^2} m_K (B_K f_K^2) \left[S_0(x_c)(V_{cs}V_{cd}^*)^2 + S_0(x_t)(V_{ts}V_{td}^*)^2 + S_0(x_c, x_t)(V_{cs}V_{cd}^* V_{ts}V_{td}^*) \right]. \quad (5.60)$$

where $x_c = m_c^2/m_W^2$. In contrast to the case of $M_{B\bar{B}}$ (5.40), in the neutral K system, $M_{K\bar{K}}$ is dominated by the charm quark. The reason is that, of the three relevant CKM combinations, the top-related one is highly suppressed: $|V_{td}^* V_{ts}| \sim \lambda^5$ compared to $|V_{cd}^* V_{cs}| \simeq \lambda$. Thus, Δm_K is dominated by the charm quark. We used this fact when writing down Eq. (5.2). The pure charm contribution to $\mathcal{I}m(M_{12})$ is, however, highly suppressed, and the top quark is dominant in $\mathcal{R}e(\epsilon_K)$.

5.B.3 CP violation in interference of decays with and without mixing

CP violation in interference of decays with and without mixing corresponds to

$$\frac{\mathcal{I}m(\lambda_f)}{1 + |\lambda_f|^2} \neq 0. \quad (5.61)$$

It can be extracted from the CP asymmetry in decays into final CP eigenstates:

$$\mathcal{A}_{f_{CP}}(t) \equiv \frac{\Gamma[\overline{B}^0(t) \to f_{CP}] - \Gamma[B^0(t) \to f_{CP}]}{\Gamma[\overline{B}^0(t) \to f_{CP}] + \Gamma[B^0(t) \to f_{CP}]} = \mathcal{I}m(\lambda_{f_{CP}}) \sin(\Delta m\ t). \quad (5.62)$$

The last equality holds when the effects of CP violation in decay are negligible, $|\overline{A}_{f_{CP}}/A_{f_{CP}}| \simeq 1$, and the effects of CP violation in mixing are small. $|q/p| \simeq 1$. In this case, $\lambda_{f_{CP}}$ is a pure phase.

Using Eq. (5.44), we obtain for $|\Gamma_{12}/M_{12}| \ll 1$,

$$\mathcal{I}m(\lambda_{f_{CP}}) = \mathcal{I}m\left(\frac{M_{12}^*}{|M_{12}|}\frac{\overline{A}_{f_{CP}}}{A_{f_{CP}}}\right) = -\sin(\phi_M + 2\phi_1). \quad (5.63)$$

The phase ϕ_M is defined in Eq. (5.43), while the phase ϕ_1 is defined in Eq. (5.41), and we assume that a_2 can be neglected.

$B \to \psi K_S$

We give here an example of the SM contribution to CP violation in the interference of decays with and without mixing in the $B \to \psi K_S$ mode. This is often called "the golden mode" with regard to CP violation as its theoretical calculation is uniquely clean of hadronic uncertainties. In fact, the CP asymmetry can be translated into a value of $\sin 2\beta$ [β is defined in Eq. (3.75)] with a theoretical uncertainty smaller than one percent.

For the neutral B meson system, $|\Gamma_{B\bar{B}}/M_{B\bar{B}}| \ll 1$ holds. From Eq. (5.40) we obtain

$$\frac{M_{B\bar{B}}^*}{|M_{B\bar{B}}|} = \frac{V_{tb}^* V_{td}}{V_{tb} V_{td}^*}. \quad (5.64)$$

The $B \to \psi K$ decay proceeds via a $\bar{b} \to \bar{c}c\bar{s}$ transition:

$$A_{\psi K} = (V_{cb}^* V_{cs}) T_{\psi K} + (V_{ub}^* V_{us}) P_{\psi K}^u. \tag{5.65}$$

The second term is CKM and loop suppressed, and can be safely neglected. Since B^0 decays into ψK^0 while $\overline{B}^0$ decays into $\psi \overline{K}^0$, an additional phase from $K^0 - \overline{K}^0$ mixing, $(V_{cd}^* V_{cs})/(V_{cd} V_{cs}^*)$, enters the calculation of $\overline{A}_{\psi K_S}/A_{\psi K_S}$:

$$\frac{\overline{A}_{\psi K_S}}{A_{\psi K_S}} = -\frac{V_{cb} V_{cd}^*}{V_{cb}^* V_{cd}}. \tag{5.66}$$

Combining Eq. (5.64) and Eq. (5.66), we obtain

$$\lambda_{\psi K_S} = -e^{-2i\beta} \implies \mathcal{I}m(\lambda_{\psi K_S}) = \sin 2\beta. \tag{5.67}$$

This demonstrate the power of CP asymmetries in measuring CKM parameters. The experimental measurement of $\mathcal{I}m(\lambda_{\psi K_S})$ translates directly into the value of a CKM parameter, β, without any hadronic parameters. A crucial role is played by the CP symmetry of the strong interactions. The size and the phase of the amplitude $T_{\psi K}$ cannot be calculated, but it is the same in the CP conjugate amplitudes $\overline{A}_{\psi K_S}$ and $A_{\psi K_S}$ and therefore cancels out when their ratio is taken.

5.B.4 Homework

Question 5.B.4: Condition for CP violation

Using Eq. (5.41), show that in order to observe CP violation, $\Gamma(B \to f) \neq \Gamma(\overline{B} \to \overline{f})$, we need two amplitudes with different weak and strong phases.

Question 5.B.5: Mixing formalism

In this question, you are asked to develop the general formalism of meson mixing.

1. Show that the mass and width differences are given by

$$4(\Delta m)^2 - (\Delta\Gamma)^2 = 4(4|M_{12}|^2 - |\Gamma_{12}|^2), \qquad \Delta m \Delta\Gamma = 4\mathrm{Re}(M_{12}\Gamma_{12}^*), \tag{5.68}$$

   and that

$$\left|\frac{q}{p}\right| = \left|\frac{\Delta m - i\Delta\Gamma/2}{2M_{12} - i\Gamma_{12}}\right|. \tag{5.69}$$

2. When CP is a good symmetry all mass eigenstates must also be CP eigenstates. Show that CP invariance requires

$$\left|\frac{q}{p}\right| = 1. \tag{5.70}$$

3. In the limit $\Gamma_{12} \ll M_{12}$ show that

$$\Delta m = 2|M_{12}|, \qquad \Delta\Gamma = 2|\Gamma_{12}|\cos\theta, \qquad \left|\frac{q}{p}\right| = 1. \tag{5.71}$$

4. Derive Eqs. (5.47).

5. Derive Eq. (**??**).

6. Show that when $\Delta\Gamma = 0$ and $|q/p| = 1$

$$\begin{aligned} \Gamma(B \to X\ell^-\bar{\nu})[t] &= e^{-\Gamma t}\sin^2(\Delta m t/2), \\ \Gamma(B \to X\ell^+\nu)[t] &= e^{-\Gamma t}\cos^2(\Delta m t/2). \end{aligned} \tag{5.72}$$

Question 5.B.6: $B \to \pi^+\pi^-$ and CP violation

One of the interesting decays to consider is $B \to \pi\pi$. Here we only briefly discuss it.

1. First assume that there is only tree level decay amplitude (that is, neglect penguin amplitudes). Draw the Feynman diagram of the amplitude, paying special attention to its CKM dependence.

2. In that case, which angle of the unitarity triangle is the time dependent CP asymmetry, Eq. (**??**), sensitive to?

3. Can you estimate the error introduced by neglecting the penguin amplitude? (Note that one can use isospin to reduce this error. Again, you are encouraged to read about it in one of the reviews.)

Question 5.B.7: B decays and CP violation

Consider the decays $\bar{B}^0 \to \psi K_S$ and $\bar{B}^0 \to \phi K_S$. Unless explicitly noted, we always work within the framework of the standard model.

1. $\bar{B}^0 \to \psi K_S$ is a tree-level process. Write down the underlying quark decay. Draw the tree level diagram. What is the CKM dependence of this diagram? In the Wolfenstein parametrization, what is the weak phase of this diagram?

2. Write down the underlying quark decay for $B^0 \to \phi K_S$. Explain why there is no tree level diagram for $B^0 \to \phi K_S$.

3. The leading one loop diagram for $B^0 \to \phi K_S$ is a gluonic penguin diagram. As we have discussed, there are several diagrams and only their sum is finite. Draw a representative diagram with an internal top quark. What is the CKM dependence of the diagram? In the Wolfenstein parametrization, what is the weak phase of the diagram?

4. Next we consider the time dependent CP asymmetries. We define as usual

$$\lambda_f \equiv \frac{\bar{A}_f}{A_f}\frac{q}{p}, \qquad A_f \equiv A(B^0 \to f), \qquad \bar{A}_f \equiv A(\bar{B}^0 \to f). \tag{5.73}$$

In our case we neglect subleading diagrams and then we have $|\lambda| = 1$ and thus

$$a_f \equiv \frac{\Gamma(\bar{B}^0(t) \to f) - \Gamma(B^0(t) \to f)}{\Gamma(\bar{B}^0(t) \to f) + \Gamma(B^0(t) \to f)} = -\mathrm{Im}\lambda_f \, \sin(\Delta m_B \, t) \tag{5.74}$$

Both $a_{\psi K_S}$ and $a_{\phi K_S}$ measure the same angle of the unitarity triangle. That is, in both cases, $\mathrm{Im}\lambda_f = \sin 2x$ where x is one of the angles of the unitarity triangle. What is x? Explain.

5. Experimentally,

$$\mathrm{Im}\lambda_{\psi K_S} = 0.68(3), \qquad \mathrm{Im}\lambda_{\phi K_S} = 0.47(19). \tag{5.75}$$

Comment about these two results. In particular, do you think these two results are in disagreement?

6. Assume that in the future we will find

$$\mathrm{Im}\lambda_{\psi K_S} = 0.68(1), \qquad \mathrm{Im}\lambda_{\phi K_S} = 0.32(3). \tag{5.76}$$

That is, that the two results are not the same. Below are three possible "solutions". For each solution explain if you think it could work or not. If you think it can work, show how. If you think it cannot, explain why.

   (a) There are standard model corrections that we neglected.

   (b) There is a new contribution to $B^0 - \bar{B}^0$ mixing with a weak phase that is different from the SM one.

   (c) There is a new contribution to the gluonic penguin with a weak phase that is different from the SM one.

Question 5.B.8: Decay of mass eigenstates

Derive Eq. (**??**). The idea is to understand that when we talk about mass eigenstates, we are talking about "late times," $t \gg x\Gamma$ so that the $\sin(\Delta m t)$ term can be averaged out.

Chapter 6

Neutrinos

6.1 Introduction

6.1.1 The Lagrangian

In this section, we study the $d = 5$ terms and their implications. There is a single class of such terms, involving two $SU(2)$-doublet lepton fields and two $SU(2)$-doublet scalar fields:

$$\mathcal{L}_{\rm SM+5} = \mathcal{L}_{\rm SM} + \frac{Z^\nu_{ij}}{\Lambda}\phi\phi L_i L_j, \tag{6.1}$$

where Z^ν is a symmetric, complex 3×3 matrix of dimensionless couplings.

6.1.2 The neutrino spectrum

The dimension five terms lead to a Majorana neutrino mass matrix:

$$m_\nu = \frac{v^2}{\Lambda}\frac{Z^\nu}{2}. \tag{6.2}$$

The matrix m_ν can be diagonalized by a unitary transformation:

$$V_{\nu L} m_\nu V_{\nu L}^T = \hat{m}_\nu = {\rm diag}(m_1, m_2, m_3). \tag{6.3}$$

We denote the corresponding neutrino mass eigenstates by ν_1, ν_2, ν_3.

While the individual neutrino mass eigenvalues are not known, two mass-squared differences are experimentally known:

$$\begin{aligned}
\Delta m^2_{21} &\equiv m_2^2 - m_1^2 = (7.5 \pm 0.2)\times 10^{-5}\ {\rm eV}^2, \\
\Delta m^2_{32} &\equiv m_3^2 - m_2^2 = \pm(2.3 \pm 0.1)\times 10^{-3}\ {\rm eV}^2.
\end{aligned} \tag{6.4}$$

In addition, there is an experimental upper bound on the mass, $m < 2$ eV, from tritium decay.

The effective low energy Lagrangian of Eq. (6.1) where, by definition, $\Lambda \gg v$, predicts that the neutrinos are much lighter than the charged fermions:

$$m_{1,2,3} \sim v^2/\Lambda \ll v \sim m_{e,\mu,\tau}. \tag{6.5}$$

The fact that experiments find that the neutrinos are indeed lighter by at least six orders of magnitude than the lightest charged fermion (the electron) makes the notion that neutrino masses are generated by $d = 5$ terms very plausible.

Clearly, the SM cannot be a valid theory above the Planck scale, $\Lambda \lesssim M_{\rm Pl}$. We thus expect that $m_i \gtrsim v^2/M_{\rm Pl} \sim 10^{-5}$ eV. If the relevant scale is, for example, the coupling unification scale, then we expect $m_\nu \sim 10^{-2}$ eV.

Conversely, an experimental lower bound on neutrino masses provides an upper bound on the scale of new physics. As we will see in Section 6.2, at least one of the neutrino has a mass

$$m \geq \sqrt{\Delta m^2_{32}} \simeq 0.05 \text{ eV}. \tag{6.6}$$

We conclude that the SM cannot be a valid theory above the scale

$$\Lambda \lesssim 10^{15} \text{ GeV}. \tag{6.7}$$

This proves that the SM cannot be valid up to the Planck scale. The upper bound is intriguingly close to the GUT scale.

6.1.3 The neutrino interactions

The addition of the dimension-five terms leads to significant changes in the phenomenology of the lepton sector. The modifications can be understood by re-writing the neutrino-related terms in the mass basis. The renormalizable SM gives

$$\mathcal{L}_{\text{SM},\nu} = i\overline{\nu_\alpha}\partial\!\!\!/\nu_\alpha - \frac{g}{2c_W}\overline{\nu_\alpha}Z\!\!\!/\nu_\alpha - \frac{g}{\sqrt{2}}\left(\overline{\ell_{L\alpha}}W\!\!\!\!/^-\nu_\alpha + \text{h.c.}\right), \tag{6.8}$$

where $\alpha = e, \mu, \tau$. The Lagrangian of Eq. (6.1) gives

$$\begin{aligned}\mathcal{L}_{\text{SM}+5,\nu} \;=\;& i\overline{\nu_i}\partial\!\!\!/\nu_i - \frac{g}{2c_W}\overline{\nu_i}Z\!\!\!/\nu_i - \frac{g}{\sqrt{2}}\left(\overline{\ell_{L\alpha}}W\!\!\!\!/^-U_{\alpha i}\nu_i + \text{h.c.}\right) + m_i\nu_i\nu_i \\ &+\frac{2m_i}{v}h\nu_i\nu_i + \frac{m_i}{v^2}hh\nu_i\nu_i,\end{aligned} \tag{6.9}$$

where $\alpha = e, \mu, \tau$ denotes the charged lepton mass eigenstates, while $i = 1, 2, 3$ denotes the neutrino mass eigenstates. Here $m_{1,2,3}$ are real, and U is unitary. Starting from an arbitrary interaction basis, the matrix U is given by

$$U = V_{eL}V^\dagger_{\nu L}. \tag{6.10}$$

While V_{eL} and $V_{\nu L}$ are basis-dependent, the combination $V_{eL}V^\dagger_{\nu L}$ is not.

Table 6.1: The neutrino interactions

interaction	force carrier	coupling
NC weak	Z^0	$e/(2s_W c_W)$
CC weak	$W^\pm$	$gU/\sqrt{2}$
Yukawa	h	$2m/v$

The most significant change from (6.8) to (6.9), beyond the fact that the neutrinos acquire Majorana masses, is that the leptonic charged current interactions are no longer universal. Instead, they involve the mixing matrix U.

The neutrinos thus have three types of interactions, mediated by massive bosons. These interactions are summarized in Table 6.1.

6.1.4 Accidental symmetries

The dimension-five terms in Eq. (6.1) break the $U(1)_e \times U(1)_\mu \times U(1)_\tau$ of the SM. All that remains as an accidental symmetry is $U(1)_B$. This symmetry is, however, anomalous and broken by non-perturbative effects. In addition, it is broken by dimension-six terms.

The counting of flavor parameters in the quark sector remains unchanged: 6 quark masses and 4 mixing parameters, of which 1 is imaginary. How many physical flavor parameters are involved in the lepton sector? The Lagrangian of Eq. (6.1) involves the 3×3 matrix Y^e (9 real and 9 imaginary parameters), and the symmetric 3×3 matrix Z^ν (6 real and 6 imaginary parameters). The kinetic and gauge terns have a $U(3)_L\times U(3)_E$ accidental global symmetry, that is completely broken by the Y^e and Z^ν terms. Thus, the number of physical parameters is $(15_R+15_I)-2\times(3_R+6_I) = 9_R+3_I$. Six of the real parameters are the three charged lepton masses $m_{e,\mu,\tau}$ and the three neutrino masses $m_{1,2,3}$. We conclude that the 3×3 unitary matrix U depends on three real mixing angles and three phases.

6.1.5 The lepton mixing parameters

As can be seen in Eq. (6.9), the lepton mixing matrix U determines the strength of the couplings of the W boson to lepton pairs:

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix}. \qquad (6.11)$$

The present status of our knowledge of the absolute values of the various entries in the lepton mixing matrix can be summarized as follows (we quote here the 3σ ranges):

$$|U| = \begin{pmatrix} 0.79-0.85 & 0.51-0.59 & 0.13-0.18 \\ 0.20-0.54 & 0.42-0.73 & 0.58-0.81 \\ 0.21-0.55 & 0.41-0.73 & 0.57-0.80 \end{pmatrix}. \tag{6.12}$$

Why does the lepton mixing matrix U depend on three phases, while the quark mixing matrix V depends on only a single phase? The reason for this difference lies in the fact that the Lagrangian of Eq. (6.1) leads to Majorana masses for neutrinos. Consequently, there is no freedom in changing the mass basis by redefining the neutrino phases, as such redefinition will introduce phases into the neutrino mass terms. While redefinitions of the six quark fields allowed us to remove five non-physical phases from V, redefinitions of the three charged lepton fields allows us to remove only three non-physical phases from U. The two additional physical phases in U are called "Majorana phases," since they appear as a result of the (assumed) Majorana nature of neutrinos. They affect lepton number violating processes.

A convenient parametrization of U is the following:

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \times \mathrm{diag}(1, e^{i\alpha_1}, e^{i\alpha_2}), \tag{6.13}$$

where $\alpha_{1,2}$ are the Majorana phases, $s_{ij} \equiv \sin\theta_{ij}$ and $c_{ij} \equiv \cos\theta_{ij}$. We describe the experimental determination of the lepton mixing parameters in Section 6.2.

6.2 Neutrino masses

6.2.1 Neutrino oscillations

In experiments, neutrinos are produced and detected by charged current weak interactions. Thus, the states that are relevant to productions and detection are the $SU(2)_L$-doublet partners of the charged lepton mass eigenstates, e, μ, τ, namely

$$\nu_e, \quad \nu_\mu, \quad \nu_\tau. \tag{6.14}$$

On the other hand, the eigenstates of free propagation in space-time are the mass eigenstates,

$$\nu_1, \quad \nu_2, \quad \nu_3. \tag{6.15}$$

In general, these interaction eigenstates are different from the mass eigenstates:

$$|\nu_\alpha\rangle = U^*_{\alpha i}|\nu_i\rangle \quad (\alpha = e, \mu, \tau, \; i = 1, 2, 3). \tag{6.16}$$

Consequently, flavor is not conserved during propagation in space-time and, in general, we may produce ν_α but detect $\nu_{\beta\neq\alpha}$.

The probability $P_{\alpha\beta}$ of producing neutrinos of flavor α and detecting neutrinos of flavor β is calculable in terms of

- The neutrino energy E;
- The distance between source and detector L;
- The mass squared difference $\Delta m^2_{ij} \equiv m_i^2 - m_j^2$ ($P_{\alpha\beta}$ is independent of the absolute mass scale);
- The parameters of the mixing matrix U (mixing angles and phase) ($P_{\alpha\beta}$ is independent of the Majorana phases).

Starting from Eq. (6.16), we can write the expression for the time evolved $|\nu_\alpha(t)\rangle$ (where $|\nu_\alpha(0)\rangle = |\nu_\alpha\rangle$):

$$|\nu_\alpha(t)\rangle = U^*_{\alpha i}|\nu_i(t)\rangle, \tag{6.17}$$

where

$$|\nu_i(t)\rangle = e^{-iE_it}|\nu_i(0)\rangle. \tag{6.18}$$

Thus, the probability of a state that is produced as ν_α to be detected as ν_β is given by

$$\begin{aligned} P_{\alpha\beta} &= |\langle\nu_\beta|\nu_\alpha(t)\rangle|^2 \\ &= |\langle\nu_\beta|\nu_i\rangle\langle\nu_i|\nu_\alpha(t)\rangle|^2 \\ &= \delta_{\alpha\beta} - 4\sum_{j>i}\mathcal{R}e\left(U_{\alpha i}U^*_{\beta i}U^*_{\alpha j}U_{\beta j}\right)\sin^2\left(\frac{\Delta m^2_{ij}L}{4E}\right) \\ &\quad +2\sum_{j>i}\mathcal{I}m\left(U_{\alpha i}U^*_{\beta i}U^*_{\alpha j}U_{\beta j}\right)\sin\left(\frac{\Delta m^2_{ij}L}{2E}\right). \end{aligned} \tag{6.19}$$

If we apply this calculation to the two generation vase, where there is a single mixing angle (and no relevant phase) and a single mass-squared difference,

$$\begin{aligned} U &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \\ \Delta m^2 &= m_2^2 - m_1^2, \end{aligned} \tag{6.20}$$

we obtain, for $\alpha \neq \beta$,

$$P_{\alpha\beta} = \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{4E}\right). \tag{6.21}$$

This expression depends on two parameters that are related to the experimental design, E and L, and two that are parameters of the Lagrangian, Δm^2 and θ. To be sensitive to the theoretical

Table 6.1: Neutrino oscillation experiments.

Source	E[MeV]	K[km]		Δm^2[eV^2]
Solar (VO)	1	10^8	$\Longrightarrow$	$10^{-11} - 10^{-9}$
Reactor	1	10^2	$\Longrightarrow$	$10^{-5} - 10^{-3}$
Atmospheric	10^3	10^{1-4}	$\Longrightarrow$	$10^{-5} - 1$
Source	n_0[cm^{-3}]	r_0[cm]		Δm^2[eV^2]
Solar (MSW)	6×10^{25}	7×10^9	$\Longrightarrow$	$10^{-9} - 10^{-5}$

parameters, one has to design the experiment appropriately:

$$\begin{aligned} \Delta m^2 L/E \ll 1 \quad & P_{\alpha\beta} \to 0, \\ \Delta m^2 L/E \sim 1 \quad & P_{\alpha\beta} \text{ sensitive to } \Delta m^2, \theta, \\ \Delta m^2 L/E \gg 1 \quad & P_{\alpha\beta} \to \tfrac{1}{2} \sin^2 2\theta. \end{aligned} \tag{6.22}$$

We learn that to allow observation of neutrino oscillations, Nature needs to provide $\sin^2 2\theta$ that is not too small. To probe small Δm^2, we need experiments with large L/E. Indeed, given natural neutrino sources as well as reactors, we can probe a rather large range of Δm^2; see the list in Table 6.1.

6.2.2 The MSW effect

The Mikheyev-Smirnov-Wolfenstein (MSW) effect procides yet another way to probe neutrino mixing and masses. Consider the two neutrino case. In vacuum, in the mass basis (ν_1, ν_2), the Hamiltonian can be written as

$$\mathcal{H} = p + \begin{pmatrix} \frac{m_1^2}{2E} & \\ & \frac{m_2^2}{2E} \end{pmatrix}. \tag{6.23}$$

In the interaction basis (ν_e, ν_a), where ν_a is a combination of ν_μ and ν_τ, we have

$$\mathcal{H} = p + \frac{m_1^2 + m_2^2}{4E} + \begin{pmatrix} -\frac{\Delta m^2}{4E}\cos 2\theta & \frac{\Delta m^2}{4E}\sin 2\theta \\ \frac{\Delta m^2}{4E}\sin 2\theta & \frac{\Delta m^2}{4E}\cos 2\theta \end{pmatrix}. \tag{6.24}$$

In matter (that is, in an (e, p, n) plasma), in the interaction basis,

$$\mathcal{H} = p + V_a + \frac{m_1^2 + m_2^2}{4E} + \begin{pmatrix} (V_e - V_a) - \frac{\Delta m^2}{4E}\cos 2\theta & \frac{\Delta m^2}{4E}\sin 2\theta \\ \frac{\Delta m^2}{4E}\sin 2\theta & \frac{\Delta m^2}{4E}\cos 2\theta \end{pmatrix}. \tag{6.25}$$

All active neutrinos have the same (universal) neutral current interactions. In contrast, in a plasma that has electrons but neither muons nor tau-leptons, only ν_e has charged current interactions with matter:

$$V_e - V_a = \sqrt{2} G_F n_e, \tag{6.26}$$

where n_e is the electron number density in the plasma. Thus, omitting the part in the Hamiltonian that is proportional to the unit matrix in flavor space (which plays no role in the oscillations), we have

$$\mathcal{H} \sim \begin{pmatrix} \sqrt{2}G_F n_e - \frac{\Delta m^2}{4E}\cos 2\theta & \frac{\Delta m^2}{4E}\sin 2\theta \\ \frac{\Delta m^2}{4E}\sin 2\theta & \frac{\Delta m^2}{4E}\cos 2\theta \end{pmatrix}. \tag{6.27}$$

We learn that the mixing angle that relates the flavor eigenstates (ν_e, ν_a) to the mass eigenstates in matter (ν_1^m, ν_2^m) depends on the matter density:

$$\tan 2\theta_m = \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta - 2\sqrt{2}G_F n_e E}. \tag{6.28}$$

For example, in case of very large electron density, $\sqrt{2}G_F n_e \gg \Delta m^2/(2E)$, we have $\theta_m \simeq \pi/2$, which means that ν_e is very close to the heavier mass eigenstate ν_2^m.

Things become even more complicated for a neutrino propagating in a varying density $n_e(x)$. The mixing angle is then changing, $\theta_m = \theta_m(n_e(x))$:

$$\tan 2\theta_m(x) = \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta - 2\sqrt{2}G_F n_e(x) E}. \tag{6.29}$$

In particular, as $n_e(x)$ decreases, so does $\theta_m(x)$. Defining

$$n_e^R = \frac{\Delta m^2 \cos 2\theta}{2\sqrt{2}G_F E}, \tag{6.30}$$

we have

$$\begin{aligned} n_e \gg n_e^R &\implies \theta_m \approx \pi/2, \\ n_e = n_e^R &\implies \theta_m = \pi/4, \\ n_e = 0 &\implies \theta_m = \theta. \end{aligned} \tag{6.31}$$

We conclude that, for a small θ, ν_2^m propagating along a decreasing n_e is mostly ν_e above n_e^R and mostly ν_a for n_e below n_e^R.

The propagation in varying density allows yet another interesting effect, and that is $\nu_1^m \leftrightarrow \nu_2^m$ transitions. The source of this effect is the fact that $e^{-iH(t)t} \neq e^{-i\int H(t')dt'}$, which means that the instantaneous mass eigenstates are not the eigenstates of time evolution. However, for *slowly* varying density, $\dot{H}t \ll H$, we have $e^{-i\int H(t')dt'} = e^{(-iHt+\dot{H}t^2+\cdots)} \approx e^{-iH(t)t}$, and the $\nu_1^m \leftrightarrow \nu_2^m$ transitions can be neglected. The condition for neglecting these transitions is known as *the adiabatic condition*:

$$\frac{1}{n}\frac{dn}{dx} \ll \frac{\Delta m^2}{E}\frac{\sin^2 2\theta}{\cos 2\theta}. \tag{6.32}$$

We now describe the characteristics of ν_e production and propagation in the Sun. The electron density in the Sun can be parameterized as $n_e(x) \approx 2n_0 \exp(-x/r_0)$, where the relevant parameters are given in Table 6.1. Consider the case where $n_e^{\rm prod} \gg n_e^R$. Then, according to Eq. (6.31), we have at the production point $\nu = \nu_2^m$ ($\theta_m = \pi/2$). Further assume that the propagation is adiabatic at $n_e \sim n_e^R$ (Eq. (6.32) is fulfilled at this point). Then, at the resonance point we still have $\nu = \nu_2^m$ ($\theta_m = \pi/4$). Finally, as the neutrino arrives to the surface of the Sun, it is still ν_2^m, but now, according to Eq. (6.31), we have $\theta_m = \theta$, and the neutrino is simply the heavy mass eigenstate. Being a mass eigenstate, it does not oscillate along its propagation to Earth. We conclude that for solar ν_e's with energy in the range

$$\Delta m^2 G_F n_e^{\rm prod} \ll E \ll \frac{\Delta m^2}{\frac{1}{n}\frac{dn}{dx}} \frac{\sin^2 2\theta}{\cos 2\theta}, \tag{6.33}$$

the probability of being detected as ν_e is given by

$$P_{ee}^{\rm MSW} = |\langle \nu_e | \nu_2 \rangle|^2 = \sin^2\theta. \tag{6.34}$$

It is highly sensitive to θ and provides the only way to probe small mixing angles. Indeed, for $\Delta m^2 \sim 10^{-4}$ eV, solar neutrinos would have allowed probing a mixing angle as small as $\sin^2\theta \sim 10^{-4}$.

On the other hand, for solar ν_e's with energy in the range

$$E \ll \Delta m^2 \;\cos 2\theta \; G_F n_e^{\rm prod}, \tag{6.35}$$

namely $n_e^{\rm prod} \ll n_e^R$, the produced state is $\nu = \sin\theta \; \nu_2^m + \cos\theta \; \nu_1^m$. Approaching the surface of the Sun, $\nu = \sin\theta \; \nu_2 + \cos\theta \; \nu_1$ and $P_{ee}(R_\odot) = 1$. Along the propagation to Earth, the neutrino is subject to vacuum oscillations, with the final result [see Eq. (6.22)]

$$P_{ee}^{\rm VO} = 1 - \frac{1}{2}\sin^2 2\theta. \tag{6.36}$$

Note that $P_{ee}^{\rm MSW} < \frac{1}{2}$ is possible, while $P_{ee}^{\rm VO} > \frac{1}{2}$. For solar neutrinos, the transition between those subject to the MSW effect, Eq. (6.34), and those subject to vacuum oscillations, Eq. (6.36), occurs at $E \sim$ MeV.

Examining Table 6.1, we conclude that, if $\theta \not\ll 1$, neutrino masses in the entire theoretically interesting range, 10^{-11} eV$^2 \lesssim \Delta m^2 \lesssim$ eV^2 could be discovered. For $10^{-2} \lesssim \theta \ll 1$, neutrino masses could still be discovered via the adiabatic MSW effect for $\Delta m^2 \sim 10^{-5}$ eV^2.

6.2.3 Experimental results

Neutrino flavor transitions have been observed for solar, atmospheric, reactor and accelerator neutrinos. Five flavor parameters – two mass-squared differences and the three mixing angles – have been measured:

$$\begin{aligned}\Delta m_{21}^2 &= (7.5 \pm 0.2) \times 10^{-5}\ {\rm eV}^2, \\ |\Delta m_{32}^2| &= (2.3 \pm 0.1) \times 10^{-3}\ {\rm eV}^2,\end{aligned}$$

$$
\begin{aligned}
\sin^2\theta_{12} &= 0.31 \pm 0.01, \\
\sin^2\theta_{23} &= 0.42 \pm 0.03, \\
\sin^2\theta_{13} &= 0.026 \pm 0.002.
\end{aligned} \tag{6.37}
$$

Note that the convention for naming the neutrino mass eigenstates is as follows. The two states separated by the smaller mass-squared difference are called ν_2 and ν_1, with ν_2 the heavier among the two. The mass eigenstate separated from these two by the larger mass-squared difference is called ν_3. It could be heavier ("normal hierarchy") or lighter ("inverted hierarchy") than the other two.

Also note the following questions that are still open:

- The absolute mass scale of the neutrinos is still unknown. On one extreme, they could be quasi-degenerate and as heavy as parts of eV. On the other extreme, they could be hierarchical, with the lightest possibly massless.
- It is not known whether the spectrum has normal or inverted hierarchy.
- None of the three phases has been measured.
- There is no experimental answer to the question of whether the neutrinos are Dirac or Majorana particles.

6.3 The νSM

Neutrino masses and mixing are an experimental fact. This means that the SM must be extended. In Section 4.1 we have seen that the addition of non-renormalizable, dimension-five terms to the SM Lagrangian gives neutrinos masses and, furthermore, explains why they are much lighter than the charged fermions. In this section we provide the simplest example of a full high energy theory that would generate at low energy the dimension-five terms. We call this extension, which amounts to adding heavy gauge-singlet fermions to the SM, the νSM.

6.3.1 Defining the Seesaw Standard Model

The SM+N model is defined as follows:

(*i*) The symmetry is a local

$$SU(3)_C \times SU(2)_L \times U(1)_Y. \tag{6.38}$$

(*ii*) The pattern of spontaneous symmetry breaking is as follows:

$$SU(3)_C \times SU(2)_L \times U(1)_Y \to SU(3)_C \times U(1)_{\rm EM} \quad (Q_{\rm EM} = T_3 + Y). \tag{6.39}$$

(*iii*) There are three fermion generations ($i = 1, 2, 3$), each consisting of six different representations:

$$Q_{Li}(3,2)_{+1/6}, \quad U_{Ri}(3,1)_{+2/3}, \quad D_{Ri}(3,1)_{-2/3}, \quad L_{Li}(1,2)_{-1/2}, \quad E_{Ri}(1,1)_{-1}, \quad N_{Ri}(1,1)_0. \tag{6.40}$$

There is a single scalar multiplet:

$$\phi(1,2)_{+1/2}. \tag{6.41}$$

6.3.2 The Lagrangian

The SM+N has the same gauge group, the same pattern of spontaneous symmetry breaking, and the same scalar content as the SM. In the fermion sector, all the SM representations are included. The only difference is the addition of the fermionic N_{Ri} fields. Since the imposed symmetry is the same, all the terms that appear in the SM Lagrangian appear also in the SM+N Lagrangian. The SM+N Lagrangian has, however, several additional terms. These are all the terms that involve the N_{Ri} fields. We can write:

$$\mathcal{L}_{\rm SM+N} = \mathcal{L}_{\rm SM} + \mathcal{L}_N. \tag{6.42}$$

Our task now is to find the specific form of the $\mathcal{L}_N$. We note the following points in this regard:

1. Given that the N_R fields are singlets of the gauge group, we have $D^\mu N_R = \partial^\mu N_R$.
2. Since the N_R fields carry no conserved charge, they can have Majorana mass terms.
3. The combination $\overline{L_L} N_R$ transforms as $(1,2)_{+1/2}$ under the gauge group, and can thus have a Yukawa coupling to the scalar doublet.

We thus obtain the most general form for the renormalizable terms in $\mathcal{L}_N$:

$$\mathcal{L}_N = i\overline{N_{Ri}}\partial\!\!\!/ N_{Ri} - \left(M^N_{ij} N_i N_j + Y^\nu_{ij}\overline{L_{Li}}\tilde{\phi}N_{Rj} + {\rm h.c.}\right). \tag{6.43}$$

Here M^N is a symmetric 3×3 complex matrix, with entries of mass dimension 1, and Y^ν is a general 3×3 complex matrix of dimensionless Yukawa couplings.

6.3.3 The spectrum

As concerns the spectrum of this theory, clearly the bosonic spectrum remains unchanged from the SM. As concerns the fermions, we note that, since the N_R fields are singlets of the full gauge group, they are also singlets of the unbroken subgroup, namely they transform as $(1)_0$ under $SU(3)_C \times U(1)_{\rm EM}$. This means that also the spectrum of the charged fermions (quarks and charged leptons) remains unchanged from the SM.

As concerns the neutrinos (the ν_L components of the $SU(2)$-doublet leptons and the N_R fields), taking into account the spontaneous symmetry breaking, we find the following mass terms in $\mathcal{L}_N$:

$$\mathcal{L}_N, \text{mass} = -M^N_{ij} N_i N_j - \frac{Y^\nu_{ij} v}{\sqrt{2}} \overline{\nu_{Li}} N_{Rj} + \text{h.c.}. \tag{6.44}$$

This gives a 6×6 neutrino mass matrix, that can be decomposed into four 3×3 blocks as follows:

$$M_\nu = \begin{pmatrix} 0 & m_D \\ m_D^T & M^N \end{pmatrix}, \quad m_D = (v/\sqrt{2}) Y^\nu. \tag{6.45}$$

To obtain the six neutrino mass eigenstates, we need to diagonalize M_ν.

We can always use a unitary transformation to bring M^N to a diagonal and real form:

$$M^N \ \to \ U_N^T M^N U_N = \hat{M}_N = \text{diag}(M_1, M_2, M_3). \tag{6.46}$$

Unlike the SM, which has a single dimensionful parameter, v, the SM+N has four dimensionful parameters: v, M_1, M_2, M_3. We now make an important assumption, inspired by both phenomenology and theoretical model building: We assume that the eigenvalues of M^N are much larger than the electroweak breaking scale:

$$M_{1,2,3} \gg v. \tag{6.47}$$

Then, we can perform the diagonalization to leading order in v/M_i. First, we use the unitary matrix K,

$$K = \begin{pmatrix} 1 & m_D \hat{M}_N^{-1} \\ -m_D \hat{M}_N^{-1} & 1 \end{pmatrix}, \tag{6.48}$$

where we omitted terms of order v^2/M_i^2, to block-diagonalize M_ν:

$$K M_\nu K^T = \begin{pmatrix} -m_D^T \hat{M}_N^{-1} m_D & 0 \\ 0 & \hat{M}_N \end{pmatrix}, \tag{6.49}$$

The lower-right block is already diagonalized. The upper-left block,

$$m_\nu = m_D^T \hat{M}_N^{-1} m_D, \tag{6.50}$$

can be diagonalized by a further unitary transformation:

$$V_{\nu L}^T m_\nu V_{\nu L} = \hat{m}_\nu = \text{diag}(m_1, m_2, m_3). \tag{6.51}$$

We thus learn the following points:

1. There are three heavy Majorana neutrinos of masses M_1, M_2, M_3. We call these states N_1, N_2, N_3. These mass eigenstates are approximately $SU(2)$-singlet states, but have a small, $\mathcal{O}(v/M_i)$, $SU(2)$-doublet component. The masses are, by assumption, much larger than the electroweak scale.

Table 6.1: The SM+N particles

particle	spin	color	Q	mass
$W^\pm$	1	(1)	± 1	$\frac{1}{2}gv$
Z^0	1	(1)	0	$\frac{1}{2}\sqrt{g^2+g'^2}v$
A^0	1	(1)	0	0
g	1	(8)	0	0
h	0	(1)	0	$\sqrt{2\lambda}v$
e, μ, τ	1/2	(1)	-1	$y_{e,\mu,\tau}v/\sqrt{2}$
ν_1, ν_2, ν_2	1/2	(1)	0	$m_{1,2,3}$
N_1, N_2, N_2	1/2	(1)	0	$M_{1,2,3}$
u, c, t	1/2	(3)	$+2/3$	$y_{u,c,t}/\sqrt{2}$
d, s, b	1/2	(3)	$-1/3$	$y_{d,s,b}/\sqrt{2}$

2. There are three light neutrinos of masses m_1, m_2, m_3 of order v^2/M_i. We call these states ν_1, ν_2, ν_3. These mass eigenstates are approximately $SU(2)$-doublet states, but have a small, $\mathcal{O}(v/M_i)$, $SU(2)$-singlet component. The masses are, by the same assumption, much smaller than the electroweak scale.

The details of the spectrum of the SM+N are summarized in Table 6.1.

There are three different mass scales:

- The masses of all bosons and of the charged fermions are of order v.
- The masses of the (approximately) singlet neutrinos are heavy, of order m_N.
- The masses of the (approximately) doublet neutrinos are light, of order v^2/m_N.

Furthermore, the heavier the gauge-singlet neutrinos are, the lighter the $SU(2)_L$-doublet neutrinos. For this reason, the mechanism that generates light neutrino masses via their Yukawa couplings to heavy neutrinos is called "the see-saw mechanism." It arises naturally in various extensions of the SM, such as $SO(10)$ grand unified theories (GUT), and left-right symmetric (LRS) models.

If the singlet neutrinos are very heavy, then they cannot be produced directly in experiments. (Given that they are gauge-singlets, it would be difficult to produce them even if it were kinematically possible to do so.) They can thus be *integrated out* from the theory. This would leave the Standard Model as the effective low energy theory, with non-renormalizable terms suppressed by m_N, the mass scale of the heavy neutrinos. The most important non-renormalizable terms generated by integrating out the N_R fields are the dimension-five terms

$$\mathcal{L}_{d=5} = \frac{Z^\nu_{ij}}{\Lambda}\phi\phi L_i L_j, \tag{6.52}$$

where

$$Z^\nu_{ij}/\Lambda = \left[Y^\nu (M^N)^{-1} Y^{\nu T}\right]_{ij}. \tag{6.53}$$

Thus, the leading terms in the low energy effective theory of the SM+N model are those of Eq. (6.1).

We emphasize the following points:

1. We do not study here the interactions of the N_R fields. By assumption, they are heavy and therefore cannot be experimentally studied.

2. With three or more N_{Ri} fields, the matrix Z^ν is a general symmetric 3×3 matrix of complex, dimensionless couplings. If there were only two (one) N_{Ri} fields, the matrix Z^ν would have one (two) zero eigenvalues. Thus, three is the minimal number required to generate at low energy the most general $\mathcal{L}_{\rm SM+5}$, which is the reason that we defined the SM+N in this way.

3. The Lagrangian $\mathcal{L}_{\rm SM+5}$ can come from other high-energy theories, such as one where we add to the SM a heavy $(1,3)_{-1}$ scalar field.

Chapter 7

Conection to cosmology

7.1 Baryogenesis

7.1.1 The baryon asymmetry

Observations indicate that the number of baryons in the Universe is unequal to the number of antibaryons. To the best of our understanding, all the structures that we see in the Universe – stars, galaxies, and clusters – consist of matter (baryons and electrons) and there is no antimatter (antibaryons and positrons) in appreciable quantities. Since various considerations suggest that the Universe has started from a state with equal numbers of baryons and antibaryons, the observed baryon asymmetry must have been generated dynamically, a scenario that is known by the name of *baryogenesis*.

The baryon asymmetry of the Universe is determined to be

$$\eta \equiv \left.\frac{n_B - n_{\bar{B}}}{n_\gamma}\right|_0 = (6.21 \pm 0.16) \times 10^{-10}, \tag{7.1}$$

where $n_B, n_{\bar{B}}$ and n_γ are the number densities of, respectively, baryons, antibaryons and photons, and a subscript 0 implies "at present time."

The value of the baryon asymmetry of the Universe is inferred in two independent ways. The first way is via big bang nucleosynthesis. This chapter in cosmology predicts the abundances of the light elements, D, ^{3}He, ^{4}He, and ^{7}Li. These predictions depend on a single parameter, which is η. The second way is from measurements of the cosmic microwave background radiation. A larger η would enhance the odd peaks in the spectrum. The fact that the two determinations agree gives much confidence in the value of the baryon asymmetry. A consistent theory of baryogenesis should thus explain $n_B \approx 10^{-9} n_\gamma$ and $n_{\bar{B}} = 0$.

7.1.2 Sakharov conditions

Three conditions that are required to dynamically generate a baryon asymmetry were formulated by Sakharov:

- Baryon number violation: This condition is required in order to evolve from an initial state with $\eta = 0$ to a state with $\eta \neq 0$.

- C and CP violation: If either C or CP were conserved, then processes involving baryons would proceed at precisely the same rate as the C- or CP-conjugate processes involving antibaryons, with the overall effect that no baryon asymmetry is generated.

- Out of equilibrium dynamics: In chemical equilibrium, there are no asymmetries in quantum numbers that are not conserved (such as B, by the first condition).

These necessary ingredients are all present in the Standard Model. Quantitatively, however, the SM fails to explain the observed asymmetry:

- Baryon number is violated in the SM, and the resulting baryon number violating processes are fast in the early Universe. The violation is due to the triangle anomaly, and leads to processes that involve nine left-handed quarks (three of each generation) and three left-handed leptons (one from each generation). A selection rule is obeyed:

  $$\Delta B = \Delta L = \pm 3n. \tag{7.2}$$

  At zero temperature, the amplitude of the baryon number violating processes if proportional to $e^{-8\pi^2/g^2}$, which is too small to have any observable effect. At high temperatures, however, these transitions become unsuppressed.

- The weak interactions of the SM violate C maximally and violate CP via the Kobayashi-Maskawa mechanism. As argued in Section 7.1.3, the KM mechanism introduces a suppression factor of order 10^{-20} into the SM contribution to the baryon asymmetry. Since there are practially no kinematic enhancement factors in the thermal bath, it is impossible to generate $\eta \sim 10^{-9}$ with such a small amount of CP violation. Consequently, baryogenesis implies that there must exist new sources of CP violation, beyond the KM phase of the SM.

- Within the Standard Model, departure from thermal equilibrium occurs at the electroweak phase transition (EWPT). Here, the non-equilibrium condition is provided by the interactions of particles with the bubble wall, as it sweeps through the plasma. The experimental measurement of $m_h \sim 126$ GeV implies, however, that this transition is not strongly first order, as required for successful baryogenesis. Thus, a different kind of departure from thermal equilibrium is required from new physics or a modification to the electroweak phase transition.

We learn that baryogenesis requires new physics that extends the SM in at least two ways. It must introduce new sources of CP violation, and it must either provide a departure from thermal equilibrium in addition to the EWPT or modify modify the EWPT.

An attractive scenario called leptogenesis is described in Section 7.1.4.

7.1.3 The suppression of KM baryogenesis

As explained in the previous section, the three generation SM violates CP if $X_{CP} \neq 0$. The baryon asymmetry of the Universe is a CP violating observable. As such, it is proportional to X_{CP}. More precisely, it is proportional to X_{CP}/T_c^{12}, where $T_c \sim 100$ GeV is the critical temperature of the electroweak phase transition. When one puts the measured values of the quark masses and CKM parameters, one obtains that $X_{CP} \sim 10^{-20}$, and this the KM mechanism cannot account for a baryon asymmetry as large as $\mathcal{O}(10^{-10})$.

One may wonder why the suppression by X_{CP} does not apply to all CP asymmetries measured in experiments. After all, there are CP asymmetries such as $S_{\pi\pi}$ that are experimentally of order one and theoretically known to be suppressed by the KM phase ($\sin 2\alpha$) but by none of the mixing angles or small quark mass-squared differences of X_{CP}. The answer provides some insights as to how the KM mechanism operates. As concerns the mixing angles, they often cancel in the CP asymmetries which are ratios of CP violating to CP conserving rates. The physics behind the mass factors in Eq. (5.21) is that, in order to exhibit CP violation, a process has to "go through" all three flavors of each quark type, and "sense" that their masses are different from each other. Sometimes, the experiment does that for us. For example, when experimenters measure the CP asymmetry in $B \to \pi\pi$, they already distinguish the bottom, up, and down masses from the others (by identifying the B and π mass eigenstates) and thus 'get rid' of the corresponding mass factors. What remains is the $(m_t^2 - m_c^2)$ factor, This factor does appear in Δm_B and, indeed, if this factor were zero, the CP asymmetry, which is really $S_{\pi\pi} \sin(\Delta m_B t)$, would vanish. In contrast, baryogenesis is a flavor-blind process (it sums over all flavors), and is suppressed by all six mass-squared factors of Eq. (5.18).

The important conclusion of the failure of the KM mechanism to account for the baryon asymmetry is the following: *There must exist sources of CP violation beyond the KM phase of the SM.*

7.1.4 Leptogenesis

The addition of the N_{Ri} fields, with the Yukawa (Y^ν) and mass (M^N) terms of Eq. (6.43), is motivated by the seesaw mechanism for light neutrino masses. The addition of these terms implies, however, an additional intriguing consequence: The physics of the singlet fermions is likely to play a role in dynamically generating a lepton asymmetry in the Universe. The reason that *leptogenesis* is qualitatively almost unavoidable once the seesaw mechanism is invoked is that the Sakharov conditions, described in Appendix 7.1.2, are (likely to be) fulfilled:

- Lepton number violation: The Lagrangian terms (6.43) violate L because lepton number cannot be consistently assigned to the N_{Ri} fields in the presence of Y^ν and M^N. If $L(N_R)=1$, then Y^ν respects L but M^N violates it by two units. If $L(N_R)=0$, then M^N respects L but Y^ν violates it by one unit. (Remember that the fact that the SM interactions violate $B+L$ implies that the requirement for baryogenesis from new physics is $B-L$ violation, and not necessarily B violation.)

- CP violation: Since there are irremovable phase in Y^ν (once Y^e and M^N are chosen to be real), the Lagrangian terms (6.43) provides new sources of CP violation.

- Departure from thermal equilibrium: The interactions of the N_i are only of the Yukawa type. If the Y^ν couplings are small enough, these interactions can be slower than the expansion rate of the Universe, in which case the singlet fermions will decay out of equilibrium.

Thus, in the presence of the seesaw terms, leptogenesis is *qualitatively* almost unavoidable, and the question of whether it can successfully explain the observed baryon asymmetry is a *quantitative* one.

We consider leptogenesis via the decays of N_1, the lightest of the singlet fermions N_i. When the decay is into a single flavor α, $N_1 \to L_\alpha \phi$ or $\bar{L}_\alpha \phi^\dagger$, the baryon asymmetry can be written as follows:

$$Y_B = \left(\frac{135\zeta(3)}{4\pi^4 g_*}\right) \times C_{\rm sphal} \times \eta \times \epsilon. \tag{7.3}$$

The first factor is the equilibrium N_1 number density divided by the entropy density at temperature $T \gg M_1$. It is of $\mathcal{O}(4\times 10^{-3})$ when the number of relativistic degrees of freedom g_* is taken as in the SM, $g_*^{\rm SM}=106.75$. The other three factors on the right hand side of Eq. (7.3) represent the following physics aspects:

1. ϵ is the CP asymmetry in N_1 decays. For every $1/\epsilon$ N_1 decays, there is one more L than there are $\bar{L}$'s.

2. η is the efficiency factor. Inverse decay, other "washout" processes, and inefficiency in N_1 production, reduce the asymmetry by $0 \le \eta \le 1$. In particular, $\eta=0$ is the limit of N_1 in perfect equilibrium, so no asymmetry is generated.

3. $C_{\rm sphal}$ describes further dilution of the asymmetry due to fast processes which redistribute the asymmetry that was produced in lepton doublets among other particle species. These include gauge, Yukawa, and $B+L$ violating non-perturbative effects.

These three factors can be calculated, with ϵ and η depending on the Lagrangian parameters. The final result can be written (with some simplifying assumptions) as

$$Y_B \sim 10^{-3}\frac{10^{-3}\ \text{eV}}{\tilde{m}}\epsilon, \tag{7.4}$$

where ($x_j \equiv M_j^2/M_1^2$)

$$\epsilon = \frac{1}{8\pi}\frac{1}{(Y^{\nu\dagger}Y^\nu)_{11}}\sum_j \mathcal{I}m\left\{\left[(Y^{\nu\dagger}Y^\nu)_{1j}\right]^2\right\}\sqrt{x_j}\left[\frac{1}{1-x_j}+1-(1+x_j)\ln\left(\frac{1+x_j}{x_j}\right)\right], \tag{7.5}$$

and

$$\tilde{m} = \frac{(Y^{\nu\dagger}Y^\nu)_{11}v^2}{M_1}. \tag{7.6}$$

The plausible range for $\tilde{m}$ is the one suggested by the range of hierarchical light neutrino masses, $10^{-3}-10^{-1}$ eV, so we expect a rather mild washout effect, $\eta \gtrsim 0.01$. Then, to account for $Y_B \sim 10^{-10}$, we need $|\epsilon| \gtrsim 10^{-5}-10^{-6}$. Using Eq. (7.5), we learn that this condition roughly implies, for the seesaw parameters,

$$\frac{M_1}{M_2}\frac{\mathcal{I}m[(Y^{\nu\dagger}Y^\nu)_{12}^2]}{(Y^{\nu\dagger}Y^\nu)_{11}} \gtrsim 10^{-4}-10^{-5}, \tag{7.7}$$

which is quite natural.

We learn that leptogenesis is attractive not only because all the required features are qualitatively present, but also because the quantitative requirements are plausibly satisfied. In particular, $\tilde{m} \sim 0.01$ eV, as suggested by the light neutrino masses, is optimal for thermal leptogenesis as it leads to effective production of N_1's in the early Universe and only mild washout effects. Furthermore, the required CP asymmetry can be achieved in large parts of the seesaw parameter space.

Appendix

Appendix A

Lie Groups

A crucial role in model building is played by symmetries. You are already familiar with symmetries and with some of their consequences. For example, nature seems to have the symmetry of the Lorentz group which implies conservation of energy, momentum and angular momentum. In order to understand the interplay between symmetries and interactions, we need a mathematical tool called *Lie groups*. These are the groups that describe all continuous symmetries. There are many texts about Lie group. Three that are very useful for particle physics purposes are the book by Howard Georgi ("Lie Algebras in particle physics"), the book by Robert Cahn ("Semi-simple Lie algebras and their representations") and the physics report by Richard Slansky ("Group Theory for Unified Model Building", Phys. Rept. 79 (1981) 1).

A.1 Groups and representations

We start by presenting a series of definitions.

Definition: A *group* G is a set x_i (finite or infinite), with a multiplication law $\cdot$, subject to the following four requirements:

- Closure:

$$x_i \cdot x_j \in G \quad \forall\, x_i. \tag{A.1}$$

- Associativity:

$$x_i \cdot (x_j \cdot x_k) = (x_i \cdot x_j) \cdot x_k. \tag{A.2}$$

- Identity element I (or e):

$$I \cdot x_i = x_i \cdot I = x_i \quad \forall\, x_i. \tag{A.3}$$

- Inverse element x_i^{-1}:

$$x_i \cdot x_i^{-1} = x_i^{-1} \cdot x_i = I. \tag{A.4}$$

Definition: A group is *Abelian* if all its elements commute:

$$x_i \cdot x_j = x_j \cdot x_i \quad \forall\, x_i. \tag{A.5}$$

A *non-Abelian* group is a group that is not Abelian, that is, at least one pair of elements does not commute.

Let us give a few examples:

- Z_2, also known as parity, is a group with two elements, I and P, such that I is the identity and $P^{-1} = P$. This completely specifies the multiplication table. This group is finite and Abelian.

- Z_N, with N=integer, is a generalization of Z_2. It contains N elements labeled from zero until $N-1$. The multiplication law is the same as addition modulo N: $x_i x_j = x_{(i+j)\text{mod } N}$. The identity element is x_0, and the inverse element is given by $x_i^{-1} = x_{N-i}$. This group is also finite and Abelian.

- Multiplication of positive numbers. It is an infinite Abelian group. The identity is the number one and the multiplication law is just a standard multiplication.

- S_3, the group that describes permutation of 3 elements. It contains 6 elements. This group is non-Abelian. Work for yourself the 6 elements and the multiplication table.

Definition: A *representation* is a realization of the multiplication law among matrices.

Definition: Two representations are *equivalent* if they are related by a similarity transformation.

Definition: A representation is *reducible* if it is equivalent to a representation that is block diagonal.

Definition: An *irreducible* representation (irrep) is a representation that is not reducible.

Definition: An irrep that contains matrices of size $n \times n$ is said to be of *dimension* n.

Statement: Any reducible representation can be written as a direct sum of irreps, *e.g.* $D = D_1 + D_2$.

Statement: The dimension of all irreps of an Abelian group is one.

Statement: Any finite group has a finite number of irreps R_i. If N is the number of elements in the group, the irreps satisfy

$$\sum_{R_i} [\dim(R_i)]^2 = N. \tag{A.6}$$

Statement: For any group there exists a *trivial* representation such that all the matrices are just the number 1. This representation is also called the *singlet* representation. It is of particular importance for us.

Let us give some examples for the statements that we made here.

- Z_2: Its trivial irrep is $I = 1,\ P = 1$. The other irrep is $I = 1,\ P = -1$. Clearly these two irreps satisfy Eq. (A.6).
- Z_N: An example of a non-trivial irrep is $x_k = \exp(i2\pi k/N)$.
- S_3: In your homework you will work out its properties.

The groups that we are interested in are *transformation groups of physical systems*. Such transformations are associated with *unitary operators* in the Hilbert space. We often describe the elements of the group by the way that they transform physical states. When we refer to representations of the group, we mean either the appropriate set of unitary operators, or, equivalently, by the matrices that operate on the vector states of the Hilbert space.

A.2 Lie groups

While finite groups are very important, the ones that are most relevant to particle physics and, in particular, to the Standard Model, are infinite groups, in particular continuous groups, that is of cardinality $\aleph_1$. These groups are called Lie groups.

Definition: A *Lie group* is an infinite group whose elements are labeled by a finite set of N continuous real parameters α_ℓ, and whose multiplication law depends smoothly on the α_ℓ's. The number N is called the dimension of the group.

Statement: An Abelian Lie group has $N = 1$. A non-Abelian Lie group has $N > 1$.

The first example is a group we denote by $U(1)$. It represents addition of real numbers modulo 2π, that is, rotation on a circle. Such a group has an infinite number of elements that are labeled by a single continuous parameter α. We can write the group elements as $M = \exp(i\alpha)$. We can also represent it by $M = \exp(2i\alpha)$ or, more generally, as $M = \exp(iX\alpha)$ with X real. Each X generates an irrep of the group.

We are mainly interested in *compact* Lie groups. We do not define this term formally here, but we can use the $U(1)$ example to give an intuitive explanation of what it means. A group of adding with a modulo is compact, while just adding (without the modulo) would be non-compact. In the first, if you repeat the same addition a number of times, you may return to your starting point, while in the latter this would never happen. In other words, in a compact Lie group, the parameters have a finite range, while in a non-compact group, their range is infinite. (Do not confuse that with the number of elements, which is infinite in either case.) Another example is rotations and boosts: Rotations represent a compact group while boosts do not.

Statement: The elements of any compact Lie group can be written as

$$M_i = \exp(i\alpha_\ell X_\ell) \tag{A.7}$$

such that X_ℓ are Hermitian matrices that are called *generators*. (We use the standard summation convention, that is $\alpha_\ell X_\ell \equiv \sum_\ell \alpha_\ell X_\ell$.)

Let us perform some algebra before we turn to our next definition. Consider two elements of a group, A and B, such that in A only $\alpha_a \neq 0$, and in B only $\alpha_b \neq 0$ and, furthermore, $\alpha_a = \alpha_b = \lambda$:

$$A \equiv \exp(i\lambda X_a), \qquad B \equiv \exp(i\lambda X_b). \tag{A.8}$$

Since A and B are in the group, each of them has an inverse. Thus also

$$C = BAB^{-1}A^{-1} \equiv \exp(i\beta_c X_c) \tag{A.9}$$

is in the group. Let us take λ to be a small parameter and expand around the identity. Clearly, if λ is small, also all the β_c are small. Keeping the leading order terms, we get

$$C = \exp(i\beta_c X_c) \approx I + i\beta_c X_c, \qquad C = BAB^{-1}A^{-1} \approx I + \lambda^2 [X_a, X_b]. \tag{A.10}$$

In the $\lambda \to 0$ limit, we have

$$[X_a, X_b] = i\frac{\beta_c}{\lambda^2} X_c. \tag{A.11}$$

Clearly, the combinations

$$f_{abc} \equiv \lambda^{-2}\beta_c \tag{A.12}$$

should be independent of λ. Furthermore, while λ and β_c are infinitesimal, the f_{abc}-constants do not diverge. This brings us to a new set of definitions.

Definition: f_{abc} are called the *structure constants* of the group.

Definition: The commutation relations [see Eq. (A.11)]

$$[X_a, X_b] = i f_{abc} X_c, \tag{A.13}$$

constitute the *algebra* of the Lie group.

Note the following points regarding the Lie Algebra:

- The algebra defines the local properties of the group but not its global properties. Usually, this is all we care about.

- The Algebra is closed under the commutation operator.

- Similar to our discussion of groups, one can define representations of the algebra, that is, matrix representations of X_ℓ. In particular, each representation has its own dimension. (Do not confuse the dimension of the representation with the dimension of the group!)

- The generators satisfy the Jacoby identity

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0. \tag{A.14}$$

- For each algebra there is the trivial (singlet) representation which is $X_\ell = 0$ for all ℓ. The trivial representation of the algebra generates the trivial representation of the group.
- Since an Abelian Lie group has only one generator, its algebra is always trivial. Thus, the algebra of $U(1)$ is the only Abelian Lie algebra.
- Non-Abelian Lie groups have non-trivial algebras.

The example of $SU(2)$ algebra is well-known from QM courses:

$$[X_a, X_b] = i\varepsilon_{abc}X_c. \tag{A.15}$$

Usually, in QM, X is called L or S or J. The $SU(2)$ group represents non-trivial rotations in a two-dimensional complex space. Its algebra is the same as the algebra of the $SO(3)$ group, which represents rotations in the three-dimensional real space.

We should explain what we mean when we say that "the group represents rotations in a space." The QM example makes it clear. Consider a finite Hilbert space of, say, a particle with spin S. The matrices that rotate the direction of the spin are written in terms of exponent of the S_i operators. For a spin-half particle, the S_i operators are written in terms of the Pauli matrices. For particles with spin different from 1/2, the S_i operators will be written in terms of different matrices. We learn that the group represents rotations in some space, while the various representations correspond to different objects that can "live" in that space.

There are three important irreps that have special names. The first one is the trivial – or *singlet* – representation that we already mentioned. Its importance stems from the fact that it corresponds to something that is symmetric under rotations. While that might sound confusing it is really trivial. Rotation of a singlet does not change its representation. Rotation of a spin half does change its representation.

The second important irrep is the *fundamental* representation. This is the smallest irrep. For $SU(2)$, this is the spinor representation. An important property of the fundamental representation is that it can be used to get all other representations. We return to this point later. Here we just remind you that this statement is well familiar from QM. One can get spin-1 by combining two spin-1/2, and you can get spin-3/2 by combining three spin-1/2. Any Lie group has a fundamental irrep.

The third important irrep is the *Adjoint* representation. It is made out of the structure constants themselves. Think of a matrix representation of the generators. Each entry, T^c_{ij} is labelled by three indices. One is the c index of the generator itself, that runs from 1 to N, such that N depends on the group. The other two indices, i and j, are the matrix indices that run from 1 to the dimension of the representation. One can show that each Lie group has one representation where the dimension of the representation is the same as the dimension of the group. This representation is obtained by defining

$$(X_c)_{ab} \equiv -if_{abc}. \tag{A.16}$$

In other words, the structure constants themselves satisfy the algebra of their own group. In $SU(2)$, the Adjoint representation is that of spin-1. It is easy to see that the ε_{ijk} are just the set of the three 3×3 representations of spin 1.

A.3 More formal developments

Definition: A *subalgebra* M is a set of generators that are closed under commutation.

Definition: Consider an algebra L with a subalgebra M. M is an *ideal* if for any $x \in M$ and $y \in L$, $[x, y] \in M$. (For a subalgebra that is not ideal we still have $[x, y] \in L$.)

Definition: A *simple* Lie algebra is an algebra without a non-trivial ideal. (Any algebra has a trivial ideal, the algebra itself.)

Definition: A *semi-simple* Lie algebra is an algebra without a $U(1)$ ideal.

Any algebra can be written as a direct product of simple lie algebras. Thus, we can think about each of the simple algebras separately. You are familiar with this. For example, consider the hydrogen atom. We can think about the Hilbert space as a direct product of the spin of the electron and the spin of the proton.

A useful example is that of the $U(2)$ group, which is not semi-simple:

$$U(2) = SU(2) \times U(1). \tag{A.17}$$

A $U(2)$ transformation corresponds to a rotation in two-dimensional complex space. Think, for example, about the rotation of a spinor. It can be separated into two: The trivial rotation is just a $U(1)$ transformation, that is, a phase multiplication of the spinor. The non-trivial rotation is the $SU(2)$ transformation, that is, an internal rotation between the two spin components.

Definition: The *Cartan subalgebra* is the largest subset of generators whose matrix representations can all be diagonalized at once.

Obviously, these generators all commute with each other and thus they constitute a subalgebra.

Definition: The number of generators in the Cartan subalgebra is called the *rank* of the algebra.

Let us consider a few examples. Since the $U(1)$ algebra has only a single generator, it is of rank one. $SU(2)$ is also rank one. You can make one of its three generators, say S_z, diagonal, but not two of them simultaneously. $SU(3)$ is rank two. We later elaborate on $SU(3)$ in much more detail. (We have to, because the Standard Model has an $SU(3)$ symmetry.)

Our next step is to introduce the terms roots and weights. We do that via an example. Consider the $SU(2)$ algebra. It has three generators. We usually choose S_3 to be in the Cartan subalgebra, and we can combine the two other generators, S_1 and S_2, to a raising and a lowering operator, $S^{\pm} = S_1 \pm iS_2$. Any representation can be defined by the eigenvalues under the operation of the generators in the Cartan subalgebra, in this case S_3. For example, for the spin-1/2 representation, the eigenvalues are $-1/2$ and $+1/2$; For the spin-1 representation, the eiganvalues are -1, 0, and

+1. Under the operation of the raising (S^+) and lowering (S^-) generators, we "move" from one eigenstate of S_3 to another. For example, for a spin-1 representation, we have $S^+|-1\rangle \propto |0\rangle$.

Let us now consider a general Lie group of rank n. Any representation is characterized by the possible eigenvalues of its eigenstates under the operation of the Cartan subalgebra: $|e_1, e_2..., e_n\rangle$. We can assemble all the operators that are not in the Cartan subalgebra into "lowering" and "raising" operators. That is, when they act on an eigenstate they either move it to another eigenstate or annihilate it.

Definition: The *weight vectors* (weights) of a representation are the possible eigenvalues of the generators in the Cartan subalgebra.

Definition: The *roots* of the algebra are the various ways in which the generators move a state between the possible weights.

Statement: The weights completely describe the representation.

Statement: The roots completely describe the Lie algebra.

Note that both roots and weights live in an n-dimensional vector space, where n is the rank of the group. The number of roots is the dimension of the group. The number of weights is the dimension of the irrep.

Let us return to our $SU(2)$ example. The vector space of roots and weights is one-dimensional. The three roots are $0, \pm 1$. The trivial representation has only one weight, zero; The fundamental has two, $\pm 1/2$; The adjoint has three, $0, \pm 1$ (the weights of the adjoint representations are just the roots); and so on.

A.4 $SU(3)$

In this section we discuss the $SU(3)$ group. It is more complicated than $SU(2)$. It allows us to demonstrate few aspects of Lie groups that cannot be demonstrated with $SU(2)$. Of course, it is also important since it is relevant to particle physics.

$SU(3)$ is a generalization of $SU(2)$. It may be useful to think about it as rotations in three-dimensional complex space. Similar to $SU(2)$, the full symmetry of the rotations is called $U(3)$, and it can be written as a direct product of simple groups, $U(3) = SU(3) \times U(1)$. The $SU(3)$ algebra has eight generators. (There are nine independent Hermitian 3×3 matrices. They can be separated to a unit matrix, which corresponds to the $U(1)$ part, and eight traceless matrices, which correspond to the $SU(3)$ part.)

Similar to the use of the Pauli matrices for the fundamental representation of $SU(2)$, the fundamental representation of $SU(3)$ is usually written in terms of the Gell-Mann matrices,

$$X_a = \lambda_a/2, \tag{A.18}$$

with

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\lambda_7 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{A.19}$$

We would like to emphasize the following points:

1. The Gell-Mann matrices are traceless, as they should.

2. There are three $SU(2)$ subalgebras. One of them is manifest and it is given by λ_1, λ_2 and λ_3. Can you find the other two?

3. It is manifest that $SU(3)$ is of rank two: λ_3 and λ_8 are in the Cartan subalgebra.

Having explicit expressions of fundamental representation in our disposal, we can draw the weight diagram. In order to do so, let us recall how we do it for the fundamental (spinor) representation of $SU(2)$. We have two basis vectors (spin-up and spin-down); we apply S_z on them and obtain the two weights, $+1/2$ and $-1/2$. Here we follow the same steps. We take the three vectors,

$$(1, 0, 0,)^T, \quad (0, 1, 0)^T, \quad (0, 0, 1)^T, \tag{A.20}$$

and apply to them the two generators in the Cartan subalgebra, X_3 and X_8. We find the three weights

$$\left(+\frac{1}{2}, +\frac{1}{2\sqrt{3}}\right), \qquad \left(-\frac{1}{2}, +\frac{1}{2\sqrt{3}}\right), \qquad \left(0, -\frac{1}{\sqrt{3}}\right). \tag{A.21}$$

We can plot this in a weight diagram in the $X_3 - X_8$ plane. Please do it.

Once we have the weights we can get the roots. They are just the combination of generators that move us between the weights. Clearly, the two roots that are in the Cartan are at the origin. The other six are those that move us between the three weights. It is easy to find that they are

$$\left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right), \qquad (\pm 1, 0)\,. \tag{A.22}$$

Again, it is a good idea to plot it. This root diagram is also the weight diagram of the Adjoint representation.

A.5 Dynkin diagrams

The $SU(3)$ example allows us to obtain more formal results. In the case of $SU(2)$, it is clear what are the raising and lowering operators. The generalization to groups with higher rank is as follows.

Definition: A *positive (negative) root* is a root whose first non-zero component is positive (negative). A raising (lowering) operator correspond to a positive (negative) root.

Definition: A *simple root* is a positive root that is not the sum of other positive roots.

Statement: Every rank-k algebra has k simple roots. Which ones they are is a matter of convention, but their relative lengths and angles are fixed.

In fact, it can be shown that the simple roots fully describe the algebra. It can be further shown that there are only four possible angles and corresponding relative length between simple roots:

angle	90°	120°	135°	150°
relative length	$1:1$	$1:1$	$1:\sqrt{2}$	$1:\sqrt{3}$.

(A.23)

The above rules can be visualized using Dynkin diagrams. Each simple root is described by a circle. The angle between two roots is described by the number of lines connecting the circles:

(A.24)

where the solid circle in a link represent the largest root.

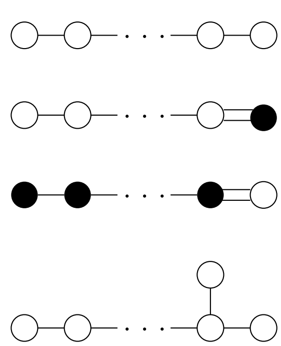
There are seven classes of Lie groups. Four classes are infinite and three classes, called the exceptional groups, have each only a finite number of Lie groups. below you can find all the sets. The number of circles is the rank of the group. Note that different names for the infinite groups are used in the physics and mathematics communities. Below we give both names, but we use only the physics name from now on.

SU$(k+1)$ $[A_k]$

Sp$(2k)$ $[B_k]$

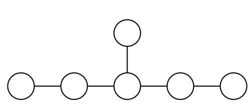
SO$(2k+1)$ $[C_k]$

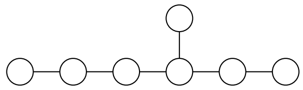
SO$(2k)$ $[D_k]$

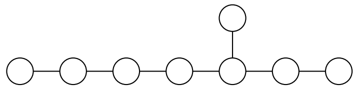


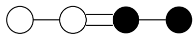
(A.25)

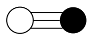
E_6



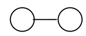
E_7 

E_8 

F_4 

G_2 

(A.26)

Consider, for example, $SU(3)$. The two simple roots are equal in length and have an angle of 120° between them. Thus, the Dynkin diagram is just  .

Dynkin diagrams provide a very good tool to tell us also about what are the subalgebras of a given algebra. We do not describe the procedure in detail here, and you are encouraged to read it for yourself in one of the books. One simple point to make is that removing a simple root always corresponds to a subalgebra. For example, removing simple roots you can see the following breaking pattern:

$$E_6 \to SO(10) \to SU(5) \to SU(3) \times SU(2). \tag{A.27}$$

You may find such a breaking pattern in the context of Grand Unified Theories (GUTs).

Finally, we would like to mention that the algebras of some small groups are the same. For example, the algebras of $SU(2)$ and $SO(3)$ are the same, as are those of $SU(4)$ and $SO(6)$.

A.6 Naming representations

How do we name a representation? In the context of $SU(2)$, which is rank one, there are three different ways to do so.

(i) We denote a representation by its highest weight. For example, spin-0 denotes the singlet representation, spin-1/2 refers to the fundamental representation, where the highest weight is 1/2, and spin-1 refers to the adjoint representation, where the highest weight is 1.

(ii) We can define the representation according to the dimension of the representation-matrices. Then the singlet representation is denoted by 1, the fundamental by 2, and the adjoint by 3.

(iii) We can name the representation by the number of times we can apply S_- to the highest weight without annihilating it. In this notation, the singlet is denoted as (0), the fundamental as (1), and the adjoint as (2).

Before we proceed, let us explain in more detail what we mean by "annihilating the state". Let us examine the weight diagram. In $SU(2)$, which is rank-one, this is a one dimensional diagram. For example, for the fundamental representation, it has two entries, at $+1/2$ and $-1/2$. We now take the highest weight (in our example, $+1/2$), and move away from it by applying the root that corresponds to the lowering operator, -1. When we apply it once, we move to the lowest weight, $-1/2$. When we apply it once more, we move out of the weight diagram, and thus "annihilate the state". Thus, for the spin-1/2 representation, we can apply the root corresponding to S_- once to the highest weight before moving out of the weight diagram, and – in the naming scheme (iii) – we call the representation (1).

We are now ready to generalize this to general Lie algebras. Either of the methods (ii) and (iii) are used. Method (ii) is straightforward, but somewhat problematic. For example, for $SU(3)$, the singlet, fundamental and adjoint representations are denoted by, respectively, 1, 3, and 8. The problem lies in the fact that there could be several different representations with the same dimension, in which case they are distinguihsed by other ways (*e.g.* m and m', or m_1 and m_2).

To use the scheme (iii), we must order the simple roots in a well-defined (even if arbitrary) order. Then we have a unique highest weight. We denote a representation of a rank-k algebra as a k-tuple, such that the first entry is the maximal number of times that we can apply the first simple root on the highest weight before the state is annihilated, the second entry refers to the maximal number of times that we can apply the second simple root on the highest weight before annihilation, and so on. Take again $SU(3)$ as an example. We order the Cartan subalgebra as X_3, X_8 and the two simple roots as

$$S_1 = \left(+\frac{1}{2}, +\frac{\sqrt{3}}{2}\right), \qquad S_2 = \left(+\frac{1}{2}, -\frac{\sqrt{3}}{2}\right). \tag{A.28}$$

Consider the fundamental representation where the highest weight can be chosen to be $\left(+1/2, +1/(2\sqrt{3})\right)$. Subtracting S_1 twice or subtracting S_2 once from the highest weight would annihilate it. Thus the fundamental representation is denoted by $(1, 0)$. You can work out the case of the adjoint representation and find that it should be denoted as $(1, 1)$. In fact, it can be shown that any pair of non-negative integers forms a different irrep. (For $SU(2)$ with the naming scheme (iii), any non-negative integer defines a different irrep.)

From now on we limit our discussion to $SU(N)$.

Statement: For any $SU(N)$ algebra, the fundamental representation is $(1, 0, 0, ..., 0)$.

Statement: For any $SU(N \geq 3)$ algebra, the adjoint representation is $(1, 0, 0, ..., 1)$.

Definition: The *conjugate representation* is the one where the order of the k-tuple is reversed.

For example, $(0, 1)$ is the conjugate of the fundamental representation, which is usually called the anti-fundamental representation. Note that some representations are self–conjugate, *e.g.*, the adjoint representation. An irrep and its conjugate have the same dimension. In the naming scheme (ii), they are called m and $\bar{m}$.

A.7 Particle representations

We now return to the notion that the groups that we are dealing with are transformation groups of physical states. These physical states are often just particles. For example, when we talk about the $SU(2)$ group that is related to the spin transformations, the physical system that is being transformed is often that of a single particle with well-defined spin. In this context, particle physicists often abuse the language by saying that the particle is, for example, in the spin-1/2 representation of $SU(2)$. What they mean is that, as a state in the Hilbert space, it transforms by the spin operator in the 1/2 representation of $SU(2)$. Similarly, when we say that the proton and the neutron form a doublet of isospin-$SU(2)$ (we later define the isospin group), we mean that we represent p by the vector-state $(1,0)^T$ and n by the vector-state $(0,1)^T$, so that the appropriate representation of the isospin generators is by the 2×2 Pauli matrices. In other words, we loosely speak on "particles in a representation" when we mean "the representation of the group generators acting on the vector states that describe these particles." Now, that we explained how physicists abuse the language, we feel free to do so ourselves; We will often talk about "particles in a representation."

How many particles there are in a given irrep? Let us consider a few examples.

- Consider an (α) representation of $SU(2)$. It has

$$N = \alpha + 1, \tag{A.29}$$

particles. The singlet (0), fundamental (1) and adjoint (2) representations have, respectively, 1, 2, and 3 particles.

- Consider an (α,β) representation of $SU(3)$. It has

$$N = (\alpha+1)(\beta+1)\,\frac{\alpha+\beta+2}{2} \tag{A.30}$$

particles. The singlet $(0,0)$, fundamental $(1,0)$ and adjoint $(1,1)$ representations have, respectively, 1, 3, and 8 particles.

- Consider an (α,β,γ) representation of $SU(4)$. It has

$$N = (\alpha+1)(\beta+1)(\gamma+1)\,\frac{\alpha+\beta+2}{2}\,\frac{\beta+\gamma+2}{2}\,\frac{\alpha+\beta+\gamma+3}{3} \tag{A.31}$$

particles. The singlet $(0,0,0)$, fundamental $(1,0,0)$ and adjoint $(1,0,1)$ representations have, respectively, 1, 4, and 15 particles. Note that there is no $\alpha+\gamma+2$ factor. Only a consecutive sequence of the label integers appears in any factor.

- The generalization to any $SU(N)$ is straightforward. It is easy to see that the fundamental of $SU(N)$ is an N and the adjoint is N^2-1.

In $SU(2)$, the number of particles in a representation is unique. In a general Lie group, however, the case may be different. Yet, it is often used to identify irreps. For example, in $SU(3)$ we usually call the fundamental 3, and the adjoint 8. For the anti-fundamental we use $\bar{3}$. In cases where there are several irreps with the same number of particles we often use a prime to distinguish them. For example, in $SU(3)$, both $(4, 0)$ and $(2, 1)$ contain 15 particles. We denote them by 15 and $15'$.

Two more definitions: For an $SU(N)$ group, a *real* representation is a one that is equal to its conjugate one. $SU(2)$ has only real irreps. The adjoint of any $SU(N)$ is real, while the fundamental for $N \geq 3$ is complex.

A.8 Combining representations

When we study spin, we learn how to combine $SU(2)$ representations. The canonical example is to combine two spin-1/2 to generate a singlet (spin-0) and a triplet (spin-1). We need to learn how to combine representations in $SU(N > 2)$ as well. The basic idea is, just like in $SU(2)$, that we need to find all the possible ways to combine the indices and then assign it to the various irreps. That way we know what irreps are in the product representation and the corresponding CG-coefficients. This is explained in many textbooks and we do not explain it any further here.

Often, however, all we want to know is what irreps appear in the product representation, without the need to get all the CG-coefficients. There is a simple way to do just this for a general $SU(N)$. This method is called *Young Tableaux*, or Young Diagrams. The details of the method are well explained in the PDG, pdg.lbl.gov/2007/reviews/youngrpp.pdf.

With this comment we conclude our very brief introduction to Lie groups. We are now ready to start the physics part of the course.

questions

Question A.8.9: S_3

In this question we study the group S_3. It is the simplest finite non-Abelian group. You can think about it as all possible permutation of three elements. The group has 6 elements. Thinking about the permutations we see that we get the following representation of the group:

$$
() = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$
$$
(13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad (23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
$$

$$(123) \quad = \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad (321) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{A.32}$$

The names are instructive. For example, (12) represents exchanging the first and second elements. (123) and (321) are cyclic permutation to the right or left.

1. Write explicitly the 6×6 multiplication table for the group.

2. Show that the group is non-Abelian. Hint, it is enough to find one example.

3. Z_3 is a sub group of S_3. Find the three generators that correspond to Z_3.

4. In class we mentioned the following theorem for finite groups

$$\sum_{R_i} [\dim(R_i)]^2 = N, \tag{A.33}$$

   where N is the number of elements in the group and R_i are all the irreps. Based on this, proof that the representation in Eq. (A.32) is reducible.

5. The representation in Eq. (A.32) is reducible. Write it explicitly in a $(1+2)$ block diagonal representation. (Hint: find a vector which is an eigenvector of all the above matrices.)

6. In the last item you found a two dimensional and a one dimensional representations of S_3. Based on (A.33) you know that there is only one more representation and that it is one dimensional. Find it.

Question A.8.10: Lie algebras

Consider two general elements of a Lie groups,

$$A \equiv \exp(i\lambda X_a), \qquad B \equiv \exp(i\lambda X_b). \tag{A.34}$$

where X_i is a generator. We think about λ as a small parameter. Then, consider a third element

$$C = BAB^{-1}A^{-1} \equiv \exp(i\beta_c X_c). \tag{A.35}$$

Expand C in powers of λ and show that at lowest order you get the Lie algebra

$$[X_a, X_b] = i f_{abc} X_c, \qquad f_{abc} \equiv \frac{\beta_c}{\lambda^2}. \tag{A.36}$$

Question A.8.11: Dynkin diagrams

1. Draw the Dynkin diagram of SO(10).

2. What is the rank of SO(10)?

3. How many generators there are for SO(10)? (We did not proof a general formula for the number of generators for SO(N). It should be simple for you to find such a formula using your understanding of rotations in real N-dimensional spaces.)

4. Based on the Dynkin diagram show that SO(10) has the following subalgebras

$$\mathrm{SO}(8), \qquad \mathrm{SU}(5), \qquad \mathrm{SU}(4) \times \mathrm{SU}(2), \qquad \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{SU}(2). \tag{A.37}$$

   In each case show which simple root you can remove from the SO(10) Dynkin diagram.

Question A.8.12: SU(3)

1. The three Gell–Mann matrices, $a\lambda_1$, $a\lambda_2$ and $a\lambda_3$ satisfy an SU(2) algebra, where a is a constant. What is a?

2. Does this fact mean that SU(3) is not a simple Lie group?

3. There are two other independent combinations of Gell–Mann matrices that satisfy SU(2) algebras. What are they? Hint: Look at the root diagram.

Question A.8.13: representations

Here we practice finding the number of degrees of freedom in a given irrep.

1. In SU(5), how many particles there are in the following irreps

$$(1, 0, 0, 0), \quad (0, 1, 0, 0), \quad (1, 1, 0, 0). \tag{A.38}$$

2. In SU(3) how many particles there are in the following irreps

$$(1, 0), \quad (2, 0), \quad (1, 1), \quad (3, 0), \quad (1, 2), \quad (2, 2). \tag{A.39}$$

Question A.8.14: Combining irreps

Here we are going to study the use of Young Tableaux. The details of the method can be found in the PDG, pdg.lbl.gov/2007/reviews/youngrpp.pdf (there is a link in the website of the course). Study the algorithm and do the following calculations. Make sure you check that the number of particles on both sides is the same. Write your answer both in the k-tuple notation and the number notation. For example, in SU(3) you should write

$$(1, 0) \times (0, 1) = (0, 0) + (1, 1), \qquad 3 \times \bar{3} = 1 + 8. \tag{A.40}$$

1. In SU(3) calculate
$$3 \times 3, \qquad 3 \times 8, \qquad \overline{10} \times 8. \tag{A.41}$$

2. Given that the quarks are $SU(3)_C$ triplets, 3, the anti-quarks are $\bar{3}$ and the gluons are color octets, 8, which of the following could be an observable bound state?
$$q\bar{q}, \qquad qq, \qquad qg, \qquad gg, \qquad q\bar{q}g, \qquad qqq. \tag{A.42}$$
Note that an observable bound state must be a color singlet.

3. Find what is $\bar{5}$ and 10 in SU(5) in a k-tuple notation.

4. Calculate is SU(5)
$$\bar{5} \times \bar{5}, \qquad 10 \times 10, \qquad \bar{5} \times 10. \tag{A.43}$$

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