Introduction to Astrophysics
Tutorial 8: Dynamical Friction

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We consider the case of a mass M moving through a population of stars each of mass m.

1 Encounter with a Single Star

First we look at the encounter of the mass M with a single star of mass m. Let $\mathbf{x_M}, \mathbf{v_M}$ and $\mathbf{x_m}, \mathbf{v_m}$ denote the positions and velocities of M and m respectively. Then, the separation vector $\mathbf{r} = \mathbf{x_m} - \mathbf{x_M}$ (with $\mathbf{V} = \dot{\mathbf{r}}$), obeys:

$$\frac{mM}{m+M}\ddot{\mathbf{r}} = -\frac{GMm}{r^2}\hat{\mathbf{e}}_\mathbf{r}$$

This is the EOM of a fictitious particle (the "reduced particle") of mass 1 in the Keplerian potential of a fixed body of mass $m + M$.

Denote

$$\Delta\mathbf{V} = \Delta\mathbf{v_m} - \Delta\mathbf{v_M}$$

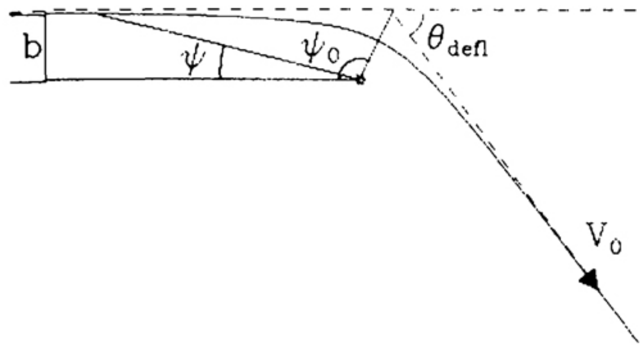
From momentum conservation we have:

$$m \cdot \Delta\mathbf{v_m} + M \cdot \Delta\mathbf{v_M} = 0$$

so the change in velocity of M (due to drag) is given by:

$$\Delta\mathbf{v_M} = -\left(\frac{m}{m+M}\right)\Delta\mathbf{V}$$

We thus wish to find an expression for $\Delta\mathbf{V}$, which is the change in velocity of the reduced particle, following the encounter.



Denote $\mathbf{V}(t \to -\infty) = \mathbf{V_0}$ and b the impact parameter. Then

$$L = bV_0$$

is the size of the conserved angular momentum. For a Keplerian potential, such as this, we have:

$$\frac{1}{r} = C\cos(\psi - \psi_0) + \frac{G(M+m)}{b^2V_0^2}$$

Differentiating with respect to time gives:

$$\begin{aligned} \dot{r} &= Cr^2\dot{\psi}\sin(\psi - \psi_0) \\ &= CbV_0\sin(\psi - \psi_0) \end{aligned}$$

where we have used the conservation of angular momentum $r^2\dot{\psi} = bV_0$. At $t \to -\infty$ we have from this:

$$-V_0 = CbV_0\sin(-\psi_0)$$

and from the previous expression:

$$0 = C\cos\psi_0 + \frac{G(M+m)}{b^2V_0^2}$$

Eliminating C gives:

$$\tan\psi_0 = -\frac{bV_0^2}{G(m+M)}$$

which allows us to find the deflection angle:

$$\theta_{defl} = 2\psi_0 - \pi$$

We thus have:

$$\begin{aligned} |\Delta\mathbf{V}_\perp| &= V_0\sin\theta_{defl} \\ &= \frac{2bV_0^3}{G(m+M)}\left[1 + \frac{b^2V_0^2}{G^2(m+M)^2}\right]^{-1} \\ \left|\Delta\mathbf{V}_\parallel\right| &= V_0 - V_0\cos\theta_{defl} \\ &= 2V_0\left[1 + \frac{b^2V_0^2}{G^2(m+M)^2}\right]^{-1} \end{aligned}$$

where $\Delta\mathbf{V}_{\|}$ points in the opposite direction of $\mathbf{V_0}$. We can now finally find:

$$\begin{aligned}
|\Delta\mathbf{V}_{\mathbf{M}\perp}| &= \frac{2bmV_0^3}{G\left(m+M\right)^2}\left[1+\frac{b^2V_0^2}{G^2\left(m+M\right)^2}\right]^{-1} \\
\left|\Delta\mathbf{V}_{\mathbf{M}\|}\right| &= \frac{2mV_0}{m+M}\left[1+\frac{b^2V_0^2}{G^2\left(m+M\right)^2}\right]^{-1}
\end{aligned}$$

where $\Delta\mathbf{V}_{\mathbf{M}\|}$ points in the same direction as $\mathbf{V_0}$.

2 Homogeneous Sea of Stars

If M is traveling through a homogeneous sea of stars then from symmetry $\sum\Delta\mathbf{V}_{\mathbf{M}\perp}=0$ but $\sum\Delta\mathbf{V}_{\mathbf{M}\|}\neq 0$ since all of these vectors point in the same direction. The mass M suffers a steady deceleration, known as DYNAMICAL FRICTION. This is caused by M deflecting smaller m's, creating an over-density behind it, which slows it down.

The distribution function $f\left(\mathbf{v}\right)$ gives the number of stars encountered with $\mathbf{v_m}\pm d^3\mathbf{v_m}$ and $b\pm db$, thus the rate that M encounters these stars is:

$$2\pi bdb\cdot V_0\cdot f\left(\mathbf{v_m}\right)d^3\mathbf{v_m}$$

and so the net change of $\mathbf{v_M}$ due to these encounters is:

$$\begin{aligned}
\left.\frac{d\mathbf{v_M}}{dt}\right|_{\mathbf{v_m}} &= \mathbf{V_0}\cdot f\left(\mathbf{v_m}\right)d^3\mathbf{v_m}\int_0^{b_{max}}\frac{2mV_0}{m+M}\left[1+\frac{b^2V_0^2}{G^2\left(m+M\right)^2}\right]^{-1}2\pi bdb \\
&= 2\pi\ln\left(1+\Lambda^2\right)G^2m\left(M+m\right)f\left(\mathbf{v_m}\right)d^3\mathbf{v_m}\frac{\mathbf{v_m}-\mathbf{v_M}}{\left|\mathbf{v_m}-\mathbf{v_M}\right|^3}
\end{aligned}$$

where:

$$\Lambda=\frac{b_{max}V_0^2}{G\left(M+m\right)}$$

Typically Λ is very large, so that we can approximate $\frac{1}{2}\ln\left(1+\Lambda^2\right)=\ln\Lambda$. We also replace V_0 in the expression for Λ by a typical speed v_{typ} (note that neither b_{max} nor v_{typ} are accurately defined, but will serve to get order of magnitude estimates).

In this approximation, and taking $\ln\Lambda=const$ the equation above states that the stars that have velocity $\mathbf{v_m}$ exert a force on M that acts parallel to $\mathbf{v_m}-\mathbf{v_M}$ and is inversely proportional to the square of this vector. So, integrating the acceleration $\left.\frac{d\mathbf{v_M}}{dt}\right|_{\mathbf{v_m}}$ on all $\mathbf{v_m}$'s is equivalent to the problem of finding the gravitational field at the point with position vector $\mathbf{v_M}$ that, is generated by the "mass density" $\rho\left(\mathbf{v_m}\right)=4\pi\ln\left(\Lambda\right)Gm\left(M+m\right)f\left(\mathbf{v_m}\right)$. If the stars move isotropically, the density distribution is spherical and by Newton's theorems, the total acceleration of M is simply equal to $\frac{G}{v_M^2}$ times the total "mass" that lies at $v_m<v_M$. Hence for an isotropic distribution of stellar velocities:

$$\boxed{\frac{d\mathbf{v_M}}{dt}=-16\pi^2\ln\left(\Lambda\right)G^2m\left(M+m\right)\int_0^{v_M}f\left(v_m\right)v_m^2dv_m\frac{\mathbf{v_M}}{v_M^3}}$$

This is called the CHANDRASEKHAR DYNAMICAL FRICTION FORMULA. Note that only stars moving slower than v_M contribute to the force, and that this is a frictional drag force (i.e. with direction always opposite to $\mathbf{v_M}$).

2.1 The Small v_M Case

If v_M is small enough, we may replace $f(v_m)$ in the integral with $f(0)$ to find:

$$\frac{d\mathbf{v_M}}{dt} = -\frac{16}{3}\pi^2 \ln(\Lambda) G^2 m (M+m) f(0) \mathbf{v_M}$$

which has the form of a constant times $\mathbf{v_m}$, much like the Stokes formula for friction in a fluid.

2.2 The Maxwellian Distribution Case

If f describes a Maxwellian distribution of velocities with dispersion σ, $f = \frac{n_0}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{v^2}{2\sigma^2}\right)$ (with n_0 the total number density of stars), then:

$$\frac{d\mathbf{v_M}}{dt} = -\frac{16\pi^2 \ln(\Lambda) G^2 (M+m) n_0 m}{v_M^3}\left[erf(X) - \frac{2X}{\sqrt{\pi}} e^{-X^2}\right]\mathbf{v_M}$$

with $X = \frac{v_m}{\sqrt{2}\sigma}$.

In the limit $M \gg m$, this can be written as:

$$\frac{d\mathbf{v_M}}{dt} = -\frac{4\pi \ln(\Lambda) G^2 \rho M}{v_M^3}\left[erf(X) - \frac{2X}{\sqrt{\pi}} e^{-X^2}\right]\mathbf{v_M}$$

where $\rho = n_0 m$ is the mass density. In this case, the frictional drag is independent of the mass m of each individual star, but depends only on the mass density ρ. Also, acceleration is proportional to M and the drag force is thus proportional to M^2.

3 Example: The Decay in the Orbit of a Globular Cluster

Consider a globular cluster orbiting in a galaxy. Dynamic friction will cause it to loose energy and spiral towards the center. We'd like to estimate the time t_{fric} it will take a cluster starting at radius r_i to spiral down to the center of the galaxy.

Assume the galaxy has a flat velocity curve $v_c = const$ with dispersion $\sigma = v_c/\sqrt{2}$, and that the density profile is:

$$\rho(r) = \frac{v_c^2}{4\pi G r^2}$$

For a cluster at $v = v_c$, the dynamical friction is:

$$\begin{aligned} F &= -\frac{4\pi \ln(\Lambda) G^2 M^2 \rho}{v_c^2}\left[erf(1) - \frac{2}{\sqrt{\pi}} e^{-1}\right] \\ &\approx -0.428 \ln\Lambda \frac{GM^2}{r^2} \end{aligned}$$

using the values $b_{max} = 2kpc$, $M = 10^6 M_\odot$, $v_{typ} = v_c = 250km/s$, which also give $\ln\Lambda \approx 10$. This tangential force causes the cluster to loose angular momentum (per unit mass) at a rate:

$$\begin{aligned} \frac{dL}{dt} &= \frac{Fr}{M} \\ &\approx -0.428 \ln\Lambda \frac{GM}{r} \end{aligned}$$

Substituting the angular momentum per unit mass $L = rv_c$ (since we assume the cluster velocity remains constant at v_c) gives:

$$r\frac{dr}{dt} = -0.428 \frac{GM}{v_c} \ln\Lambda$$

Integrating with initial condition $r\,(t = 0) = r_i$ gives:

$$\begin{aligned} t_{fric} &= \frac{1.17}{\ln\Lambda} \frac{r_i^2 v_c}{GM} \\ &= \frac{2.64 \cdot 10^{11}}{\ln\Lambda} \left(\frac{r_i}{2kpc}\right)^2 \left(\frac{v_c}{250km/s}\right) \left(\frac{10^6 M_\odot}{M}\right) yr \end{aligned}$$

(one would think that very low mass clusters would survive longer, however these are more vulnerable to tidal disruption)

Applying this to our nearest large galaxy, M31, indicates that all clusters within $r \sim 3kpc$ should have spiraled in by now, implying that the bright nucleus of M31 is composed of debris of 20-30 massive globular clusters.