Quantum Mechanics II Problem Set #9 due 12/2/98

1 Lorentz Transformations for Fields

For a field Ψ an infinitesimal Lorentz transformation can be written as

$$\Psi \to (1 - i\boldsymbol{\theta}\mathbf{L} - i\boldsymbol{\beta}\mathbf{K})\Psi,$$

where the generators of rotations and boosts are defined as

$$L^i = \frac{1}{2}\epsilon^{ijk}J^{jk}, \qquad K^i = J^{0i}$$

in terms of the generalized angular momentum $J^{\mu\nu}$. Note that the Lorentz group, being "noncompact" (due to the boosts), has no faithful, finite-dimensional representations that are unitary. But this does not matter here, since Ψ is not a wavefunction, but a classical field.

1. Use the well known commutation relations for $\mathbf{L}$ and $\mathbf{K}$ to show that the linear combinations $\mathbf{J}_\pm \equiv \frac{1}{2}(\mathbf{L} \pm i\mathbf{K})$ commute with one another and separately satisfy the commutation relations of angular momentum.

2. Explain why this implies that all finite-dimensional representations of the Lorentz group correspond to a pair of integers or half integers (j_+, j_-), corresponding to pairs of representations of the rotation group.

3. Using the fact that $\mathbf{J} = \boldsymbol{\sigma}/2$ in the spin 1/2 representation of angular momentum, write explicitly the transformation laws of the 2-component objects transforming according to the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representation of the Lorentz group. Show that these correspond precisely to the transformations of the Weyl spinors ψ_L and ψ_R given in the lectures.

4. Prove the identity $\boldsymbol{\sigma}^T = -\sigma_2\boldsymbol{\sigma}\sigma_2$.

5. Using this identity show that $\psi' \equiv \psi_L^T\sigma_2$ transforms in the unitarily equivalent form
$$\psi' \to \psi'(1 + i\boldsymbol{\theta}\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\frac{\boldsymbol{\sigma}}{2}).$$

6. Using this law, we can represent the object that transforms as $(\frac{1}{2}, \frac{1}{2})$ as a 2×2 matrix that has the ψ_R transformation law on the left and, simultaneously, the transposed ψ_L transformation on the right. Parameterize this matrix as
$$\begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix}$$
and show that the object V^μ transforms like a 4-vector.

2 Chiral Symmetry

1. Show that the *chirality operator* defined as $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ has the following properties:
$$\gamma^{5\dagger} = \gamma^5, \qquad (\gamma^5)^2 = 1_4, \qquad \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0.$$

2. Find its representation in the Dirac-Pauli representation where
$$\gamma^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$
and in the Weyl (or chiral) representation where
$$\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

3. Show that the operators $P_\pm \equiv \frac{1}{2}(1_4 \pm \gamma^5)$ have the appropriate properties to be projection operators, that is,
$$P_\pm^2 = P_\pm, \qquad P_+ + P_- = 1_4, \qquad P_\pm P_\mp = 0.$$
$P_R = P_+$ and $P_L = P_-$ are called the right- and left-hand projection operators.

4. Consider the left- and right-handed fields
$$\psi_L = P_L\psi, \qquad \psi_R = P_R\psi,$$
where ψ is a Dirac field. Find expressions for the conjugated fields $\overline{\psi_L}$ and $\overline{\psi_R}$ in terms of $\overline{\psi}$ and $P_\pm$ and write the Lagrangian of the massless Dirac field
$$\mathcal{L} = i\overline{\psi}\partial\!\!\!/\psi$$
in terms of the *chiral* fields $\psi_L, \psi_R, \overline{\psi_L}$ and $\overline{\psi_R}$.

5. Show that this Lagrangian has a *chiral symmetry*, i.e. it is invariant under the chiral transformations:
$$\delta\psi_L = i\epsilon_L\psi_L, \qquad \delta\psi_R = i\epsilon_R\psi_R,$$
where ϵ_L and ϵ_R are two independent infinitesimal parameters. Rewrite these transformations in the form:
$$\delta\psi = i(\epsilon - \epsilon_5\gamma^5)\psi,$$
and express ϵ, ϵ_5 in terms of ϵ_L, ϵ_R.

6. Generalize your result for a Lagrangian of N massless Dirac fields and compute the conserved currents and charges.

7. Finally, show that for massive fermions the *handedness* is not a good quantum number, since the mass term $m\overline{\psi}\psi$ spoils the chiral symmetry.