1 Boosting a Photon

Suppose that observer $\mathcal{O}$ sees a photon ($m_\gamma = 0$) with momentum $\mathbf{p}$ in the y-direction and polarization vector in the z-direction. A second observer $\mathcal{O}'$ moves relative to the first with velocity $\mathbf{v}$ in the z-direction. How does $\mathcal{O}'$ describe the same photon?

Hint: Consider first an infinitesimal boost.

2 Classical Electromagnetism

Classical electromagnetism (with no sources) follows form the action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \tag{1}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

1. Derive Maxwell's equations as the Euler-Lagrange equations of this action, treating the components A_μ as the dynamical variables. Write the equations in standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$.

2. Construct the energy-momentum tensor for this theory. Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu,$$

leads to an energy-momentum tensor $\tilde{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{1}{2}(E^2 + B^2), \qquad \mathbf{S} = \mathbf{E} \times \mathbf{B}.$$

3. Show that the action (1) stays invariant under the symmetry transformation $A_\mu \to A'_\mu = \lambda A_\mu$ and $x_\mu \to x'_\mu = x_\mu / \lambda$, where λ is a real parameter, and find the conserved current and charge.

3 The Complex Scalar Field

Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

$$S = \int d^4x (\partial_\mu \phi^\star \partial^\mu \phi - m^2 \phi^\star \phi).$$

It is easiest to analyze this theory by considering $\phi(x)$ and $\phi^\star(x)$, rather than the real and imaginary parts of $\phi(x)$, as the basic dynamical variables.

1. Find the conjugate momenta to $\phi(x)$ and $\phi^\star(x)$ and the canonical commutation relations. Show that the Hamiltonian is

   $$H = \int d^3x (\pi^\star \pi + \nabla \phi^\star \nabla \phi + m^2 \phi^\star \phi).$$

   Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

2. Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m.

3. Rewrite the conserved charge

   $$Q = \int d^3x \frac{i}{2}(\phi^\star \pi^\star - \pi \phi)$$

   in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

4. Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_a(x)$, where $a = 1, 2$. Show that there are now four conserved charges, one given by the generalization of part (c), and the others three given by

   $$Q_j = \int d^3x \frac{i}{2}(\phi_a^\star (\sigma^j)_{ab} \pi_b^\star - \pi_a (\sigma^j)_{ab} \phi_b),$$

   where σ^j are the Pauli sigma matrices. Show that these three charges have the same commutation relations of angular momentum $SU(2)$. Generalize these results to the case of n identical complex scalar fields.