Quantum Mechanics II Problem Set #6 due 7/1/98

I. Generalized Angular Momentum

A rotation in the 4d-Minkowski space is generated by the generalized angular momentum $J^{\mu\nu}$, which has six independent components, three for rotations and three for boosts. The $J^{\mu\nu}$ satisfy the commutation relation of the $SO(3,1)$ Lie algebra:

$$i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu}, \tag{1}$$

where $\eta^{00} = -1$ and $\eta^{11} = \eta^{22} = \eta^{33} = 1$ and all other components are zero. Indices may be lowered or raised by contraction with $\eta_{\mu\nu}$ or $\eta^{\mu\nu}$:

$$\omega_{\sigma\rho} \equiv \eta_{\mu\sigma} \omega^{\mu}{}_{\rho}, \quad \omega^{\mu}{}_{\rho} \equiv \eta^{\mu\sigma} \omega_{\sigma\rho}.$$

The scalar product of two four-vectors $A^\mu = (A^0, \vec{A})$ and $B^\nu = (B^0, \vec{B})$ is defined by

$$A \cdot B = \vec{A} \cdot \vec{B} - A^0 B^0 = A_\mu B^\mu = A^\mu B_\mu = A_\mu B_\nu \eta^{\mu\nu} = A^\mu B^\nu \eta_{\mu\nu}.$$

1. Using the commutation relations (1) show explicitly that the angular momentum three vector $\vec{J} = (J_1, J_2, J_3) \equiv (J^{23}, J^{31}, J^{12})$ obeys the $SU(2)$ algebra.
2. Verify that $\eta_{\mu\nu}\eta^{\mu\nu} = 4$.
3. Prove that $\partial_\mu x_\nu = \eta_{\mu\nu}$.
4. Use this to show that the orbital momentum operator $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$ satisfy the commutation relations of the $SU(2)$ Lie algebra.

Note that for a scalar particle $J_{\mu\nu} = L_{\mu\nu}$, but in general we will have to add a spin angular momentum $S_{\mu\nu}$ which separately obeys (1).