1 Angular momentum and the $SU(2)$ algebra

The group of (continuous) rotations in 3d-Euclidean space is an example of an infinite group. It is a *connected Lie group*, as all transformations can be described by a finite set of real continuous parameters, e.g. the three Euler-angles. Moreover it is a non-Abelian group, since the three generators, which we identify with the angular momentum operators J_1, J_2 and J_3 do *not* commute, but satisfy the $SU(2)$ algebra

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{1}$$

Let us introduce the non-Hermitian step-operators defined by

$$J_\pm \equiv J_1 \pm iJ_2.$$

1. Show that the operator $J^2 = J_1^2 + J_2^2 + J_3^2$ commutes with all the generators J_i. Such an operator is called invariant or Casimir operator.
2. Derive the commutation relations between J_+, J_- and J_3.
3. Prove that $J_\pm J_\mp = J^2 - J_3(J_3 \mp 1)$.
4. Let $|a,b\rangle$ be the joint eigenstate of J^2 and J_3 with eigenvalues a and b, respectively:

   $$J^2|a,b\rangle = a\ |a,b\rangle, \qquad J_3|a,b\rangle = b\ |a,b\rangle.$$

   Prove that $a \geq b^2$.
5. Show that $J_\pm|a,b\rangle$ are also joint eigenfunctions of J^2 and J_3 and find their eigenvalues.
6. Using this conclude that there must be states such that $J_\pm|a,b_\pm\rangle = 0$. What values take b_+ and b_-?
7. If we set $j \equiv b_+$, prove that $a = j(j+1)$ and that j is either an integer or half-odd-integral.
8. Now let us set also $m \equiv b$ in agreement with the usual notation for angular momentum. Find the matrix representation of J_1, J_2 and J_3 for $|j,m\rangle$. Write down the explicit matrices for $j = 1/2$, $j = 1$ and $j = 3$.

2 Adjoint representation

The *adjoint representation* is defined by the generators:

$$(T_a)_{ab} \equiv -i f_{abc}. \tag{2}$$

1. Prove that the T_a's indeed fulfill the group algebra.
2. Show explicitly that the spin 1 representation of $SU(2)$ you obtained above is equivalent to the adjoint representation (2), with the *structure constants* f_{abc} given by the completely antisymmetric tensor ϵ_{abc}.

3 Direct sums and products

Assume we have two representations D_1 and D_2 of the *same* Lie group G. If $\{|i\rangle\}(i = 1,..,n)$ is an orthonormal basis in the space on which D_1 acts and $\{|x\rangle\}(x = 1,...,m)$ is an orthonormal basis for D_2, we can identify the product $\{|i\rangle|x\rangle\}$ with the orthonormal basis in an $n \times m$ dimensional space, called the *direct product space*. On this space, we can define the action of the *direct product representation* $D_1 \otimes D_2$ by

$$(D_1 \otimes D_2)\{|i\rangle|x\rangle\} = \{D_1|i\rangle D_2|x\rangle\}.$$

1. Using the expansion of a representation D in terms of the generators X_a given by $D(\alpha_a) = \exp(i \sum_a \alpha_a X_a)$, show by differentiating that the matrix elements of the generators of the direct product representation are:

$$[X_a(D_1 \otimes D_2)]_{ix,jy} = [X_a(D_1)]_{ij}\delta_{xy} + \delta_{ij}[X_a(D_2)]_{xy}.$$

2. As an example consider both D_1 and D_2 to be the spin 1/2 representation of $SU(2)$. Compute the generators of $D_1 \otimes D_2$ and convince yourself that they fulfill the $SU(2)$ algebra.
3. *For extra credit:* Show for the above example that $D_1 \otimes D_2$ is a reducible representation and find its decomposition into a direct product representation. *Hint:* Look for a common eigenvector to the three generators of $D_1 \otimes D_2$.