Quantum Mechanics II Problem Set #10 due 19/2/98

1 Majorana Fermions

You have seen in class that one can write a relativistic equation (the *Weyl equation*) for a massless 2-component fermion field that transforms as the upper two components of a Dirac spinor (Ψ_L). Call such a 2-component field $\chi_a(x)$, where $a = 1, 2$.

1. Show that it is possible to write an equation for $\chi(x)$ as a massive field in the following way:

$$i\bar{\sigma}^\mu \partial_\mu \chi - im\sigma^2 \chi^* = 0$$

   That is, show, first, that this equation is relativistically invariant and, second, that it implies the Klein-Gordon equation, $(\partial^2 + m^2)\chi = 0$. This form of the fermion mass is called a Majorana mass term.

2. Does the Majorana equation follow from a Lagrangian? The mass term would seem to be the variation of $(\sigma_2)_{ab}\chi(x)^*_a\chi(x)^*_b$; however, since σ_2 is antisymmetric, this expression would vanish if $\chi(x)$ were an ordinary c-number field. When we go to quantum field theory, we know that $\chi(x)$ will become an anticommuting quantum field. Therefore, it makes sense to develop its classical theory by considering $\chi(x)$ as a classical anticommuting field, that is, as a field that takes as values *Grassmann numbers* which satisfy

$$\alpha\beta = -\beta\alpha \qquad \text{for any} \quad \alpha, \beta.$$

   Note that this relation implies that $\alpha^2 = 0$. A Grassmann field $\xi(x)$ can be expanded in a basis of functions as

$$\xi(x) = \sum_n \alpha_n \phi_n(x),$$

   where the $\phi_n(x)$ are orthogonal c-number functions and the α_n are a set of independent Grassmann numbers. Define the complex conjugate of a product of Grassmann numbers to reverse their order:

$$(\alpha\beta)^* \equiv \beta^*\alpha^* = -\alpha^*\beta^*$$

   This rule imitates the Hermitian conjugation of quantum fields. Show that the classical action,

$$S = \int d^4x \left[\chi^\dagger i\bar{\sigma}\partial\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right],$$

   (where $\chi^\dagger = (\chi^*)^T$) is real ($S^* = S$), and that varying this S with respect to χ and χ^* yields the Majorana equation.

3. Let us write a 4-component Dirac field as:

$$\Psi(x) = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix},$$

and recall that the lower components of Ψ transform in a way equivalent by a unitary transformation to the complex conjugate of the representation of Ψ_L. In this way, we can rewrite the 4-component Dirac field in terms of two 2-component spinors:

$$\Psi_L(x) = \chi_1(x), \qquad \Psi_R(x) = i\sigma_2\chi_2^*(x).$$

Rewrite the Dirac Lagrangian and the action in terms of χ_1 and χ_2 and note the form of the mass term.

4. Show that the above action has a global symmetry. Compute the divergences of the currents

$$J^\mu = \chi^\dagger\bar{\sigma}_\mu\chi, \qquad J^\mu = \chi_1^\dagger\bar{\sigma}_\mu\chi_1 - \chi_2^\dagger\bar{\sigma}_\mu\chi_2,$$

for the theories of parts 1.2 and 1.3, respectively, and relate your results to the symmetries of these theories.

5. Quantize the Majorana theory of parts 1.1 and 1.2. That is, promote $\chi(x)$ to a quantum field satisfying the canonical anticommutation relation

$$\left\{\chi_a(\mathbf{x}), \chi_b^\dagger(\mathbf{y})\right\} = \delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$

construct a Hermitian Hamiltonian, and find a representation of the canonical commutation relations that diagonalizes the Hamiltonian in term of of a set of creation and anhilation operators. (*Hint: Compare $\chi(x)$ to the two upper components of the quantized Dirac field.*)

2 Gordon Identity

Derive the *Gordon identity,*

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma_{\mu\nu}q_\nu}{2m}\right]u(p),$$

where $q = (p' - p)$ and $u(p)$ satisfies $(\gamma_\mu p^\mu - m)\, u(p) = 0$.