THE DIRAC FIELD

This part of the course is based on ref. [1]

Having exhaustively treated the simplest relativistic field equation, we now move on to the second simplest, the Dirac equation. You may already be familiar with the Dirac equation in its original incarnation, that is, a single-particle quantum-mechanical wave equation. In this chapter, our viewpoint will be quite different. First we will rederive the Dirac equation as a *classical* relativistic field equation, with special emphasis on its relativistic invariance. Then in section 5, we will quantize the Dirac field in a manner similar to that used for the Klein-Gordon field.

1. Lorentz Invariance in Wave Equations

First we must address a question that we swept over in our discussion of the Klein-Gordon field: What do we mean when we say that an equation is "relativistically invariant"? A reasonable definition is the following: If ϕ is a field or a collection of fields and $\mathcal{D}$ is some differential operator, then the statement "$\mathcal{D}\phi = 0$ is relativistically invariant" means that if $\phi(x)$ satisfies this equation, and we perform a rotation or boost to a different frame of reference, then the transformed field, in the new frame of reference, satisfies the same equation. Equivalently, we can imagine physically rotating or boosting all particles or fields by a common angle or velocity; again, the equation $\mathcal{D}\phi = 0$ should be true after the transformation. We will adopt this "active" point of view toward transformations in the following analysis.

The Lagrangian formulation of field theory makes it especially easy to discuss Lorentz invariance. An equation of motion is automatically Lorentz invariant by the above definition if it follows from a Lagrangian that is Lorentz *scalar*. This is an immediate consequence of the principle of least action: If boosts leave the Lagrangian unchanged, the boost of an extremum in the action will be another extremum.

As an example, consider the Klein-Gordon theory. We can write an arbitrary Lorentz transformation as

$$x^\mu \;\to\; x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{1.1}$$

for some 4×4 matrix Λ. What happens to the Klein-Gordon field under this transformation? Think of the field ϕ as measuring the local value of some quantity that is distributed through space. If there is an accumulation of this quantity at $x = x_0$, $\phi(x)$ will have a maximum at x_0. If we now transform the original distribution by a boost, the new distribution will have a maximum at $x = \Lambda x_0$. The corresponding transformation of the field is

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x). \tag{1.2}$$

That is, the transformed field, evaluated at the boosted point, gives the same value as the original field evaluated at the point before boosting.

We should check that this transformation leaves the form of the Klein-Gordon Lagrangian unchanged. According to (1.2), the mass term $\frac{1}{2}m^2\phi^2(x)$ is simply shifted to the point $(\Lambda^{-1}x)$. The transformation of $\partial_\mu\phi(x)$ is

$$\partial_\mu\phi(x) \to \partial_\mu\left(\phi(\Lambda^{-1}x)\right) = (\Lambda^{-1})^\mu{}_\nu(\partial_\nu\phi)(\Lambda^{-1}x). \tag{1.3}$$

Since the metric tensor $g^{\mu\nu}$ is Lorentz invariant, the matrices Λ^{-1} obey the identity

$$(\Lambda^{-1})^\rho{}_\nu(\Lambda^{-1})^\sigma{}_\nu g^{\mu\nu} = g^{\rho\sigma}. \tag{1.4}$$

Under this relation, we can compute the transformation law of the kinetic term of the Klein-Gordon Lagrangian:

$$\begin{aligned}
(\partial_\mu\phi(x))^2 &\to g^{\mu\nu}(\partial_\mu\phi'(x))(\partial_\nu\phi'(x)) \\
&= g^{\mu\nu}[(\Lambda^{-1})^\rho{}_\mu\partial_\rho\phi][(\Lambda^{-1})^\sigma{}_\nu\partial_\sigma\phi](\Lambda^{-1}x) \\
&= g^{\rho\sigma}(\partial_\rho\phi)(\partial_\sigma\phi)(\Lambda^{-1}x) \\
&= (\partial_\mu\phi)^2(\Lambda^{-1}x).
\end{aligned} \tag{1.5}$$

Thus, the whole Lagrangian is simply transformed as a scalar:

$$\mathcal{L}(x) \to \mathcal{L}(\Lambda^{-1}x). \tag{1.6}$$

The action, S, formed by integrating $\mathcal{L}$ over spacetime, is Lorentz invariant. A similar calculation shows that the equation of motion is invariant:

$$\begin{aligned}
(\partial^2 + m^2)\phi'(x) &= [(\Lambda^{-1})^\nu{}_\mu\partial_\nu(\Lambda^{-1})^{\sigma\mu}\partial_\sigma + m^2]\phi(\Lambda^{-1}x) \\
&= (g^{\nu\sigma}\partial_\nu\partial_\sigma + m^2]\phi(\Lambda^{-1}x) \\
&= 0.
\end{aligned} \tag{1.7}$$

The transformation law (1.2) used for ϕ is the simplest possible transformation law for a field. It is the only possibility for a field that has just one component. But we know examples of multiple-component fields that transform in more complicated ways. The most familiar case is that of a vector field, such as the 4-current density $j^\mu(x)$ or the vector potential $A^\mu(x)$. In this case, the quantity that is distributed in spacetime also carries an orientation, which must be rotated or boosted. The orientation must be rotated forward as the point of evaluation is changed:

$$\begin{aligned} \text{under 3} - \text{dimensional rotations,} \quad & V^i(x) \to R^{ij} V^j(R^{-1}x); \\ \text{under Lorentz transformations,} \quad & V^\mu(x) \to \Lambda^\mu{}_\nu V^\nu(\Lambda^{-1}x). \end{aligned} \tag{1.8}$$

Tensors of arbitrary rank can be built out of vectors by adding more indices, with correspondingly more factors of Λ in the transformation law. Using such vector and tensor fields we can write a variety of Lorentz-invariant equations, for example, Maxwell's equations,

$$\partial^\mu F_{\mu\nu} = 0 \quad \text{or} \quad \partial^2 A_\nu - \partial_\nu \partial^\mu A_\mu = 0, \tag{1.9}$$

which follow from the Lagrangian

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}(F_{\mu\nu})^2 = -\frac{1}{4}(\partial\mu A_\nu - \partial_\nu A_\mu)^2. \tag{1.10}$$

In general, any equation in which each term has the same set of uncontracted Lorentz indices will naturally be invariant under Lorentz transformations.

This method of tensor notation yields a large class of Lorentz-invariant equations, but it turns out that there are still more. How do we find them? We could try to systematically find all possible transformations laws for a field. Then it would not be hard to write invariant Lagrangians. For simplicity, we will restrict our attention to linear transformations, so that, if Φ_a is an n component multiplet, the Lorentz transformation law is given by an $n \times n$ matrix $M(\Lambda)$:

$$\Phi_a(x) \to M_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x). \tag{1.11}$$

It can be shown that the most general nonlinear transformation laws can be built from these linear transformations, so there is no advantage in considering transformations more general than (1.11). In the following discussion, we will suppress the change in the field argument and write the transformation (1.11) in the form

$$\Phi \to M(\Lambda)\Phi. \tag{1.12}$$

What are the possible allowed forms for the matrices $M(\Lambda)$? The basic restriction on $M(\Lambda)$ is found by imagining two successive transformations, Λ and Λ'. The net result must be a new Lorentz transformation Λ''; that is, the Lorentz transformations form a *group*. This gives a consistency condition that must be satisfied by the matrices $M(\Lambda)$: Under the sequence of two transformations,

$$\Phi \to M(\Lambda')M(\Lambda)\Phi = M(\Lambda'')\Phi, \tag{1.13}$$

for $\Lambda'' = \Lambda'\Lambda$. Thus the correspondence between the matrices M and the transformations Λ must be preserved under multiplication. In mathematical language, we say that the matrices M must form an n-dimensional *representation* of the Lorentz group. So our question now is rephrased in mathematical language: What are the (finite-dimensional) matrix representations of the Lorentz group?

Before answering this question for the Lorentz group, let us consider a simpler group, the rotation group in three dimensions. This group has representations of every dimensionality n, familiar in quantum mechanics as the matrices that rotate the n-component wavefunctions of particles of different spins. The dimensionality is related to the spin quantum number s by $n = 2s + 1$. The most important nontrivial representation is the two-dimensional representation, corresponding to spin 1/2. The matrices of this representation are the 2×2 unitary matrices with determinant 1, which can be expressed as

$$U = e^{-i\theta^i \sigma^i/2}, \tag{1.14}$$

where θ^i are three arbitrary parameters and σ^i are the Pauli sigma matrices.

For any continuous group, the transformations that lie infinitesimally close to the identity define a vector space, called the *Lie algebra* of the group. The basis vectors for this vector space are called the *generators* of the Lie algebra or of the group. For the rotation group, the generators are the angular momentum operators J^i, which satisfy the commutation relations

$$[J^i, J^j] = i\epsilon^{ijk} J^k. \tag{1.15}$$

The finite rotation operations are formed by exponentiating these operators: In quantum mechanics, the operator

$$R = \exp[-i\theta^i J^i] \tag{1.16}$$

gives the rotation by an angle $|\theta|$ about the axis $\hat{\theta}$. The commutation relations of the operators J^i determine the multiplication laws of these rotation operators. Thus, a set of matrices satisfying the commutation relations (1.15) produces, through exponentiation as in (1.16), a representation of the rotation group. In the example given in the previous paragraph, the representation of the angular momentum operators

$$J^i \to \frac{\sigma^i}{2}, \tag{1.17}$$

produces the representation of the rotation group given in eq. (1.14). It is generally true that one can find matrix representations of a continuous group by finding matrix representations of the generators of the group (which must satisfy the proper commutation relations), then exponentiating these infinitesimal transformations.

For our present problem, we need to know the commutation relations of the generators of the group of Lorentz transformations. For the rotation group, one can work out the commutation relations by writing the generators as differential operators; from the expression

$$\mathbf{J} = \mathbf{x} \times \mathbf{p} = \mathbf{x} \times (-i\nabla), \tag{1.18}$$

the angular momentum commutation relations (1.15) follow straightforwardly. The use of the cross product in (1.18) is special to the case of three dimensions. However, we can also write the operators as an antisymmetric tensor,

$$J^{ij} = -i(x^i \nabla^j - x^j \nabla^i), \tag{1.19}$$

so that $J^3 = J^{12}$ and so on. The generalization to four-dimensional Lorentz transformations is now quite general:

$$J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu). \tag{1.20}$$

We will soon see that these six operators generate three boosts and three rotations of the Lorentz group.

To determine the commutation rules of the Lorentz algebra, we can now simply compute the commutators of the diffrerential operators (1.20). The result is

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}). \tag{1.21}$$

Any matrices that are to represent this algebra must obey these same commutation relations.

Just to see that we have this right, let us look at one particular representation (which we will simply pull out of a hat). Consider the 4×4 matrices

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}). \tag{1.22}$$

(Here μ and ν label which of the six matrices we want, while α and β label components of matrices.) You can easily verify that these matrices satisfy the commutation relations (1.21). In fact, they are nothing but the matrices that act on ordinary Lorentz 4-vectors. To see this, parametrize an infinitesimal transformation as follows:

$$V^{\alpha} \;\rightarrow\; \left(\delta^{\alpha}{}_{\beta} - \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta}\right) V^{\beta}, \tag{1.23}$$

where V is a 4-vector and $\omega_{\mu\nu}$, an antisymmetric tensor, gives the infinitesimal angles. For example, consider the case $\omega_{12} = -\omega_{21} = \theta$, with all other components of ω equal to zero. Then eq. (1.23) becomes

$$V \;\rightarrow\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} V, \tag{1.24}$$

which is just an infinitesimal rotation in the xy-plane. You can also verify that setting $\omega_{01} = -\omega_{10} = \beta$ gives

$$V \;\rightarrow\; \begin{pmatrix} 1 & \beta & 0 & 0 \\ \beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} V, \tag{1.25}$$

an infinitesimal boost in the x-direction. The other components of ω generate the remaining boosts and rotations in a similar manner.

2. The Dirac Equation

Now that we have seen one finite-dimensional representation of the Lorentz group, the next logical step would be to develop the formalism for finding all other representations. Although this is not very difficult to do, it is hardly necessary for our purposes, since we are mainly interested in the representation(s) corresponding to spin 1/2.

We can find such a representation using a trick due to Dirac: Suppose that we had a set of four $n \times n$ matrices γ^{μ} satisfying the anticommutation relations

$$\{\gamma^{\mu}, \gamma^{\nu}\} \equiv \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} \times \mathbf{1}_{n\times n} \quad \text{(Dirac algebra)}. \tag{2.1}$$

Then we could immediately write down as n-dimensional representation of the Lorentz algebra. Here it is:

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{2.2}$$

By repeated use of (2.1), it is easy to verify that these matrices satisfy the commutation relations (1.21).

This computation goes through in any dimensionality, with Lorentz or Euclidean metric. In particular, it should work in three-dimensional Euclidean space, and in fact we can simply write

$$\gamma^j \equiv i\sigma^j \quad \text{so that } \{\gamma^i, \gamma^j\} = -2\delta^{ij}, \tag{2.3}$$

where σ^j are the Pauli sigma matrices, and the factor i in the first equation and minus sign in the second are purely conventional. The matrices representing the Lorentz algebra are then

$$S^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma^k, \tag{2.4}$$

which we recognize as the two-dimensional representation of the rotation group.

Now let us find the Dirac matrices γ^μ for four-dimensional Minkowski space. It turns out that these matrices must be at least 4×4. (There is no fourth 2×2 matrix, for example, that anticommutes with the three Pauli sigma matrices.) Further, all 4×4 representations of the Dirac algebra are unitarily equivalent. We thus need only write one explicit realization of the Dirac algebra. One representation, in 2×2 block form, is

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{2.5}$$

This representation is called the *Weyl* or *chiral* representation. We will find it an especially convenient choice, and we will use it exclusively throughout this discussion. (Be careful, however, since many textbooks choose a different representation, in which γ^0 is diagonal. Furthermore, books that use chiral representations often make a different choice of sign conventions.)

In our representation, the boost and rotation generators are

$$S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \tag{2.6}$$

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2}\epsilon^{ijk}\Sigma^k. \tag{2.7}$$

A four component field ψ that transforms under boosts and rotations according to (2.6) and (2.7) is called a *Dirac spinor*. Note that the rotations generator S^{ij} is just the three-dimensional spinor transformation matrix (2.4) replicated twice. The boost generators S^{0i} are not Hermitian, and thus our implementation of boosts is not unitary (this was also true of the vector representation (1.22)). In fact the Lorentz group, being "noncompact", has no faithful, finite-dimensional representations that are unitary. But that does not matter to us, since ψ is not a wavefunction; it is a classical field.

Now that we have the transformation law for ψ, we should look for an appropriate field equation. One possibility is simply the Klein-Gordon equation:

$$(\partial^2 + m^2)\psi = 0. \tag{2.8}$$

This works because the spinor transformation matrices (2.6) and (2.7) operate only in the "internal" space; they go right through the differential operator. But it is possible to write a stronger, first-order equation, which implies (2.8) but contains additional information. To do this we need to know one more property of the γ matrices. With a short computation you can verify that

$$[\gamma^\mu, S^{\rho\sigma}] = (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \gamma^\nu, \tag{2.9}$$

or equivalently,

$$(1 + \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma})\gamma^\mu(1 - \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}) = (1 - \frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma})^\mu{}_\nu \gamma^\nu. \tag{2.10}$$

This equation is just the infinitesimal form of

$$\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu \Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu \gamma^\nu, \tag{2.11}$$

where

$$\Lambda_{\frac{1}{2}} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \tag{2.12}$$

is the spinor representation of the Lorentz transformation Λ (compare (1.23)). Eq. (2.11) says that the gamma matrices are invariant under simultaneous rotations of their vector and spinor indices (just like the σ^i under spatial rotations). In other words, we can "take the vector index μ on γ^μ seriously," and dot γ^μ into ∂_μ to form a Lorentz invariant differential operator.

We are now ready to write down the Dirac equation. Here it is:

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0. \tag{2.13}$$

To show that it is Lorentz invariant, write down the Lorentz-transformed version of the left-hand side and calculate:

$$\begin{aligned}
[i\gamma^\mu\partial_\mu - m]\psi(x) \rightarrow & [i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m]\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}\Lambda_{\frac{1}{2}}^{-1}[i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m]\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}[i\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m]\psi(\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}[i\Lambda^\mu{}_\sigma\gamma^\sigma(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m]\psi(\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}[i\gamma^\nu\partial_\nu - m]\psi(\Lambda^{-1}x) \\
&=0.
\end{aligned} \tag{2.14}$$

To see that the Dirac equation implies the Klein-Gordon equation, act on the left with $(-i\gamma^\mu\partial_\mu - m)$:

$$\begin{aligned}
0 &= (-i\gamma^\mu\partial_\mu - m)(i\gamma^\nu\partial_\nu - m)\psi \\
&= (\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\psi \\
&= (\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu + m^2)\psi \\
&= (\partial^2 + m^2)\psi.
\end{aligned} \tag{2.15}$$

To write down a Lagrangian for the Dirac theory, we must figure out how to multiply two Dirac spinors to form a Lorentz scalar. The obvious guess, $\psi^\dagger\psi$, does not work. Under a Lorentz boost this becomes $\psi^\dagger\Lambda_{\frac{1}{2}}^\dagger\Lambda_{\frac{1}{2}}\psi$; if the boost matrix were unitary, we would have $\Lambda_{\frac{1}{2}}^\dagger = \Lambda_{\frac{1}{2}}^{-1}$ and everything would be fine. But $\Lambda_{\frac{1}{2}}$ is not unitary, because the generators (2.6) are not Hermitian.

The solution is to define

$$\bar{\psi} \equiv \psi^\dagger\gamma^0. \tag{2.16}$$

Under an infinitesimal Lorentz transformation parametrized by $\omega_{\mu\nu}$, we have $\bar{\psi} \rightarrow \psi^\dagger(1 - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger)\gamma^0$. The sum over μ and ν has six distinct nonzero terms. In the rotation terms, where μ and ν are both nonzero, $(S^{\mu\nu})^\dagger = S^{\mu\nu}$ and $S^{\mu\nu}$ commutes with γ^0. In the boost terms, where μ or ν is zero, $(S^{\mu\nu})^\dagger = -S^{\mu\nu}$ but $S^{\mu\nu}$ anticommutes with γ^0. Passing the γ^0 to the left therefore removes the dagger from $S^{\mu\nu}$, yielding the transformation law

$$\bar{\psi} \rightarrow \bar{\psi}\Lambda_{\frac{1}{2}}^{-1}, \tag{2.17}$$

and therefore the quantity $\bar{\psi}\psi$ is a Lorentz scalar. Similarly, you can show (with the aid of (2.11)) that $\bar{\psi}\gamma^\mu\psi$ is a Lorentz vector.

The correct, Lorentz invariant Dirac Lagrangian is therefore

$$\mathcal{L}_{\mathrm{Dirac}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi. \tag{2.18}$$

The Euler-Lagrange equation for $\bar{\psi}$ (or $\psi^\dagger$) immediately yields the Dirac equation in the form (2.13); The Euler-Lagrange equation for ψ gives the same equation, in Hermitian-conjugate form:

$$-i\partial_\mu \bar{\psi}\gamma^\mu - m\bar{\psi} = 0. \tag{2.19}$$

Weyl Spinors

From the block-diagonal form of the operators (2.6) and (2.7), it is apparent that the Dirac representation of the Lorentz group is *reducible*. We can form two 2-dimensional representations by considering each block separately, and writing

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \tag{2.20}$$

The two-component objects ψ_L and ψ_R are called left-handed and right-handed *Weyl Spinors*. You can easily verify that their transformation laws, under infinitesimal rotations θ and boosts β, are

$$\begin{aligned} \psi_L &\to (1 - i\theta \cdot \frac{\sigma}{2} - \beta \cdot \frac{\sigma}{2})\psi_L; \\ \psi_R &\to (1 - i\theta \cdot \frac{\sigma}{2} + \beta \cdot \frac{\sigma}{2})\psi_R. \end{aligned} \tag{2.21}$$

These transformation laws are connected by complex conjugation; using the identity

$$\sigma^2 \sigma^* = -\sigma \sigma^2, \tag{2.22}$$

it is not hard to show that the quantity $\sigma^2 \psi_L^*$ transforms like a right-handed spinor.

In terms of ψ_L and ψ_R, the Dirac equation is

$$(i\gamma^\mu \partial_\mu - m)\psi = \begin{pmatrix} -m & i(\partial_0 + \sigma \cdot \nabla) \\ i(\partial_0 - \sigma \cdot \nabla) & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0. \tag{2.23}$$

The two Lorentz group representations ψ_L and ψ_R are mixed by the mass term in the Dirac equation. But if we set $m = 0$, the equations for ψ_L and ψ_R decouple:

$$\begin{aligned} i(\partial_0 - \sigma \cdot \nabla)\psi_L &= 0; \\ i(\partial_0 + \sigma \cdot \nabla)\psi_R &= 0. \end{aligned} \tag{2.24}$$

These are called the *Weyl equations*; they are especially important when treating neutrinos and the theory of weak interactions.

It is possible to clean up this notation slightly. Define

$$\sigma^{\mu} \equiv (1, \sigma), \quad \bar{\sigma}^{\mu} \equiv (1, -\sigma), \tag{2.25}$$

so that

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}. \tag{2.26}$$

(The bar on $\bar{\sigma}$ has absolutely nothing to do with the bar on $\bar{\psi}$.) Then the Dirac equation can be written

$$\begin{pmatrix} -m & i\sigma \cdot \partial \\ i\bar{\sigma} \cdot \partial & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0, \tag{2.27}$$

and the Weyl equations become

$$i\bar{\sigma} \cdot \partial \psi_L \;=\; 0; \quad i\sigma \cdot \partial \psi_R \;=\; 0. \tag{2.28}$$

3. Free-Particle Solutions of the Dirac Equation

To get some feel for the physics of the Dirac equation, let us now discuss the plane-wave solutions. Since a Dirac field ψ obeys the Klein-Gordon equation, we know immediately that it can be written as a linear combination of plane waves:

$$\psi(x) = u(p) e^{-ip\cdot x}, \quad \text{where } p^2 = m^2. \tag{3.1}$$

For the moment we will concentrate on solutions with positive frequency, that is $p^0 > 0$. The column vector $u(p)$ must obey an additional constraint, found by plugging (3.1) into the Dirac equation:

$$(\gamma^{\mu} p_{\mu} - m) u(p) = 0. \tag{3.2}$$

It is easiest to analyze this equation in the rest frame, where $p = p_0 = (m, \mathbf{0})$; the solution for general p can then be found by boosting with $\Lambda_{\frac{1}{2}}$. In the rest frame, eq. (3.2) becomes

$$(m\gamma^0 - m) u(p_0) = m \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u(p_0) = 0, \tag{3.3}$$

and the solutions are

$$u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \tag{3.4}$$

for any numerical two-component spinor ξ. We conventionally normalize ξ so that $\xi^\dagger\xi = 1$; the factor $\sqrt{m}$ has been inserted for future convenience. We can interpret the spinor ξ by looking at the rotation generator (2.7): ξ transforms under rotations as an ordinary two-component spinor of the rotation group, and therefore determines the spin orientation of the Dirac solution in the usual way. For example, when $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the particle has spin up along the 3-direction.

Notice that after applying the Dirac equation, we are free to choose only two of the four components of $u(p)$. This is just what we want, since a spin 1/2 particle has only two physical states – spin up and spin down. (Of course we are being a bit premature in talking about *particles* and *spin*. We will *prove* that the spin angular momentum of a Dirac particle is $\hbar/2$ when we quantize the theory in section 5; for now, just notice that there are two possible solutions $u(p)$ for any momentum p.)

Now that we have the general form of $u(p)$ in the rest frame, we can obtain $u(p)$ in any other frame by boosting. Consider a boost along the 3-direction. First we should remind ourselves what the boost does to the 4-momentum vector. In infinitesimal form,

$$\begin{pmatrix} E \\ p^3 \end{pmatrix} = \left[1 + \eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \begin{pmatrix} m \\ 0 \end{pmatrix}, \tag{3.5}$$

where η is some infinitesimal parameter. For finite η we must write

$$\begin{aligned} \begin{pmatrix} E \\ p^3 \end{pmatrix} &= \exp\left[\eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \begin{pmatrix} m \\ 0 \end{pmatrix} \\ &= \left[\cosh\eta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sinh\eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \begin{pmatrix} m \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} m\cosh\eta \\ m\sinh\eta \end{pmatrix}. \end{aligned} \tag{3.6}$$

The parameter η is called the *rapidity.* It is the quantity that is additive under successive boosts.

Now apply the same boost to $u(p)$. According to eqs. (2.6) and (2.12),

$$\begin{aligned} u(p) &= \exp\left[-\frac{1}{2}\eta \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right] \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \\ &= \left[\cosh(\frac{1}{2}\eta) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \sinh(\frac{1}{2}\eta) \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right] \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \\ &= \begin{pmatrix} e^{\eta/2}(\frac{1-\sigma^3}{2}) + e^{-\eta/2}(\frac{1+\sigma^3}{2}) & 0 \\ 0 & e^{\eta/2}(\frac{1+\sigma^3}{2}) + e^{-\eta/2}(\frac{1-\sigma^3}{2}) \end{pmatrix} \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \\ &= \begin{pmatrix} \left[\sqrt{E+p^3}(\frac{1-\sigma^3}{2}) + \sqrt{E-p^3}(\frac{1+\sigma^3}{2})\right]\xi \\ \left[\sqrt{E+p^3}(\frac{1+\sigma^3}{2}) + \sqrt{E-p^3}(\frac{1-\sigma^3}{2})\right]\xi \end{pmatrix}. \end{aligned} \tag{3.7}$$

The last line can be simplified to give

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi \\ \sqrt{p \cdot \bar{\sigma}}\xi \end{pmatrix}, \tag{3.8}$$

where it is understood that in taking the square root of a matrix, we take the positive root of each eigenvalue. This expression for $u(p)$ is not only more compact, but is also valid for an arbitrary direction of $\mathbf{p}$. When working with expressions of this form, it is often useful to know the identity

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = p^2 = m^2. \tag{3.9}$$

You can verify directly that (3.8) is a solution of the Dirac equation in the form of (2.27).

In practice it is often convenient to work with specific spinors ξ. A useful choice here would be eigenstates of σ^3. For example, if $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (spin up along the 3-axis), we get

$$u(p) = \begin{pmatrix} \sqrt{E - p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E + p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \rightarrow_{\text{large boost}} \sqrt{2E}\begin{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{pmatrix}, \tag{3.10}$$

while for $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (spin down along the 3-axis) we have

$$u(p) = \begin{pmatrix} \sqrt{E + p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E - p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \rightarrow_{\text{large boost}} \sqrt{2E}\begin{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{pmatrix}. \tag{3.11}$$

In the limit $\eta \to \infty$ the states degenerate into the two-component spinors of a massless particle. (We now see the reason of the factor $\sqrt{m}$ in (3.4); it keeps the spinor expressions finite in the massless limit.)

The solutions (3.10) and (3.11) are eigenstates of the *helicity* operator,

$$h \equiv \hat{p} \cdot \mathbf{S} = \frac{1}{2}\hat{p}_i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \tag{3.12}$$

A particle with $h = +1/2$ is called *right-handed*, while one with $h = -1/2$ is called *left-handed*. The helicity of a massive particle depends on the frame of reference, since one can always boost to a frame in which its momentum is in the opposite direction (but its spin is unchanged). For a massless particle, which travels at the speed of light, one cannot perform such a boost.

The extremely simple form of $u(p)$ for a massless particle in a helicity eigenstate makes the behavior of such a particle easy to understand. One often does a mindless calculation first, then looking at helicity eigenstates in the high-energy limit to understand what has been done.

Incidentally, we are now ready to understand the origin of the notation ψ_L and ψ_R for Weyl spinors. The solutions of the Weyl equations are states of definite helicity, corresponding to left- and right-handed particles, respectively. The Lorentz invariance of the helicity (for a massless particle) is manifest in the notation of Weyl spinors, since ψ_L and ψ_R live in different representations of the Lorentz group.

It is convenient to write the normalization condition for $u(p)$ in a Lorentz invariant way. We saw above that $\psi^\dagger\psi$ is not Lorentz invariant. Similarly,

$$u^\dagger u = (\xi^\dagger\sqrt{p\cdot\sigma}, \xi^\dagger\sqrt{p\cdot\bar{\sigma}}) \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix} = 2E_{\mathbf{p}}\xi^\dagger\xi. \tag{3.13}$$

To make a Lorentz scalar we define

$$\bar{u}(p) = u^\dagger(p)\gamma^0. \tag{3.14}$$

Then by an almost identical calculation,

$$\bar{u}u = 2m\xi^\dagger\xi. \tag{3.15}$$

This will be our normalization condition, once we also require that the two-component spinor ξ be normalized as usual: $\xi^\dagger\xi = 1$. It is also conventional to choose basis spinors ξ^1 and ξ^2 (such as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$) that are orthogonal. For a massless particle, eq. (3.15) is trivial, so we must write the normalization condition in the form of (3.13).

Let us summarize our discussion so far. The general solution of the Dirac equation can be written as a linear combination of plane waves. The positive-frequency waves are of the form

$$\psi(x) = u(p)e^{-ip\cdot x}, \quad p^2 = m^2, \quad p^0 > 0. \tag{3.16}$$

There are two linearly independent solutions for $u(p)$,

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}, \quad s = 1, 2 \tag{3.17}$$

which we normalize according to

$$\bar{u}^r(p)u^s(p) = 2m\delta^{rs} \quad \text{or} \quad u^{r\dagger}(p)u^s(p) = 2E_{\mathbf{p}}\delta^{rs}. \tag{3.18}$$

In exactly the same way, we can find the negative-frequency solutions:

$$\psi(x) = v(p)e^{+ip\cdot x}, \quad p^2 = m^2, \quad p^0 > 0. \tag{3.19}$$

(Note that we have chosen to put the + sign into the exponent rather than having $p^0 < 0$.) There are two linearly independent solutions for $v(p)$,

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s \end{pmatrix}, \quad s = 1, 2 \tag{3.20}$$

where η^s is another basis of two-component spinors. These solutions are normalized according to

$$\bar{v}^r(p)v^s(p) = -2m\delta^{rs} \quad \text{or} \quad v^{r\dagger}(p)v^s(p) = +2E_{\mathbf{p}}\delta^{rs}. \tag{3.21}$$

The u's and v's are also orthogonal to each other:

$$\bar{u}^r(p)v^s(p) = \bar{v}^r(p)u^s(p) = 0. \tag{3.22}$$

Be careful, since $u^{r\dagger}(p)v^s(p) \neq 0$ and $v^{r\dagger}(p)u^s(p) \neq 0$. However, note that

$$u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p}) = v^{r\dagger}(-\mathbf{p})u^s(\mathbf{p}) = 0, \tag{3.23}$$

where we have changed the sign of the three momentum in one factor of each spinor product.

Spin Sums

In evaluating Feynman diagrams, we will often wish to sum over the polarization states of a fermion. We can derive the relevant completeness relations with a simple calculation:

$$\begin{aligned} \sum_{s=1,2} u^s(p)\bar{u}^s(p) &= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \left(\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}}, \xi^{s\dagger}\sqrt{p\cdot\sigma}\right) \\ &= \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma} \end{pmatrix} \\ &= \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix}. \end{aligned} \tag{3.24}$$

In the second line we have used

$$\sum_{s=1,2} \xi^s\xi^{s\dagger} = \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.25}$$

Thus we arrive at the desired formula,

$$\sum_s u^s(p)\bar{u}^s(p) = \gamma \cdot p + m. \tag{3.26}$$

Similarly,

$$\sum_s v^s(p)\bar{v}^s(p) = \gamma \cdot p - m. \tag{3.27}$$

The combination $\gamma \cdot p$ occurs so often that Feynman introduced the notation $\not{p} \equiv \gamma^\mu p_\mu$. We will use this notation frequently from now on.

4. Dirac Matrices and Dirac Field Bilinears

We saw in section 2 that the quantity $\bar{\psi}\psi$ is a Lorentz scalar. It is also easy to show that $\bar{\psi}\gamma^\mu\psi$ is a 4-vector – we used this fact in writing down the Dirac Lagrangian (2.18). Now let us ask a more general question: Consider the expression $\bar{\psi}\Gamma\psi$, where Γ is any 4×4 constant matrix. Can we decompose this expression into terms that have definite transformation properties under the Lorentz group? The answer is yes, if we write Γ in terms of the following basis of sixteen 4×4 matrices, defined as antisymmetric combinations of γ-matrices:

$$\begin{aligned}
&1 \quad (1)\\
&\gamma^\mu \quad (4)\\
&\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] \equiv \gamma^{[\mu}\gamma^{\nu]} \equiv -i\sigma^{\mu\nu} \quad (6)\\
&\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} \quad (4)\\
&\gamma^{\mu\nu\rho\sigma} = \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]} \quad (1)
\end{aligned} \tag{4.1}$$

where the number in parenthesis gives the number of matrices of each type (summing up to sixteen). The Lorentz transformation properties of these matrices are easy to determine. For example,

$$\begin{aligned}
\bar{\psi}\gamma^{\mu\nu}\psi &\to (\bar{\psi}\Lambda_{\frac{1}{2}}^{-1})(\frac{1}{2}[\gamma^\mu, \gamma^\nu])(\Lambda_{\frac{1}{2}}\psi)\\
&= \frac{1}{2}\bar{\psi}(\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}\Lambda_{\frac{1}{2}}^{-1}\gamma^\nu\Lambda_{\frac{1}{2}} - \Lambda_{\frac{1}{2}}^{-1}\gamma^\nu\Lambda_{\frac{1}{2}}\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}})\psi\\
&= \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\bar{\psi}\gamma^{\alpha\beta}\psi.
\end{aligned} \tag{4.2}$$

Each set of matrices transforms as an antisymmetric tensor of successively higher rank.

The last two sets of matrices can be simplified by introducing an additional gamma matrix,

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma. \tag{4.3}$$

Then $\gamma^{\mu\nu\rho\sigma} = -i\epsilon^{\mu\nu\rho\sigma}\gamma^5$ and $\gamma^{\mu\nu\rho} = -i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5$. The matrix γ^5 has the following properties, all of which can be verified using (4.3) and the anticommutation relations (2.1):

$$(\gamma^5)^\dagger = \gamma^5; \tag{4.4}$$

$$(\gamma^5)^2 = 1; \tag{4.5}$$

$$\{\gamma^5, \gamma^\mu\} = 0. \tag{4.6}$$

This last property implies that $[\gamma^5, S^{\mu\nu}] = 0$. Thus the Dirac representation must be reducible, since eigenvectors of γ^5 whose eigenvalues are different transform without mixing (this criterion for reducibility is known as Schur's lemma). In our basis,

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.7}$$

in block-diagonal form. So a Dirac spinor with only left- (right-) handed components is an eigenstate of γ^5 with eigenvalues $-1(+1)$, and indeed these spinors do transform without mixing, as we saw explicitly in section 2.

Let us rewrite our table of 4×4 matrices and introduce some standard terminology:

1	scalar	1
γ^μ	vector	4
$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$	tensor	6
$\gamma^\mu\gamma^5$	pseudo-vector	4
γ^5	pseudo-scalar	1

The terms *pseudo-vector* and *pseudo-scalar* arise from the fact that these quantities transform as a vector and a scalar, respectively, under continuous Lorentz transformations, but with an additional sign change under parity transformations (as we will discuss in section 6).

From the vector and pseudo-vector matrices we can form two currents out of the Dirac field bilinears:

$$j^{\mu}(x) = \bar{\psi}(x)\gamma^{\mu}\psi(x); \quad j^{\mu 5}(x) = \bar{\psi}(x)\gamma^{\mu}\gamma^{5}\psi(x). \tag{4.8}$$

Let us compare the divergences of these currents, assuming that ψ staisfies the Dirac equation:

$$\begin{aligned} \partial_{\mu} j^{\mu} &= (\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi + \bar{\psi}\gamma^{\mu}\partial_{\mu}\psi \\ &= (im\bar{\psi})\psi + \bar{\psi}(-im\psi) \\ &= 0. \end{aligned} \tag{4.9}$$

Thus j^{μ} is always conserved if $\psi(x)$ satisfies the Dirac equation. When we couple the Dirac field to the electromagnetic field, j^{μ} will become the electric current density. Similarly, one can compute

$$\partial_{\mu} j^{\mu 5} = 2im\bar{\psi}\gamma^{5}\psi. \tag{4.10}$$

If $m = 0$, this current (often called the *axial vector current*) is also conserved. It is then useful to form the linear combinations

$$j^{\mu}_{L} = \bar{\psi}\gamma^{\mu}\left(\frac{1-\gamma^{5}}{2}\right)\psi, \quad j^{\mu}_{R} = \bar{\psi}\gamma^{\mu}\left(\frac{1+\gamma^{5}}{2}\right)\psi. \tag{4.11}$$

When $m = 0$, these are the electric current densities of left-handed and right-handed particles, respectively, and are separately conserved.

The two currents $j^{\mu}(x)$ and $j^{\mu 5}(x)$ are the Noether currents corresponding to the two transformations

$$\psi(x) \to e^{i\alpha}\psi(x) \quad \text{and} \quad \psi(x) \to e^{i\alpha\gamma^{5}}\psi(x). \tag{4.12}$$

The first of these is a symmetry of the Dirac Lagrangian (2.18). The second, called a *chiral transformation*, is a symmetry of the derivative term in $\mathcal{L}$ but not the mass term; thus, Noether's theorem confirms that the axial vector current is conserved only if $m = 0$.

Products of Dirac bilinears obey interchange relations, known as *Fierz identities*. We will discuss only the simplest of these. This simplest identity is most easily written in terms of two-component Weyl spinors introduced in eq. (2.20). The core of the relation is the identity for 2×2 matrices σ^{μ} defined in eq. (2.25):

$$(\sigma^{\mu})_{\alpha\beta}(\sigma_{\mu})_{\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}. \tag{4.13}$$

One can understand this relation by noting that the indices α, γ transform in the Lorentz representation of ψ_L, while β, δ transform in the separate representation of ψ_R, and the whole quantity must be a Lorentz invariant. Alternatively, one can just verify the 16 components of (4.13) explicitly.

By sandwiching identity (4.13) between the right-handed portions (i.e., lower half) of Dirac spinors u_1, u_2, u_3, u_4, we find the identity

$$\begin{aligned}(\bar{u}_{1R}\sigma^{\mu}u_{2R})(\bar{u}_{3R}\sigma_{\mu}u_{4R}) &= 2\epsilon_{\alpha\gamma}\bar{u}_{1R\alpha}\bar{u}_{3R\gamma}\epsilon_{\beta\delta}u_{2R\beta}u_{4R\delta} \\ &= -(\bar{u}_{1R}\sigma^{\mu}u_{4R})(\bar{u}_{3R}\sigma_{\mu}u_{2R}).\end{aligned} \tag{4.14}$$

This nontrivial relation says that the product of bilinears in (4.14) is antisymmetric under the interchange of labels 2 and 4, and also under the interchange of 1 and 3. Identity (4.13) also holds for $\bar{\sigma}^{\mu}$, and so we also find

$$(\bar{u}_{1L}\bar{\sigma}^{\mu}u_{2L})(\bar{u}_{3L}\bar{\sigma}_{\mu}u_{4L}) = -(\bar{u}_{1L}\bar{\sigma}^{\mu}u_{4L})(\bar{u}_{3L}\bar{\sigma}_{\mu}u_{2L}). \tag{4.15}$$

It is sometimes useful to combine the Fierz identity (4.14) with the identity linking σ^{μ} and $\bar{\sigma}^{\mu}$:

$$\epsilon_{\alpha\beta}(\sigma^{\mu})_{\beta\gamma} = (\bar{\sigma}^{\mu T})_{\alpha\beta}\epsilon_{\beta\gamma}. \tag{4.16}$$

This relation is also straightforward to verify explicitly. By the use of (4.16), (4.15), and the relation

$$\bar{\sigma}^{\mu}\sigma_{\mu} = 4, \tag{4.17}$$

we can, for example, simplify horrible products of bilinears such as

$$\begin{aligned}(\bar{u}_{1L}\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\lambda}u_{2L})(\bar{u}_{3L}\bar{\sigma}_{\mu}\sigma_{\nu}\bar{\sigma}_{\lambda}u_{4L}) &= 2\epsilon_{\alpha\gamma}\bar{u}_{1L\alpha}\bar{u}_{3L\gamma}\epsilon_{\beta\delta}(\sigma^{\nu}\bar{\sigma}^{\lambda}u_{2R})_{\beta}(\sigma_{\nu}\bar{\sigma}_{\lambda}u_{4R})_{\delta} \\ &= 2\epsilon_{\alpha\gamma}\bar{u}_{1L\alpha}\bar{u}_{3L\gamma}\epsilon_{\beta\delta}u_{2R\beta}(\sigma^{\lambda}\bar{\sigma}^{\nu}\sigma_{\nu}\bar{\sigma}_{\lambda}u_{4R})_{\delta} \\ &= 2\cdot(4)^2\cdot\epsilon_{\alpha\gamma}\bar{u}_{1L\alpha}\bar{u}_{3L\gamma}\epsilon_{\beta\delta}u_{2R\beta}u_{4R\delta} \\ &= (\bar{u}_{1L}\bar{\sigma}^{\mu}u_{2L})(\bar{u}_{3L}\bar{\sigma}_{\mu}u_{4L}).\end{aligned} \tag{4.18}$$

There are also Fierz rearrangement identities for 4-component Dirac spinors and 4×4 Dirac matrices.

5. Quantization of the Dirac Field

We are now ready to construct the quantum theory of the free Dirac field. From the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi, \tag{5.1}$$

we see that the canonical momentum conjugate to ψ is $i\psi^\dagger$, and thus the Hamiltonian is

$$H = \int d^3x \bar{\psi}(-i\gamma \cdot \nabla + m)\psi = \int d^3x \psi^\dagger[-i\gamma^0\gamma \cdot \nabla + m\gamma^0]\psi. \tag{5.2}$$

If we define $\alpha = \gamma^0\gamma$, $\beta = \gamma^0$, you may recognize the quantity in brackets as the Dirac Hamiltonian of one-particle quantum mechanics:

$$h_D = -i\alpha \cdot \nabla + m\beta. \tag{5.3}$$

How Not to Quantize the Dirac Field:
A Lesson in Spin and Statistics

To quantize the Dirac field in analogy with the Klein-Gordon field we would impose the canonical commutation relations

$$[\psi_a(\mathbf{x}), \psi_b^\dagger(\mathbf{y})] = \delta^{(3)}(\mathbf{x} - \mathbf{y})\delta_{ab}, \qquad \text{(equal times)} \tag{5.4}$$

where a and b denote the spinor components of ψ. This already looks peculiar: If $\psi(x)$ were real-valued, the left-hand side would be antisymmetric under $\mathbf{x} \leftrightarrow \mathbf{y}$, while the right-hand side is symmetric. But ψ is complex, so we do not have a contradiction yet. In fact, we will soon find that much worse problems arise when we impose commutation relations on the Dirac field. But it is instructive to see how far we can get, in order to better understand the relation between spin and statistics. So let us press on; just remember that the next few pages will eventually turn out to be a blind alley.

Our first task is to find a representation of the commutation relations in terms of creation and annihilation operators that diagonalizes H. From the form of the Hamiltonian (5.2), it will clearly be helpful to expand $\psi(x)$ in a basis of eigenfunctions of h_D. We know these eigenfunctions already from our calculations in section 3. There we found that

$$[i\gamma^0\partial_0 + i\gamma \cdot \nabla - m]u^s(p)e^{-ip\cdot x} = 0, \tag{5.5}$$

so $u^s(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}$ are eigenfunctions of h_D with eigenvalues $E_\mathbf{p}$. Similarly, the functions $v^s(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}$ (or equivalently $v^s(-\mathbf{p})e^{+i\mathbf{p}\cdot\mathbf{x}}$) are eigenfunctions of h_D with eigenvalues $-E_\mathbf{p}$. These form a complete set of eigenfunctions, since for any $\mathbf{p}$ there are two u's and two v's, giving us four eigenvectors of the 4×4 matrix h_D.

Expanding ψ in this basis, we obtain

$$\psi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} e^{i\mathbf{p}\cdot\mathbf{x}} \sum_{s=1,2} \left(a_\mathbf{p}^s u^s(\mathbf{p}) + b_{-\mathbf{p}}^s v^s(-\mathbf{p}) \right), \tag{5.6}$$

where $a^s_{\mathbf{p}}$ and $b^s_{-\mathbf{p}}$ are operator coefficients. (For now we work in the Schrödinger picture, where ψ does not depend on time.) Postulate the commutation relations

$$[a^r_{\mathbf{p}}, a^{s\dagger}_{\mathbf{q}}] = [b^r_{\mathbf{p}}, b^{s\dagger}_{\mathbf{q}}] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})\delta^{rs}. \tag{5.7}$$

It is easy then to verify the commutation relations (5.4) for ψ and $\psi^\dagger$:

$$\begin{aligned}
[\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})] &= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e^{i(\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y})} \\
&\quad \times \sum_{r,s} \Big([a^r_{\mathbf{p}}, a^{s\dagger}_{\mathbf{q}}] u^r(\mathbf{p}) \bar{u}^s(\mathbf{q}) + [b^r_{-\mathbf{p}}, b^{s\dagger}_{-\mathbf{q}}] v^r(-\mathbf{p}) \bar{v}^s(-\mathbf{q}) \Big) \gamma^0 \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \\
&\quad \times \big[(\gamma^0 E_{\mathbf{p}} - \boldsymbol{\gamma}\cdot\mathbf{p} + m) + (\gamma^0 E_{\mathbf{p}} + \boldsymbol{\gamma}\cdot\mathbf{p} - m) \big] \gamma^0 \\
&= \delta^{(3)}(\mathbf{x} - \mathbf{y}) \times \mathbf{1}_{4\times 4}.
\end{aligned} \tag{5.8}$$

In the second step we have used the spin sum completeness relations (3.26) and (3.27).

We are now ready to write H in terms of the a's and b's. After another short calculation (making use of the orthogonality relations (3.18), (3.21) and (3.23)), we find

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(E_{\mathbf{p}} a^{s\dagger}_{\mathbf{p}} a^s_{\mathbf{p}} - E_{\mathbf{p}} b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} \right). \tag{5.9}$$

Something is terribly wrong with the second term: By creating more and more particles with $b^\dagger$, we can lower the energy indefinitely. (It would not have helped to rename $b \leftrightarrow b^\dagger$, since doing so would ruin the commutation relation (5.8).)

We seem to be in rather deep trouble, but again let's press on, and investigate the causality of this theory. To do this we should compute $[\psi(x), \psi^\dagger(y)]$ (or more conveniently, $[\psi(x), \bar{\psi}(y)]$) at non-equal times and hope to get zero outside the light-cone. First we must switch to the Heisenberg picture and restore the time dependence of ψ and $\bar{\psi}$. Using the relations

$$e^{iHt} a^s_{\mathbf{p}} e^{-iHt} = a^s_{\mathbf{p}} e^{-iE_{\mathbf{p}}t}, \quad e^{iHt} b^s_{\mathbf{p}} e^{-iHt} = b^s_{\mathbf{p}} e^{+iE_{\mathbf{p}}t}, \tag{5.10}$$

we immediately have

$$\begin{aligned}
\psi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + b^s_{\mathbf{p}} v^s(p) e^{ip\cdot x} \right); \\
\bar{\psi}(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^{s\dagger}_{\mathbf{p}} \bar{u}^s(p) e^{ip\cdot x} + b^{s\dagger}_{\mathbf{p}} \bar{v}^s(p) e^{-ip\cdot x} \right).
\end{aligned} \tag{5.11}$$

We can now calculate the general commutator:

$$
\begin{aligned}
[\psi_a(x), \bar{\psi}_b(y)] &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_s \left(u_a^s(p)\bar{u}_b^s(p) e^{-ip\cdot(x-y)} + v_a^s(p)\bar{v}_b^s(p) e^{ip\cdot(x-y)} \right) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left((\not{p} + m)_{ab} e^{-ip\cdot(x-y)} + (\not{p} - m)_{ab} e^{ip\cdot(x-y)} \right) \\
&= (i\not{\partial}_x + m)_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) \\
&= (i\not{\partial}_x + m)_{ab} [\phi(x), \phi(y)].
\end{aligned}
\tag{5.12}
$$

Since $[\phi(x), \phi(y)]$ (the commutator of a real Klein-Gordon field) vanishes outside the light-cone, this quantity does also.

There is something odd, however, about this solution to the causality problem. Let $|0\rangle$ be the state that is annihilated by all the $a_{\mathbf{p}}^s$ and $b_{\mathbf{p}}^s$: $a_{\mathbf{p}}^s |0\rangle = b_{\mathbf{p}}^s |0\rangle = 0$. Then

$$
\begin{aligned}
[\psi_a(x), \bar{\psi}_b(y)] &= \langle 0|[\psi_a(x), \bar{\psi}_b(y)]|0\rangle \\
&= \langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle - \langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle,
\end{aligned}
\tag{5.13}
$$

just as for the Klein-Gordon field. But in the Klein-Gordon case, we got one term of the commutator from each of these two pieces: the propagation of a particle from y to x was canceled by the propagation of an antiparticle from x to y outside the light-cone. Here both terms come from the first piece, $\langle 0|\psi(x)\bar{\psi}(y)|0\rangle$, since the second piece is zero. The cancellation is between positive-energy particles and negative-energy particles, both propagating from y to x.

This observation can actually lead us to a resolution of the negative-energy problem. One of the assumptions we made in quantizing the Dirac theory must have been incorrect. Let us therefore forget about the postulated relations (5.4) and (5.7), and see whether we can find a way for positive energy particles to propagate in both directions. We will also have to drop our definition of the vacuum $|0\rangle$ as the state that is annihilated by all $a_{\mathbf{p}}^s$ and $b_{\mathbf{p}}^s$. We will, however, retain the expression (5.11) for $\psi(x)$ and $\bar{\psi}(x)$ as Heisenberg operators, since if $\psi(x)$ and $\bar{\psi}(x)$ solve the Dirac equation, they must be decomposable into such plane-wave solutions.

First consider the propagation amplitude $\langle 0|\psi(x)\bar{\psi}(y)|0\rangle$, which is to represent a positive-energy particle propagating from y to x. In this case we want the (Heisenberg) state $\bar{\psi}(y)|0\rangle$ to be made up of only positive-energy or negative-frequency components (since a Heisenberg state $\Psi_H = e^{+iHt}\Psi_S$). Thus only the $a_{\mathbf{p}}^{s\dagger}$ term of $\bar{\psi}(y)$ can contribute, which means that $b^{s\dagger}_{\mathbf{p}}$ must annihilate the vacuum. Similarly, $\langle 0|\,\psi(x)$ can contain only positive-frequency components. Thus we have

$$\begin{aligned}\langle 0|\psi(x)\bar{\psi}(y)|0\rangle &= \langle 0|\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_r a^r_{\mathbf{p}} u^r(p)e^{-ip\cdot x} \\ &\qquad \times \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}\sum_s a^{s\dagger}_{\mathbf{q}}\bar{u}^s(q)e^{iq\cdot y}\,|0\rangle\,.\end{aligned} \tag{5.14}$$

We can say something about the matrix element $\langle 0|a^r_{\mathbf{p}}a^{s\dagger}_{\mathbf{q}}|0\rangle$ even without knowing how to interchange $a^r_{\mathbf{p}}$ and $a^{s\dagger}_{\mathbf{q}}$, by using translational and rotational invariance. If the ground state $|0\rangle$ is to be invariant under translations, we must have $|0\rangle = e^{i\mathbf{P}\cdot\mathbf{x}}\,|0\rangle$. Furthermore, since $a^{s\dagger}_{\mathbf{q}}$ creates momentum $\mathbf{q}$, we can use

$$e^{-i\mathbf{P}\cdot\mathbf{x}}a_{\mathbf{p}}e^{i\mathbf{P}\cdot\mathbf{x}} = a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}, \quad e^{-i\mathbf{P}\cdot\mathbf{x}}a^\dagger_{\mathbf{p}}e^{i\mathbf{P}\cdot\mathbf{x}} = a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}, \tag{5.15}$$

to compute

$$\begin{aligned}\langle 0|a^r_{\mathbf{p}}a^{s\dagger}_{\mathbf{q}}|0\rangle &= \langle 0|a^r_{\mathbf{p}}a^{s\dagger}_{\mathbf{q}}e^{i\mathbf{P}\cdot\mathbf{x}}|0\rangle \\ &= e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\langle 0|e^{i\mathbf{P}\cdot\mathbf{x}}a^r_{\mathbf{p}}a^{s\dagger}_{\mathbf{q}}|0\rangle \\ &= e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\langle 0|a^r_{\mathbf{p}}a^{s\dagger}_{\mathbf{q}}|0\rangle.\end{aligned} \tag{5.16}$$

This says that if $\langle 0|a^r_{\mathbf{p}}a^{s\dagger}_{\mathbf{q}}|0\rangle$ is to be nonzero, $\mathbf{p}$ must equal $\mathbf{q}$. Similarly, it can be shown that rotational invariance of $|0\rangle$ implies $r = s$. (This should be intuitively clear, and can be checked after we discuss the angular momentum operator later in this chapter.) From these considerations we conclude that the matrix element can be written

$$\langle 0|a^r_{\mathbf{p}}a^{s\dagger}_{\mathbf{q}}|0\rangle = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs}A(\mathbf{p}), \tag{5.17}$$

where $A(\mathbf{p})$ is so far undetermined. Note, however, that if the norm of a state is always positive (as it should be in self-respecting Hilbert space), $A(\mathbf{p})$ must be greater than zero. We can now go back to (5.14), and write

$$\begin{aligned}\langle 0|\psi(x)\bar{\psi}(y)|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\sum_s u^s(p)\bar{u}^s(p)A(\mathbf{p})e^{-ip\cdot(x-y)} \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(\not{p}+m)A(\mathbf{p})e^{-ip\cdot(x-y)}.\end{aligned} \tag{5.18}$$

This expression is properly invariant under boosts only if $A(\mathbf{p})$ is a Lorentz scalar, i.e., $A(\mathbf{p}) = A(p^2)$. Since $p^2 = m^2$, A must be a constant. So finally we obtain

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = (i\not{\partial}_x + m)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)}\cdot A. \tag{5.19}$$

Similarly, in the amplitude $\langle 0|\bar{\psi}(y)\psi(x)|0\rangle$, we want the only contributions to be from the positive-frequency terms of $\bar{\psi}(y)$ and the negative-frequency terms of $\psi(x)$. So $a^s_{\mathbf{p}}$ still annihilates the vacuum, but $b^s_{\mathbf{p}}$ does not. Then by arguments identical to those given above, we have

$$\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle = -(i\not{\partial}_x + m)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{ip\cdot(x-y)}\cdot B, \tag{5.20}$$

where B is another positive constant. The minus sign is important; it comes from the completeness relation (3.27) for $\sum v\bar{v}$ and the sign of x in the exponential factor. It implies that we cannot have $\langle 0|[\psi(x),\bar{\psi}(y)]|0\rangle = 0$ outside the light-cone: the two terms (5.19) and (5.20) would indeed cancel if $A = -B$, but this is impossible since A and B must both be positive.

The solution, however, is now at hand. By setting $A = B = 1$, it is easy to obtain (outside the light-cone)

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = -\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle. \tag{5.21}$$

That is, the spinor fields *anti*commute at spacelike separation. This is enough to preserve causality, since all reasonable observables (such as energy, charge and particle number) are built out of an *even* number of spinor fields; for any such observables $\mathcal{O}_1$ and $\mathcal{O}_2$, we still have $[\mathcal{O}_1(x),\mathcal{O}_2(y)] = 0$ for $(x-y)^2 < 0$.

And remarkably, postulating *anti*commutation relations for the Dirac field solves the negative energy problem. The equal-time anticommutation relations will be

$$\begin{aligned}\{\psi_a(\mathbf{x}),\psi^\dagger_b(\mathbf{y})\} &= \delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{ab};\\ \{\psi_a(\mathbf{x}),\psi_b(\mathbf{y})\} &= \{\psi^\dagger_a(\mathbf{x}),\psi^\dagger_b(\mathbf{y})\} = 0.\end{aligned} \tag{5.22}$$

We can expand $\psi(\mathbf{x})$ in terms of $a^s_{\mathbf{p}}$ and $a^{s\dagger}_{\mathbf{p}}$ as before (eq. (5.6)). The creation and annihilation operators must now obey

$$\{a^r_{\mathbf{p}}, a^{s\dagger}_{\mathbf{q}}\} = \{b^r_{\mathbf{p}}, b^{s\dagger}_{\mathbf{q}}\} = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs} \tag{5.23}$$

(with all other anticommutators equal to zero) in order that (5.22) be satisfied. Another computation gives the Hamiltonian,

$$H = \int \frac{d^3p}{(2\pi)^3}\sum_s (E_{\mathbf{p}}a^{s\dagger}_{\mathbf{p}}a^s_{\mathbf{p}} - E_{\mathbf{p}}b^{s\dagger}_{\mathbf{p}}b^s_{\mathbf{p}}), \tag{5.24}$$

which is the same as before; $b_{\mathbf{p}}^{s\dagger}$ still creates negative energy. However, the relation $\{b_{\mathbf{p}}^r, b_{\mathbf{q}}^{s\dagger}\} = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs}$ is symmetric between $b_{\mathbf{p}}^r$ and $b_{\mathbf{q}}^{s\dagger}$. So let us simply redefine

$$\tilde{b}_{\mathbf{p}}^s \equiv b_{\mathbf{p}}^{s\dagger}; \quad \tilde{b}_{\mathbf{p}}^{s\dagger} \equiv b_{\mathbf{p}}^s. \tag{5.25}$$

These of course obey exactly the same anticommutation relations, but now the second term in the Hamiltonian is

$$-E_{\mathbf{p}} b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s = +E_{\mathbf{p}} \tilde{b}_{\mathbf{p}}^{s\dagger} \tilde{b}_{\mathbf{p}}^s \; - \; (\text{const}). \tag{5.26}$$

If we choose $|0\rangle$ to be the state that is annihilated by $a_{\mathbf{p}}^s$ and $\tilde{b}_{\mathbf{p}}^s$, then all excitations of $|0\rangle$ have positive energy.

What happened? To better understand this trick, let us abandon the field theory for a moment and consider a theory with a single pair of b and $b^\dagger$ operators obeying $\{b, b^\dagger\} = 1$ and $\{b, b\} = \{b^\dagger, b^\dagger\} = 0$. Choose a state $|0\rangle$ such that $b\,|0\rangle = 0$. Then $b^\dagger\,|0\rangle$ is a new state; call it $|1\rangle$. This state satisfies $b\,|1\rangle = |0\rangle$ and $b^\dagger\,|1\rangle = 0$. So b and $b^\dagger$ act on a Hilbert space of only two states, $|0\rangle$ and $|1\rangle$. We might say that $|0\rangle$ represents an "empty" state, and that $b^\dagger$ "fills" the state. But we could equally well call $|1\rangle$ the empty state and say that $b = \tilde{b}^\dagger$ fills it. The two descriptions are completely equivalent, until we specify some observable that allows us to distinguish the states physically. In our case the correct choice is to take the state of lower energy to be the empty one. And it is less confusing to put the dagger on the operator that creates positive energy. This is exactly what we have done.

Note, by the way, that since $(\tilde{b}^\dagger)^2 = 0$, the state cannot be filled twice. More generally, the anticommutation relations imply that any multiparticle state is antisymmetric under the interchange of two particles: $a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger\,|0\rangle = -a_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger\,|0\rangle$. Thus we conclude that if the ladder operatots obey *anticommutation* relations, the corresponding particles obey *Fermi-Dirac* statistics.

We have just shown that in order to insure that the vacuum has only positive-energy excitations, we must quantize the Dirac field with anticommutation relation; under these conditions the particles associated with the Dirac field obey Fermi-Dirac statistics. This conclusion is part of a more general result, first derived by Pauli: Lorentz invariance, positive energies, positive norms, and causality together imply that particles of integer spin obey Bose-Einstein statistics, while particles of half-odd-integer spin obey Fermi-Dirac statistics.

The Quantized Dirac Field

Let us now summarize the results of the quantized Dirac theory in a systematic way. Since the dust has settled, we should clean up our notation: From now on we will write $\tilde{b}_{\mathbf{p}}$ (the operator that lowers the energy of a state) simply as $b_{\mathbf{p}}$, and $\tilde{b}^{s\dagger}_{\mathbf{p}}$ as $b^\dagger_{\mathbf{p}}$. All the expressions we will need in our later work are listed below; corresponding expressions above, where they differ, should be forgotten.

First we write the field operators:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} v^s(p) e^{ip\cdot x} \right); \tag{5.27}$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(b^s_{\mathbf{p}} \bar{v}^s(p) e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}} \bar{u}^s(p) e^{ip\cdot x} \right). \tag{5.28}$$

The creation and annihilation operators obey the anticommutation rules

$$\{a^r_{\mathbf{p}}, a^{s\dagger}_{\mathbf{q}}\} = \{b^r_{\mathbf{p}}, b^{s\dagger}_{\mathbf{q}}\} = (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs}, \tag{5.29}$$

with all other anticommutation relations equal to zero. The equal-time anticommutation relations for ψ and $\psi^\dagger$ are then

$$\begin{aligned} \{\psi_a(\mathbf{x}), \psi^\dagger_b(\mathbf{y})\} &= \delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{ab}; \\ \{\psi_a(\mathbf{x}), \psi_b(\mathbf{y})\} &= \{\psi^\dagger_a(\mathbf{x}), \psi^\dagger_b(\mathbf{y})\} = 0. \end{aligned} \tag{5.30}$$

The vacuum $|0\rangle$ is defined to be the state such that

$$a^s_{\mathbf{p}} |0\rangle = b^s_{\mathbf{p}} |0\rangle = 0. \tag{5.31}$$

The Hamiltonian can be written

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\mathbf{p}} \left(a^{s\dagger}_{\mathbf{p}} a^s_{\mathbf{p}} + b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} \right), \tag{5.32}$$

where we have dropped the infinite constant term that comes from anticommuting $b^s_{\mathbf{p}}$ and $b^{s\dagger}_{\mathbf{p}}$. From this we see that the vacuum is the state of lowest energy, as desired. The momentum operator is

$$\mathbf{P} = \int d^3x \; \psi^\dagger(-i\nabla)\psi = \int \frac{d^3p}{(2\pi)^3} \sum_s \mathbf{p} \left(a^{s\dagger}_{\mathbf{p}} a^s_{\mathbf{p}} + b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} \right). \tag{5.33}$$

Thus both $a^{s\dagger}_{\mathbf{p}}$ and $b^{s\dagger}_{\mathbf{p}}$ create particles with energy $+E_{\mathbf{p}}$ and momentum $\mathbf{p}$. We will refer to the particles created by $a^{s\dagger}_{\mathbf{p}}$ as *fermions* and to those created by $b^{s\dagger}_{\mathbf{p}}$ as *antifermions.*

The one-particle states,

$$|\mathbf{p}, s\rangle = \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^{s\dagger} |0\rangle \tag{5.34}$$

are defined so that their inner product

$$\langle \mathbf{p}, r|\mathbf{q}, s\rangle = 2E_{\mathbf{p}}(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})\delta^{rs} \tag{5.35}$$

is Lorentz invariant. This implies that the operator $U(\Lambda)$ that implements Lorentz transformations on the states of the Hilbert space is unitary, even though for boosts, $\Lambda_{\frac{1}{2}}$ is not unitary.

It will be reassuring to do a consistency check, to see that $U(\Lambda)$ implements the right transformation on $\psi(x)$. So calculate

$$U\psi(x)U^{-1} = U \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a_{\mathbf{p}}^s u^s(p) e^{-ip\cdot x} + b_{\mathbf{p}}^{s\dagger} v^s(p) e^{ip\cdot x} \right) U^{-1}. \tag{5.36}$$

We can concentrate on the first term; the second is completely analogous. Eq. (5.34) implies that $a_{\mathbf{p}}^s$ transforms according to

$$U(\Lambda) a_{\mathbf{p}}^s U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda p}}{E_p}} a_{\Lambda p}^s, \tag{5.37}$$

assuming that the axis of spin quantization is parallel to the boost or rotation axis. To use this relation to evaluate (5.36), rewrite the integral as

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} a_{\mathbf{p}}^s = \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_{\mathbf{p}}} \cdot \sqrt{E_{\mathbf{p}}} a_{\mathbf{p}}^s. \tag{5.38}$$

The second term is transformed in a simple way by U, and the first is a Lorentz invariant integral. Thus, if we apply (5.37) and make the substitution $\tilde{p} = \Lambda p$, eq. (5.36) becomes

$$U(\Lambda)\psi(x)U^{-1}(\Lambda) = \int \frac{d^3\tilde{p}}{(2\pi)^3} \frac{1}{2E_{\tilde{p}}} u^s(\Lambda^{-1}\tilde{p}) \sqrt{E_{\tilde{p}}} a_{\tilde{p}}^s e^{-i\tilde{p}\cdot\Lambda x} + \cdots. \tag{5.39}$$

But $u^s(\Lambda^{-1}\tilde{p}) = \Lambda_{\frac{1}{2}}^{-1} u^s(\tilde{p})$, so indeed we have

$$U(\Lambda)\psi(x)U^{-1}(\Lambda) = \int \frac{d^3\tilde{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\tilde{p}}}} \Lambda_{\frac{1}{2}}^{-1} u^s(\tilde{p}) a_{\tilde{p}}^s e^{-i\tilde{p}\cdot\Lambda x} + \cdots = \Lambda_{\frac{1}{2}}^{-1} \psi(\Lambda x). \tag{5.40}$$

This result says that the transformed field creates and destroys particles at the point Λx, as it must. Note, however, that this transformation appears to be in the wrong direction compared to eq. (1.2), where the transformed field ϕ was evaluated at $\Lambda^{-1}x$. The difference is that in section 1 we imagined that we transformed a pre-existing field distribution that was measured by $\phi(x)$. Here, we are transforming the action of $\phi(x)$ in creating or destroying particles. The two ways of implementing the Lorentz transformation work in opposite directions. Notice, though, that the matrix acting on ψ and the transformation of the coordinate x have the correct relative orientation, consistent with eq. (1.11).

Next we should discuss the spin of the Dirac particle. We expect Dirac fermions to have spin 1/2; now we can demonstrate this property from our formalism. We have already shown that the particles created by $a_{\mathbf{p}}^{s\dagger}$ and $b_{\mathbf{p}}^{s\dagger}$ each come in two "spin" states: $s = 1, 2$. But we haven't proved yet that this "spin" has anything to do with angular momentum. To do this, we must write down the angular momentum operator.

Recall that we found the linear momentum operator in section 2 by looking for the conserved quantity associated with translational invariance. We can find the angular momentum operator is a similar way as a consequence of rotational invariance. Under a rotation (or any Lorentz transformation), the Dirac field transforms (in our original convention) according to

$$\psi(x) \to \psi'(x) = \Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x). \tag{5.41}$$

To apply Noether's theorem we must compute the change in the field at a fixed point, that is,

$$\delta\psi = \psi'(x) - \psi(x) = \Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) - \psi(x). \tag{5.42}$$

Consider for definiteness an infinitesimal rotation of coordinates by an angle θ about the z-axis. The parametrization of this transformation is given just below (1.23): $\omega_{12} = -\omega_{21} = \theta$. Using the same parameters in eq. (2.12), we find

$$\Lambda_{\frac{1}{2}} \approx 1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu} = 1 - \frac{i}{2}\theta\Sigma^3. \tag{5.43}$$

We can now compute

$$\begin{aligned}\delta\psi(x) &= (1 - \frac{i}{2}\theta\Sigma^3)\psi(t, x+\theta y, y-\theta x, z) - \psi(x) \\ &= -\theta(x\partial_y - y\partial_x + \frac{i}{2}\Sigma^3)\psi(x) \equiv \theta\Delta\psi.\end{aligned} \tag{5.44}$$

The time component of the conserved Noether current is then

$$j^0 = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)}\Delta\psi = -i\bar{\psi}\gamma^0(x\partial_y - y\partial_x + \frac{i}{2}\Sigma^3)\psi. \tag{5.45}$$

Similar expressions hold for rotations about the y- and z-axes, so the angular momentum operator is

$$\mathbf{J} = \int d^3x \; \psi^\dagger \left(\mathbf{x} \times (-i\nabla) + \frac{1}{2}\mathbf{\Sigma} \right) \psi. \tag{5.46}$$

For nonrelativistic fermions, the first term of (5.46) gives the orbital angular momentum. The second term therefore gives the spin angular momentum. Unfortunately, the division of (5.46) into spin and orbital parts is not so straightforward for relativistic fermions, so it is not simple to write a general expression for this quantity in terms of ladder operators.

To prove that a Dirac particle has spin 1/2, however, it suffices to consider particles at rest. In that case, the orbital term of (5.46) does not contribute, and we can easily write the spin term in terms of ladder operators. It is easiest to use the Schrödinger picture expression (5.6) for $\psi(x)$:

$$\begin{aligned} J_z = \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^6} \frac{1}{\sqrt{2E_\mathbf{p} 2E_{\mathbf{p}'}}} e^{-i\mathbf{p}'\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} \\ \times \sum_{r,r'} \left(a_{\mathbf{p}'}^{r'\dagger} u^{r'\dagger}(\mathbf{p}') + b_{-\mathbf{p}'}^{r'} v^{r'\dagger}(-\mathbf{p}') \right) \frac{\Sigma^3}{2} \left(a_\mathbf{p}^r u^r(\mathbf{p}) + b_{-\mathbf{p}}^{r\dagger} v^r(-\mathbf{p}) \right). \end{aligned} \tag{5.47}$$

We would like to apply this operator to the one-particle zero-momentum state $a_0^{s\dagger} |0\rangle$. This is most easily done using a trick: Since J_z must annihilate the vacuum, $J_z a_0^{s\dagger} |0\rangle = [J_z, a_0^{s\dagger}] |0\rangle$. The only nonzero term in this latter quantity has the structure $[a_\mathbf{p}^{r\dagger} a_\mathbf{p}^{r'}, a_0^{s\dagger}] = (2\pi)^3 \delta^{(3)}(\mathbf{p}) a_0^{r\dagger} \delta^{r's}$; the other three terms in the commutator either vanish or annihilate the vacuum. Thus we find

$$J_z a_0^{s\dagger} |0\rangle = \frac{1}{2m} \sum_r \left(u^{r\dagger}(0) \frac{\Sigma^3}{2} u^s(0) \right) a_0^{r\dagger} |0\rangle = \sum_r \left(\xi^{r\dagger} \frac{\sigma^3}{2} \xi^s \right) a_0^{r\dagger} |0\rangle, \tag{5.48}$$

where we have used the explicit form (3.4) of $u(0)$ to obtain the last expression. The sum over r is accomplished most easily by choosing the spinors ξ^r to be eigenstates of σ^3. We then find that for $\xi^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the one particle state is an eigenstate of J_z with eigenvalue $+1/2$, while for $\xi^s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, it is an eigenstate of J_z with eigenvalue $-1/2$. This result is exactly what we expect for electrons.

An analogous calculation determines the spin of a zero-momentum antifermion. But in this case, since the order of the b and $b^\dagger$ terms in J_z is reversed, we get an extra minus sign from evaluating $[b_\mathbf{p} b_\mathbf{p}^\dagger, b_0^\dagger] = -[b_\mathbf{p}^\dagger b_\mathbf{p}, b_0^\dagger]$. Thus for positrons, the association between the spinors ξ^s and the spin angular momentum is reversed: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponds to spin $-1/2$, while $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ corresponds to spin $+1/2$. The reversal of the sign agrees with the prediction of the Dirac hole theory. From that viewpoint, the positron is the absence of a negative-energy electron. If the missing electron had positive J_z, its absence has negative J_z.

In summary, the angular momentum of zero-momentum fermions is given by

$$J_z a_0^{s\dagger} |0\rangle = \pm \frac{1}{2} a_0^{s\dagger} |0\rangle = \pm \frac{1}{2}, \quad J_z b_0^{s\dagger} |0\rangle = \mp \frac{1}{2} b_0^{s\dagger} |0\rangle , \tag{5.49}$$

where the upper sign is for $\xi^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the lower sign is for $\xi^s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

There is one more important conserved quantity in the Dirac theory. In section 4, we saw that the current $j^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved. The charge associated with this current is

$$Q = \int d^3x \ \psi^\dagger(x)\psi(x) = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s + b_{-\mathbf{p}}^s b_{-\mathbf{p}}^{s\dagger} \right) , \tag{5.50}$$

or, if we ignore another infinite constant,

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s - b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s \right) . \tag{5.51}$$

So $a_{\mathbf{p}}^{s\dagger}$ creates fermions with charge $+1$, while $b_{\mathbf{p}}^{s\dagger}$ creates antifermions with charge -1. When we couple the Dirac field to the electromagnetic field, we will see that Q is none other than the electric charge (up to a constant factor that depends on which type of particle we wish to describe; e.g., for electrons, the electric charge is Qe).

In Quantum Electrodynamics we will use the spinor field ψ to describe electrons and positrons. The particles created by $a_{\mathbf{p}}^{s\dagger}$ are electrons; they have energy $E_{\mathbf{p}}$, momentum $\mathbf{p}$, spin 1/2 with polarization appropriate to ξ^s, and charge $+1$ (in units of e). The particles created by $b_{\mathbf{p}}^{s\dagger}$ are positrons; they have energy $E_{\mathbf{p}}$, momentum $\mathbf{p}$, spin 1/2 with polarization appropriate to ξ^s, and charge -1. The state $\psi_\alpha(x)\,|0\rangle$ contains a positron at position x, whose polarization corresponds to the spinor component chosen. Similarly, $\bar{\psi}_\alpha(x)\,|0\rangle$ is a state of one electron at position x.

The Dirac Propagator

Calculating propagation amplitudes for the Dirac field is by now a straightforward exercise:

$$\begin{aligned}\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\sum_s u_a^s(p)\bar{u}_b^s(p)e^{-ip\cdot(x-y)} \\ &= (i\not\partial_x+m)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)},\end{aligned} \tag{5.52}$$

$$\begin{aligned}\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\sum_s v_a^s(p)\bar{v}_b^s(p)e^{-ip\cdot(y-x)} \\ &= -(i\not\partial_x+m)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(y-x)}.\end{aligned} \tag{5.53}$$

Just as we did for the Klein-Gordon equation, we can construct Green's functions for the Dirac equation obeying various boundary conditions. For example, the retarded Green's function is

$$S_R^{ab}(x-y) \equiv \theta(x^0-y^0)\langle 0|\{\psi_a(x),\bar{\psi}_b(y)\}|0\rangle. \tag{5.54}$$

It is easy to verify that

$$S_R(x-y) = (i\not\partial_x+m)D_R(x-y), \tag{5.55}$$

since on the right-hand side the term involving $\partial_0\theta(x^0-y^0)$ vanishes. Using (5.55) and the fact that $\not\partial\not\partial = \partial^2$, we see that S_R is a Green's function of the Dirac operator:

$$(i\not\partial_x-m)S_R(x-y) = i\delta^{(4)}(x-y)\cdot\mathbf{1}_{4\times 4}. \tag{5.56}$$

The Green's function of the Dirac operator can also be found by Fourier transformations. Expanding $S_R(x-y)$ as a Fourier integral and acting on both sides with $(i\not\partial_x-m)$, we find

$$i\delta^{(4)}(x-y) = \int\frac{d^4p}{(2\pi)^4}(\not p-m)e^{-ip\cdot(x-y)}\tilde{S}_R(p), \tag{5.57}$$

and hence

$$\tilde{S}_R(p) = \frac{i}{\not p-m} = \frac{i(\not p+m)}{p^2-m^2}. \tag{5.58}$$

To obtain the retarded Green's function, we must evaluate the p^0 integral in (5.58) along the contour that we used for similar purposes when discussing the Klein-Gordon field. For $x^0>y^0$ we close the contour below, picking up both poles to obtain the sum of (5.52) and (5.53). For $x^0<y^0$ we close the contour above and get zero.

The Green's function with Feynman boundary conditions is defined by a contour similar to the Klein-Gordon field:

$$\begin{aligned} S_F(x-y) &= \int \frac{d^4p}{(2\pi)^4}\frac{i(\not p+m)}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)} \\ &= \begin{cases} \langle 0|\psi(x)\bar\psi(y)|0\rangle & \text{for } x^0>y^0 \text{ (close contour below)} \\ -\langle 0|\bar\psi(y)\psi(x)|0\rangle & \text{for } x^0<y^0 \text{ (close contour above)} \end{cases} \\ &\equiv \langle 0|T\psi(x)\bar\psi(y)|0\rangle, \end{aligned} \tag{5.59}$$

where we have chosen to define the time-ordered product of spinor fields with an additional minus sign when the operators are interchanged. This minus sign is extremely important in the quantum field theory of fermions.

As with the Klein-Gordon theory, the expression (5.59) for the Feynman propagator is the most useful result of this chapter. When we do perturbative calculations with Feynman diagrams, we will associate the factor $\tilde S_F(p)$ with each internal fermion line.

6. Discrete Symmetries of the Dirac Theory

In the last section we discussed the implementation of continuous Lorentz transformations on the Hilbert space of the Dirac theory. We found that for each transformation Λ there was a unitary operator $U(\Lambda)$, which induced the correct transformation on the fields:

$$U(\Lambda)\psi(x)U^{-1}(\Lambda)=\Lambda_{\frac{1}{2}}^{-1}\psi(\Lambda x). \tag{6.1}$$

In this section we will discuss the analogous operators that implement various discrete symmetries on the Dirac field.

In addition to continuous Lorentz transformations, there are two other spacetime operations that are potential symmetries of the Lagrangian: *parity* and *time reversal*. Parity, denoted by P, sends $(t,\mathbf{x})\to(t,-\mathbf{x})$, reversing the handedness of space. Time reversal, denoted by T, sends $(t,\mathbf{x})\to(-t,\mathbf{x})$, interchanging the forward and backward light-cones. Neither of these operations can be achieved by a continuous Lorentz transformation starting from the identity. Both, however, preserve the Minkowski interval $x^2=t^2-\mathbf{x}^2$. In standard terminology, the continuous Lorentz transformations are referred to as the proper orthochronous Lorentz group, $\mathbf{L}_+^\uparrow$. Then the full Lorentz group breaks up into four disconnected subsets, as shown below.

$$\begin{array}{ccccc} \mathbf{L}_+^\uparrow & (P) & \mathbf{L}_-^\uparrow = P\mathbf{L}_+^\uparrow & \text{“orthochronous”} & \\ (T) & & (T) & & \\ \mathbf{L}_+^\downarrow = T\mathbf{L}_+^\uparrow & (P) & \mathbf{L}_-^\downarrow = PT\mathbf{L}_+^\uparrow & \text{“nonorthochronous”} & \\ \text{“proper”} & & \text{“improper”} & & \end{array} \tag{6.2}$$

At the same time that we discuss P and T, it will be convenient to discuss a third (non-spacetime) discrete operation: *charge conjugation*, denoted by C. Under this operation, particles and antiparticles are interchanged.

Although any relativistic field theory must be invariant under $\mathbf{L}_+^\uparrow$, it need not be invariant under P, T, or C. What is the status of these symmetry operations in the real world? From experiment, we know that three of the forces of Nature – the gravitational, electromagnetic and strong interactions – are symmetric with respect to P, C, and T. The weak interactions violate C and P separately, but preserve CP and T. But certain rare processes (all so far observed involve neutral K mesons) also show CP and T violation. All observations indicate that the combination CPT is a perfect symmetry of Nature.

The currently accepted theoretical model of the weak interactions is the Glashow-Weinberg-Salam gauge theory. This theory violates C and P in the strongest possible way. It is actually a surprise (though not quite an accident) that C and P happen to be quite good symmetries in the most readily observed processes. On the other hand, no one knows a really beautiful theory that violates CP. In the current theory, when there are three (or more) fermion generations, there is a room for a parameter that, if nonzero, causes CP violation. But the value of this parameter is no better understood than the value of the electron mass; the physical origin of CP violation remains a mystery.

Parity

With this introduction, let us discuss the action of P, T and C on Dirac particles and fields. First consider parity. The operator P should reverse the momentum of a particle without flipping its spin. Mathematically, this means that P should be implemented by a unitary operator (properly called $U(P)$, but we'll just call it P) which, for example, transforms the state $a^s_{\mathbf{p}}\left|0\right\rangle$ into $a^s_{-\mathbf{p}}\left|0\right\rangle$. In other words, we want

$$Pa^s_{\mathbf{p}}P = \eta_a a^s_{-\mathbf{p}}, \qquad Pb^s_{\mathbf{p}}P = \eta_b b^s_{-\mathbf{p}}, \tag{6.3}$$

where η_a and η_b are possible phases. These phases are restricted by the condition that two applications of the parity operator should return observables to their original values. Since observables are built from an even number of fermion operators, this requires $\eta_a^2, \eta_b^2 = \pm 1$.

Just as a continuous Lorentz transformation is implemented on the Dirac field as the 4×4 constant matrix $\Lambda_{\frac{1}{2}}$, the parity transformation should also be represented by a 4×4

constant matrix. To find this matrix, and to determine η_a and η_b, we compute the action of P os $\psi(x)$. Using (6.3), we have

$$P\psi(x)P = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(\eta_a a^s_{-\mathbf{p}} u^s(p) e^{-ip\cdot x} + \eta_b^* b^{s\dagger}_{-\mathbf{p}} v^s(p) e^{ip\cdot x} \right). \tag{6.4}$$

Now change variables to $\tilde{p} = (p^0, -\mathbf{p})$. Note that $p \cdot x = \tilde{p} \cdot (t, -\mathbf{x})$. Also $\tilde{p} \cdot \sigma = p \cdot \bar{\sigma}$ and $\tilde{p} \cdot \bar{\sigma} = p \cdot \sigma$. This allows us to write

$$\begin{aligned} u(p) &= \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix} = \begin{pmatrix} \sqrt{\tilde{p}\cdot\bar{\sigma}}\xi \\ \sqrt{\tilde{p}\cdot\sigma}\xi \end{pmatrix} = \gamma^0 u(\tilde{p}); \\ v(p) &= \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ -\sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix} = \begin{pmatrix} \sqrt{\tilde{p}\cdot\bar{\sigma}}\xi \\ -\sqrt{\tilde{p}\cdot\sigma}\xi \end{pmatrix} = -\gamma^0 v(\tilde{p}). \end{aligned} \tag{6.5}$$

So (6.4) becomes

$$P\psi(x)P = \int \frac{d^3\tilde{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\tilde{p}}}} \sum_s \left(\eta_a a^s_{\tilde{p}} \gamma^0 u^s(\tilde{p}) e^{-i\tilde{p}(t,-\mathbf{x})} - \eta_b^* b^{s\dagger}_{\tilde{p}} \gamma^0 v^s(\tilde{p}) e^{i\tilde{p}(t,-\mathbf{x})} \right). \tag{6.6}$$

This should equal some constant matrix times $\psi(t, -\mathbf{x})$, and indeed it works if we make $\eta_b^* = -\eta_a$. This implies

$$\eta_a \eta_b = -\eta_a \eta_a^* = -1. \tag{6.7}$$

Thus we have the parity transformation of $\psi(x)$ in its final form,

$$P\psi(t,\mathbf{x})P = \eta_a \gamma^0 \psi(t, -\mathbf{x}). \tag{6.8}$$

It will be very important (for example, in writing down Lagrangians) to know how the various Dirac field bilinears transform under parity. Recall that the five bilinears are

$$\bar{\psi}\psi, \quad \bar{\psi}\gamma^\mu\psi, \quad i\bar{\psi}[\gamma^\mu, \gamma^\nu]\psi, \quad \bar{\psi}\gamma^\mu\gamma^5\psi, \quad i\bar{\psi}\gamma^5\psi. \tag{6.9}$$

The factors of i have been chosen to make all these quantities Hermitian, as you can easily verify. (Any new term that we add to the Lagrangian must be real.) First we should compute

$$P\bar{\psi}(t,\mathbf{x})P = P\psi^\dagger(t,\mathbf{x})P\gamma^0 = (P\psi(t,\mathbf{x})P)^\dagger\gamma^0 = \eta_a^* \bar{\psi}(t,-\mathbf{x})\gamma^0. \tag{6.10}$$

Then the scalar bilinear transforms as

$$P\bar{\psi}\psi P = |\eta_a|^2 \bar{\psi}(t,-\mathbf{x})\gamma^0\gamma^0\psi(t,-\mathbf{x}) = +\bar{\psi}\psi(t,-\mathbf{x}), \tag{6.11}$$

while for the vector we obtain

$$P\bar{\psi}\gamma^\mu\psi P = \bar{\psi}\gamma^0\gamma^\mu\gamma^0\psi(t,-\mathbf{x}) = \begin{cases} +\bar{\psi}\gamma^\mu\psi(t,-\mathbf{x}) & \text{for } \mu = 0 \\ -\bar{\psi}\gamma^\mu\psi(t,-\mathbf{x}) & \text{for } \mu = 1,2,3. \end{cases} \tag{6.12}$$

Note that the vector acquires the same minus sign on the spatial components as does the vector x^μ. Similarly, the transformations of the pseudo-scalar and pseudo-vector are

$$Pi\bar{\psi}\gamma^5\psi P = i\bar{\psi}(t,-\mathbf{x})\gamma^0\gamma^5\gamma^0\psi(t,-\mathbf{x}) = -i\bar{\psi}\gamma^5\psi(t,-\mathbf{x}); \tag{6.13}$$

$$P\bar{\psi}\gamma^\mu\gamma^5\psi P = \bar{\psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\psi(t,-\mathbf{x}) = \begin{cases} -\bar{\psi}\gamma^\mu\gamma^5\psi(t,-\mathbf{x}) & \text{for } \mu = 0 \\ +\bar{\psi}\gamma^\mu\gamma^5\psi(t,-\mathbf{x}) & \text{for } \mu = 1,2,3. \end{cases} \tag{6.14}$$

Just as we anticipated in section 4, the "pseudo" signifies an extra minus sign in the parity transformation. Note that the transformation properties of fermion bilinears were independent of η_a, so there would have been no loss of generality in setting $\eta_a = -\eta_b = 1$ from the beginning.

However, the relative minus sign in eq. (6.7) between the parity transformations of a fermion and an antifermion has important consequences. Consider a fermion-antifermion state, $a^{s\dagger}_{\mathbf{p}} b^{s'\dagger}_{\mathbf{q}} |0\rangle$. Applying P, we find $P(a^{s\dagger}_{\mathbf{p}} b^{s'\dagger}_{\mathbf{q}} |0\rangle) = -(a^{s\dagger}_{-\mathbf{p}} b^{s'\dagger}_{-\mathbf{q}} |0\rangle)$. Thus a state containing a fermion-antifermion pair gets an extra (-1) under parity. This information is most useful in the context of bound states, in which the fermion and antifermion momenta are integrated with the Schrödinger wavefunction to produce a system localized in space. If the spatial wavefunction is symmetric under $\mathbf{x} \to -\mathbf{x}$, the state has *odd* parity, while if it is antisymmetric under $\mathbf{x} \to -\mathbf{x}$, the state has *even* parity. The $L = 0$ bound states, for example, have odd parity; the $J = 0$ state transforms as a pseudo-scalar, while the three $J = 1$ states transform as the spatial components of a vector. These properties show up in the selection rules for decays of positronium and quark-antiquark systems.

Time Reversal

Now let us turn to the implementation of time reversal. We would like T to take the form of a unitary operator that sends $a_{\mathbf{p}}$ to $a_{-\mathbf{p}}$ (and similarly for $b_{\mathbf{p}}$) and $\psi(t,\mathbf{x})$ to $\psi(-t,\mathbf{x})$ (times some constant matrix). These properties, however, are extremely difficult to achieve, since we saw above that sending $a_{\mathbf{p}}$ to $a_{-\mathbf{p}}$ instead sends $(t,\mathbf{x})$ to $(t,-\mathbf{x})$ in the expansion of ψ. The difficulty is even more apparent when we impose the constraint that time reversal should be a symmetry of the free Dirac theory, $[T, H] = 0$. Then

$$\begin{aligned} \psi(t, \mathbf{x}) &= e^{iHt}\psi(\mathbf{x})e^{-iHt} \\ \Longrightarrow T\psi(t, \mathbf{x})T &= e^{iHt}[T\psi(\mathbf{x})T]e^{-iHt} \\ \Longrightarrow T\psi(t, \mathbf{x})T\,|0\rangle &= e^{iHt}[T\psi(\mathbf{x})T]\,|0\rangle\,, \end{aligned} \tag{6.15}$$

assuming that $H\,|0\rangle = 0$. The right-hand side is a sum of negative-frequency terms only. But if T is to reverse the time dependence of $\psi(t, \mathbf{x})$, then the left-hand side is (up to a constant matrix) $\psi(-t, \mathbf{x})\,|0\rangle = e^{-iHt}\psi(\mathbf{x})\,|0\rangle$, which is a sum of positive-frequency terms. Thus we have proved that T cannot be implemented as a linear unitary operator.

What can we do? The way out is to retain the unitarity condition $T^\dagger = T^{-1}$, but have T act on c-numbers as well as operators, as follows:

$$T(\mathrm{c-number}) = (\mathrm{c-number})^* T. \tag{6.16}$$

Then even if $[T, H] = 0$, the time dependence of all exponential factors is reversed: $Te^{+iHt} = e^{-iHt}T$. Since all time evolution in quantum mechanics is performed with such exponential factors, this effectively changes the sign of t. Note that the operation of complex conjugation is nonlinear; T is referred to as an *antilinear* or *antiunitary* operator.

In addition to reversing the momentum of a particle, T should also flip the spin. To quantify this, we must find a mathematical operation that flips a spinor ξ.

In the earlier parts of this chapter, we denoted the spin state of a fermion by a label $s = 1, 2$. In the remainder of this section, we will associate s with the physical spin component of the fermion along a specific axis. If this axis has polar coordinates θ, ϕ, the two-component spinors with spin up and spin down along this axis are

$$\xi(\uparrow) = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, \quad \xi(\downarrow) = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}. \tag{6.17}$$

Let $\xi^s = (\xi(\uparrow), \xi(\downarrow))$ for $s = 1, 2$. Also define

$$\xi^{-s} = -i\sigma^2(\xi^s)^*. \tag{6.18}$$

This quantity is the flipped spinor; from the explicit formulae,

$$\xi^{-s} = (\xi(\downarrow), -\xi(\uparrow)). \tag{6.19}$$

The form of the spin reversal relation follows more generally from the identity $\sigma^i\sigma^2 = \sigma^2(-\sigma^{i*})$. This equation implies that, if ξ satisfies $(\mathbf{n}^i\sigma^i)\xi = +\xi$ for some axis $\mathbf{n}$, then

$$(\mathbf{n}^i\sigma^i)(-i\sigma^2\xi^*) = -i\sigma^2(-\mathbf{n}^i\sigma^i)^*\xi^* = i\sigma^2(\xi^*) = -(-i\sigma^2\xi^*). \tag{6.20}$$

Notice that, with this convention for the sign flip, two successive spin flips return a spin to (-1) times the original state.

We now associate the various fermion spin states with these spinors. The electron annihilation operator $a^s_{\mathbf{p}}$ destroys an electron whose spinor $u^s(p)$ contains ξ^s. The positron annihilation operator $b^s_{\mathbf{p}}$ destroys an positron whose spinor $v^s(p)$ contains ξ^{-s}:

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma^*}\xi^{-s} \\ -\sqrt{p\cdot\bar{\sigma}^*}\xi^{-s} \end{pmatrix}. \tag{6.21}$$

As in eq. (6.19), we define

$$a^{-s}_{\mathbf{p}} = (a^2_{\mathbf{p}}, -a^1_{\mathbf{p}}), \quad b^{-s}_{\mathbf{p}} = (b^2_{\mathbf{p}}, -b^1_{\mathbf{p}}). \tag{6.22}$$

We can now work out the relation between the Dirac spinors u and v and their time reversals. Define $\tilde{p} = (p^0, -\mathbf{p})$. This vector satisfies the identity $\sqrt{\tilde{p}\cdot\sigma}\sigma^2 = \sigma^2\sqrt{p\cdot\sigma^*}$; to prove this, expand the square root as in (3.7). For some choice of spin and momentum, associated with the Dirac spinor $u^s(p)$, let $u^{-s}(\tilde{p})$ be the spinor with the reversed momentum and flipped spin. These quantities are related by

$$\begin{aligned} u^{-s}(\tilde{p}) &= \begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}(-i\sigma^2\xi^*) \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}(-i\sigma^2\xi^*) \end{pmatrix} = \begin{pmatrix} -i\sigma^2\sqrt{p\cdot\sigma^*}\xi^* \\ -i\sigma^2\sqrt{p\cdot\bar{\sigma}^*}\xi^* \end{pmatrix} \\ &= -i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}[u^s(p)]^* = -\gamma^1\gamma^3\,[u^s(p)]^*. \end{aligned} \tag{6.23}$$

Since $v^s(p)$ already contains ξ^{-s}, v^{-s} contains $\xi^{-(-s)} = -\xi^s$; this gives an extra factor (-1):

$$v^{-s}(\tilde{p}) = +\gamma^1\gamma^3\,[v^s(p)]^*. \tag{6.24}$$

Using the notation of eq. (6.22), we define the time reversal transformation of fermion annihilation operators as follows:

$$Ta^s_{\mathbf{p}}T = a^{-s}_{-\mathbf{p}}, \quad Tb^s_{\mathbf{p}}T = -b^{-s}_{-\mathbf{p}}. \tag{6.25}$$

The need for the relative minus sign will be clear in a moment; an additional overall phase will have no effect on the rest of our discussion and is omitted for simplicity. Relations (6.25) allow us to compute the action of T on the fermion field $\psi(x)$:

$$\begin{aligned}
T\psi(t,\mathbf{x})T &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s T\left(a^s_{\mathbf{p}}u^s(p)e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}v^s(p)e^{ip\cdot x}\right)T \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s \left(a^{-s}_{-\mathbf{p}}[u^s(p)]^* e^{ip\cdot x} + b^{-s\dagger}_{-\mathbf{p}}[v^s(p)]^* e^{-ip\cdot x}\right) \\
&= (-\gamma^1\gamma^3)\int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\tilde{p}}}}\sum_s \left(a^{-s}_{\tilde{p}}u^{-s}(\tilde{p})e^{i\tilde{p}(t,-\mathbf{x})} + b^{-s\dagger}_{\tilde{p}}v^{-s}(\tilde{p})e^{-i\tilde{p}(t,-\mathbf{x})}\right) \\
&= (-\gamma^1\gamma^3)\psi(-t,\mathbf{x}).
\end{aligned} \tag{6.26}$$

In the last step we used $\tilde{p}\cdot(t,-\mathbf{x}) = -\tilde{p}\cdot(-t,\mathbf{x})$. Just as for parity, the relative minus sign in the transformations of particle and antiparticle operators allowed us to derive a simple transformation law for the fermion field $\psi(x)$.

Now we can check the action of T on various bilinears. First we need

$$T\bar{\psi}T = (T\psi T)^\dagger(\gamma^0)^* = \psi^\dagger(-t,\mathbf{x})[-\gamma^1\gamma^3]^\dagger\gamma^0 = \bar{\psi}(-t,\mathbf{x})[\gamma^1\gamma^3]. \tag{6.27}$$

Then the transformation of the scalar bilinear is

$$T\bar{\psi}\psi(t,\mathbf{x})T = \bar{\psi}(\gamma^1\gamma^3)(-\gamma^1\gamma^3)\psi(-t,\mathbf{x}) = +\bar{\psi}\psi(-t,\mathbf{x}). \tag{6.28}$$

The pseudo-scalar acquires an extra minus sign when T goes through the i:

$$Ti\bar{\psi}\gamma^5\psi T = -i\bar{\psi}(\gamma^1\gamma^3)\gamma^5(-\gamma^1\gamma^3)\psi(-t,\mathbf{x}) = -i\bar{\psi}\gamma^5\psi(-t,\mathbf{x}). \tag{6.29}$$

For the vector, we must separately compute each of the four cases $\mu = 0,1,2,3$. After a bit of work you should find

$$\begin{aligned}
T\bar{\psi}\gamma^\mu\psi T &= \bar{\psi}(\gamma^1\gamma^3)(\gamma^\mu)^*(-\gamma^1\gamma^3)\psi \\
&= \begin{cases} +\bar{\psi}\gamma^\mu\psi(-t,\mathbf{x}) & \text{for } \mu = 0 \\ -\bar{\psi}\gamma^\mu\psi(-t,\mathbf{x}) & \text{for } \mu = 1,2,3. \end{cases}
\end{aligned} \tag{6.30}$$

This is exactly the transformation property we want for vectors such as the current density. You can verify that the pseudo-vector transforms in exactly the same way under time reversal.

Charge Conjugation

The last of the three discrete symmetries is the particle-antiparticle symmetry C. There will be no problem in implementing C as a unitary linear operator. Charge conjugation is conventionally defined to take a fermion with a given spin orientation into an antifermion with the same spin orientation. Thus, a convenient choice for the transformation of fermion annihilation operator is

$$C a^s_{\mathbf{p}} C = b^s_{\mathbf{p}}; \quad C b^s_{\mathbf{p}} C = a^s_{\mathbf{p}}. \tag{6.31}$$

Again, we ignore possible additional phases for simplicity.

Next we want to work out the action of C on $\psi(x)$. First we need a relation between $v^s(p)$ and $u^s(p)$. Using (6.21),

$$(v^s(p))^* = \begin{pmatrix} \sqrt{p\cdot\sigma}(-i\sigma^2\xi^*) \\ -\sqrt{p\cdot\bar\sigma}(-i\sigma^2\xi^*) \end{pmatrix}^* = \begin{pmatrix} -i\sigma^2\sqrt{p\cdot\bar\sigma^*}\xi^* \\ i\sigma^2\sqrt{p\cdot\sigma^*}\xi^* \end{pmatrix}^* = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar\sigma}\xi \end{pmatrix}. \tag{6.32}$$

That is,

$$u^s(p) = -i\gamma^2\left(v^s(p)\right)^*, \quad v^s(p) = -i\gamma^2\left(u^s(p)\right)^*. \tag{6.33}$$

If we substitute (6.33) into the expression for the fermion field operator, and then transform this operator with C, we find

$$\begin{aligned} C\psi(x)C = &\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s\left(-i\gamma^2 b^s_{\mathbf{p}}[v^s(p)]^* e^{-ip\cdot x} - i\gamma^2 a^{s\dagger}_{\mathbf{p}}[u^s(p)]^* e^{ip\cdot x}\right) \\ = &-i\gamma^2\psi^*(x) = \; -i\gamma^2(\psi^\dagger)^T == \; -i(\bar\psi\gamma^0\gamma^2)^T. \end{aligned} \tag{6.34}$$

Note that C is a linear unitary operator even though it takes $\psi \to \psi^*$.

Once again, we would like to know how C acts on fermion bilinears. First we need

$$C\bar\psi(x)C = C\psi^\dagger C\gamma^0 = (-i\gamma^2\psi)^T\gamma^0 = (-i\gamma^0\gamma^2\psi)^T. \tag{6.35}$$

Working out the transformations of bilinears is a bit tricky, and it helps to write in spinor indices. For the scalar,

$$\begin{aligned} C\bar\psi\psi C &= (-i\gamma^0\gamma^2\psi)^T(-i\bar\psi\gamma^0\gamma^2)^T \; = \; -\gamma^0_{ab}\gamma^2_{bc}\psi_c\bar\psi_d\gamma^0_{de}\gamma^2_{ea} \\ &= +\bar\psi_d\gamma^0_{de}\gamma^2_{ea}\gamma^0_{ab}\gamma^2_{bc}\psi_c = \; -\bar\psi\gamma^2\gamma^0\gamma^0\gamma^2\psi = \; +\bar\psi\psi \end{aligned} \tag{6.36}$$

.

(The minus sign in the third step is from fermion anticummutation.) The pseudo-scalar is no more difficult:

$$Ci\bar{\psi}\gamma^5\psi C = i(-i\gamma^0\gamma^2\psi)^T\gamma^5(-i\bar{\psi}\gamma^0\gamma^2)^T = i\bar{\psi}\gamma^5\psi. \tag{6.37}$$

We must do each component of the vector and pseudo-vector separately. Noting that γ^0 and γ^2 are symmetric matrices while γ^1 and γ^3 are antisymmetric, we eventually find

$$C\bar{\psi}\gamma^\mu\psi C = -\bar{\psi}\gamma^\mu\psi; \tag{6.38}$$

$$C\bar{\psi}\gamma^\mu\gamma^5\psi C = +\bar{\psi}\gamma^\mu\gamma^5\psi. \tag{6.39}$$

Although the operator C interchanges ψ and $\bar{\psi}$, it does not actually change the order of the creation and annihilation operators. Thus, if $\bar{\psi}\gamma^0\psi$ is defined to subtract the infinite constant noted above eq. (5.51), this constant does not reappear in the process of conjugation by C.

Summary of C, P, and T

The transformation properties of the various fermion bilinears under C, P and T are summarized in the table below. Here we use the shorthand $(-1)^\mu \equiv 1$ for $\mu = 0$ and $(-1)^\mu \equiv -1$ for $\mu = 1, 2, 3$.

	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$	∂_μ
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
C	$+1$	$+1$	-1	$+1$	-1	$+1$
CPT	$+1$	$+1$	-1	-1	$+1$	-1

We have included the transformation properties of the tensor bilinear, and also of the derivative operator.

Notice first that the free Dirac Lagrangian $\mathcal{L}_0 = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$ is invariant under C, P, and T separately. We can build more general quantum systems that violate any of these symmetries by adding to $\mathcal{L}_0$ some perturbation $\delta\mathcal{L}$. But $\delta\mathcal{L}$ must be Lorentz scalar, and the last line shows that all Lorentz scalar combinations of $\bar{\psi}$ and ψ are invariant under the combined symmetry CPT. Actually, it is quite generally true that one cannot build a Lorentz-invariant quantum field theory with a Hermitian Hamiltonian that violates CPT.