Neutrino Physics

Neutrinos hold great promise to teach us about new physics beyond the Standard Model. The main reason for that is the following. Electroweak symmetry breaking allows the left-handed neutrinos to mix with Standard Model singlets. (This is, of course, impossible for any charged fermions since any charged field is not an $SU(2)_{\rm L} \times U(1)_{\rm Y}$ singlet.) Then, the features of the light neutrinos may probe physics related to the singlets that can take place at a scale much higher than the electroweak breaking scale.

1. Kinematics of Neutrinos

In a certain sense, the neutrinos produced in β decay are the simplest fermions one can find in four dimensions. They are produced as left-handed particles or right-handed antiparticles. No right-handed neutrinos have been observed, which is to be contrasted with charged particles which have both left anf right helicities. Accordingly, the mathematical description of neutrinos is the simplest.

Like all fermions in four dimensions, neutrinos are defined by local fields which transform as spinor representations of the Lorentz group. The Lorentz group, generated by the three rotations J_i and the three boosts K_i, is $SO(3,1)$. If discrete symmetries are not a matter of concern, these hermitean generators can be assembled into non-unitary combinations $J_i + iK_i$ and $J_i - iK_i$ which generate two commuting $SU(2)$'s. In Minkowski space, these two groups can be related to one another in several ways, either by conjugation C$(i \to -i)$ or by parity P$(J_i \to J_i,\ K_i \to -K_i)$, so that both sets are seen to each independently sustain the combined operation of CP. At each point of space-time, local fields are taken to transform as finite dimensional (non-unitary) representations of $SU(2) \otimes SU(2)$. The scalar field transforms as $(1,1)$, the spinor fields as $(2,1)$ and $(1,2)$, the vector (gauge) fields as $(2,2)$, etc.. For later purposes, we note that the field strengths transform as $(3,1)$ and $(1,3)$.

The left-handed neutrino fields transform as the $(2,1)$ representation. Let us denote these two-component complex fields $\nu^a_L(x)$, where a denotes their flavor. Under the action of the Lorentz group, they transform as

$$\nu^a_L(x) \ \to \ \Lambda_L \nu^a_L(x), \quad \Lambda_L = e^{\frac{i}{2}(\vec{\omega}+i\vec{\chi})\cdot\vec{\sigma}}, \tag{1.1}$$

where $\vec{\omega}$ and $\vec{\chi}$ parametrize the rotations and boosts, respectively. As anticipated, it is non-unitary. Conversely, a right-handed field $\nu_R(x)$, transforming as the $(1,2)$ representation, undergoes

$$\nu_R(x) \;\rightarrow\; \Lambda_R \nu_R(x), \quad \Lambda_R = e^{\frac{i}{2}(\vec{\omega}-i\vec{\chi})\cdot\vec{\sigma}}. \tag{1.2}$$

It is easy to see that one can turn a left-handed spinor into a right-handed spinor and vice-versa rather easily. The magic of the Pauli matrices,

$$\sigma^2\sigma^i\sigma^2 = -\sigma^{i*}, \tag{1.3}$$

enables us to write

$$\sigma^2 e^{\frac{i}{2}(\vec{\omega}+i\vec{\chi})\cdot\vec{\sigma}}\sigma^2 = e^{-\frac{i}{2}(\vec{\omega}+i\vec{\chi})\cdot\vec{\sigma}^*} = \left[e^{\frac{i}{2}(\vec{\omega}-i\vec{\chi})\cdot\vec{\sigma}}\right]^*. \tag{1.4}$$

Consequently, the fields

$$\overline{\nu_L^a}(x) = \sigma_2 \nu_L^{a*}(x), \tag{1.5}$$

can be seen to transform as right-haned spinors.

The property (1.5) enables us to make a bilinear that transforms as a four-vector. Take a single $\nu_L = (2,1)$ and build a $(2,1)$ spinor out of it. Then multiply them together to get $(1,2)\otimes(2,1) = (2,2)$. One knows that the quantity $\nu_L^\dagger \nu_L$ is invariant under rotations while $\nu_L^\dagger \sigma^i \nu_L$ behaves as a three vector. Under boosts,

$$\begin{aligned} \delta(\nu_L^\dagger \nu_L) &= \chi^i \nu_L^\dagger \sigma^i \nu_L, \\ \delta(\nu_L^\dagger \sigma^i \nu_L) &= \chi^i \nu_L^\dagger \nu_L. \end{aligned} \tag{1.6}$$

Equation (1.6) compares with the transformation law of a four vector,

$$\delta V^\mu = \epsilon^\mu{}_\nu V^\nu, \tag{1.7}$$

where $\epsilon^{0i} = -\chi^i$ are the boost parameters. Thus the quantity

$$J^{\mu ab}(x) = \nu_L^{a\dagger}(x)\sigma^\mu \nu_L^b(x), \tag{1.8}$$

where $\sigma^\mu = (\sigma^0 \equiv 1, \vec{\sigma})$ denote the 2×2 unit and Pauli matrices, respectively, is a four current.

The SM left-handed neutrinos are described by the fields $\nu_L^a(x)$, the right-handed antineutrinos by $\overline{\nu_L^a}(x)$. The current we have just written above is the neutrino part of the current that couples to the Z when summed over the three known flavors of neutrinos. In the following, we will dispense with the L and R subscripts; a spinor field without a bar over it is understood to transform as $(2,1)$, one with a bar over it as $(1,2)$.

Since the derivative ∂_μ transforms as a four-vector $(2,2)$, the invariant kinetic term for neutrinos is given by

$$\mathcal{L}_{\text{kin}} = \sum_a \nu^{a\dagger}(x)\sigma^\mu \partial_\mu \nu^a(x) + \text{h.c.}. \tag{1.9}$$

One can construct other bilinear in neutrino fields. From the $SU(2)$ property that $2 \otimes 2 = 1_A \oplus 3_S$, it is easy to arrive at the following combinations:

$$\nu^{aT}(x)\sigma^2\nu^b(x); \quad \nu^{aT}(x)\sigma^2\vec{\sigma}\nu^b(x). \tag{1.10}$$

The first combination corresponds to the Lorentz scalar $(1,1)$ and the second to part of the antisymmetric tensor $(3,1)$. They have different flavor symmetry properties. Since the $\nu^a(x)$ obey Fermi statistics, the scalar combination is necessarily symmetric in the flavor indices a, b, because antisymmetrization has already taken place on the Lorentz indices. This combination is of course the Majorana mass and, being Lorentz invariant, it can appear in the Lagrange density as

$$\mathcal{L}_{\text{Maj}} = m_{(ab)}\nu^{aT}\sigma^2\nu^b + \text{h.c.}, \tag{1.11}$$

where we have indicated the flavor symmetrization by round brackets; the $m_{(ab)}$ coefficients have dimension of mass. One notes right away that if each field ν^a is assigned a global lepton number L_a, this mass term violates it by two units. It is useful at this point to contrast this situation with that of charged particle mass such as the electron. The electron field is described by both left-handed and right-handed fields $e_L(x)$ and $e_R(x)$; each can be given one unit of lepton number. The usual mass term for the electron is written as

$$\mathcal{L}_{\text{Dir}} = m e_L^\dagger e_R + \text{h.c.}, \tag{1.12}$$

and it is seen to conserve lepton number. We call this type of mass a Dirac mass to distinguish it from the above Majorana mass which violates lepton number by two units. Both types of mass terms can be described in the same formalism. For this we need left-handed fields $N_L^a(x)$, $a = 1, 2$. Their mass terms can be written as

$$\mathcal{L} = (N_L^{1T}\ N_L^{2T}) \begin{pmatrix} m_1 & m \\ m & m_2 \end{pmatrix} \begin{pmatrix} N_L^1 \\ N_L^2 \end{pmatrix} + \text{h.c.}. \tag{1.13}$$

We can characterize the different entries in the 2×2 matrix in terms of N-number. We see that $m_{1,2}$ violate $N_{1,2}$-number by two units while preserving $N_{2,1}$-number. On the other hand, the off-diagonal term m violates both N_1- and N_2-number while preserving their difference. This suggests that the off-diagonal terms of the (Majorana) mass matrix can be viewed as a Dirac mass term; indeed, if we identify $N_L^2 = \sigma^2 N_R^*$, and $N_L^1 = N_L$, then the off-diagonal contribution simply becomes

$$-2mN_R^\dagger N_L + \text{h.c.}, \tag{1.14}$$

which is nothing but the Dirac mass. For charged particles, the diagonal terms of this 2×2 matrix would violate electric charge by two units, so this formalism is not really necessary. But for neutral fermions, there is no local quantum number which stands in the way of the diagonal elements, just a global quantum number (*i.e.* lepton number). Thus for neutral particles, it makes sense to use this formalism, with the general 'mass' matrix as a complex, symmetric matrix, of the form

$$\begin{pmatrix} \text{Maj} & \text{Dir} \\ \text{Dir} & \text{Maj} \end{pmatrix}. \tag{1.15}$$

It should be clear now that if neutrinos are found to have a mass, there are two ways in which this can happen. Either the mass violates lepton number, and there need not be extra fermion degrees of freedom, or the mass preserves lepton number at the price of introducing for each massive neutrino a right-handed partner with the same lepton number. We will return to these possibilities later.

Let us now turn to the second bilinear one can construct with left-handed fields. As we have seen, it has the Lorentz quantum numbers of electromagnetic moments. Recall that an antisymmetric tensor $F_{\mu\nu}$ can be decomposed into two three-vectors $E_i = F_{0i}$ and $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$. Furthermore, these two vectors can be arranged as a complex (Hertz) vector $\vec{E} \pm i\vec{B}$, corresponding to $(3,1)$ and $(1,3)$, respectively. Alternatively, one can view the $(3,1)$ as a complex antisymmetric tensor which obeys the complex self-duality condition

$$F_{\mu\nu} = \frac{i}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}. \tag{1.16}$$

Fermi statistics demands that the flavor indices be antisymmetrized. Hence we cannot form an electromagnetic moment with only left-handed field. For charged particles, this is well understood since the magnetic moment makes a transition between left and right chiralities. The constructs of interest are of the form

$$\nu^{aT}\sigma_2\vec{\sigma}\nu^b, \quad \bar{\nu}^{aT}\sigma_2\vec{\sigma}\bar{\nu}^b. \tag{1.17}$$

One can form one particular linear combination that is CP-even and an orthogonal combination that is CP-odd, corresponding to the magnetic and electric moments, respectively. In the Lagrange density, these will appear multiplied by the field strengths and thus have dimension five. In a renormalizable field theory they will appear, if allowed at all, as a result of loop diagrams. We note that these terms violate the total lepton number by two units. However, the same statement as for the masses applies: the difference of lepton numbers can be conserved, as it is in the electron's case.

2. Standard Model Neutrinos

Within the Standard Model, there are three neutrinos. These are the $T_3 = +1/2$ components in the left-handed lepton doublets,

$$L_i^I(1,2)_{-1/2} = \begin{pmatrix} \nu_{Li} \\ \ell_{Li} \end{pmatrix}. \tag{2.1}$$

Here are some of the important properties of the Standard Model neutrinos:

(i) The SM neutrinos have neither strong nor electromagnetic interactions. This follows straightforwardly from the fact that the lepton doublets are $SU(3)_{\rm C}$ singlets and the neutrinos have $Q_{\rm EM} = T_3 + Y = 0$.

(ii) The SM neutrinos have weak interactions. Their charged current interactions are given by

$$-\mathcal{L}_{W^\pm} = \frac{g}{\sqrt{2}}\overline{\nu_{Li}}\gamma^\mu \ell^-_{Li} W^+_\mu + \text{h.c.}, \tag{2.2}$$

and their neutral current interactions are given by

$$-\mathcal{L}_{Z^0} = \frac{g}{2\cos\theta_W}\overline{\nu_{Li}}\gamma^\mu \nu_{Li} Z^0_\mu. \tag{2.3}$$

(iii) The SM neutrinos are massless. Within the SM it is simply impossible to write any renormalizable gauge-invariant Yukawa (or bare mass) terms for the neutrinos.

(iv) The SM neutrino interactions have neither mixing nor CP violation. As a consequence of the fact that the SM neutrinos are degenerate, one can choose an arbitrary mass basis for them. In particular, it is always possible to find a basis where the charged current mixing matrix is just a unit matrix.

Since neither a singlet fermion ν_R nor triplet scalar Δ are part of the SM, one cannot write any renormalizable tree-level mass terms for the neutrinos. However, everyone knows that out of $SU(2)_{\rm L}$ doublets, one can easily manufacture a triplet. Specifically, one can consider the following dimension five term in the effective Lagrangian:

$$\frac{Z_{ij}}{\langle\phi\rangle} L_{Li}^T \sigma_2 \tau_2 \vec{\tau} L_{Lj} \cdot \phi^T \tau_2 \vec{\tau} \phi, \qquad M_\nu^{\rm Majorana} = vZ. \tag{2.4}$$

The matrix Z_{ij} is a symmetric, complex 3×3 matrix. Such a term might be generated, in principle, beyond the tree level. However, it cannot be generated within the SM in perturbation theory. The reason is that the SM has an accidental global symmetry:

$$U(1)_B \times U(1)_e \times U(1)_\mu \times U(1)_\tau. \tag{2.5}$$

Total lepton number is a subgroup of (2.5). With L_i (ϕ) carrying lepton number $+1$ (0), (2.4) violates lepton number by two units. Within the SM, a global conservation law forbids neutrino masses. It is true that lepton number is an anomalous symmetry within the SM. However, $B - L$ is non-anomalous. Since (2.4) violates also $B - L$ by two units, (2.4) cannot be generated by non-perturbative contributions either.

We conclude that within the SM, neutrinos cannot get masses. If experiments reveal that neutrinos have masses, it will provide the first evidence for physics beyond the Standard Model.

3. Neutrino Masses Beyond the SM

In order to allow for massive neutrinos, the SM can be extended in two ways. In the first, the neutrino masses are of the Dirac type with preservation of some lepton number; in the second the neutrino masses are of the Majorana type with a concomitant breakdown of lepton number. In the first alternative, new fermion degrees of freedom are necessarily introduced to serve as Dirac partners of the usual neutrinos. In the second, extra fermions need not be introduced. We proceed to give an overview of the various possibilities.

3.1. Lepton number conserving extensions

If we allowed for singlet neutrinos, $N^I_m(1,1)_0$, then we could write down a Yukawa coupling that would lead to a Dirac mass matrix (we omit here the superscript I):

$$-\mathcal{L}_Y = Y^\nu_{im} L^T_{Li} \sigma_2 N_m \tau_2 \phi + \text{h.c.}, \quad M^{\text{Dirac}}_\nu = \frac{v}{\sqrt{2}} Y^\nu. \tag{3.1}$$

We assign lepton number -1 to N_m so that the Dirac mass terms (3.1) conserve lepton number. The index $m = 1,\ldots,M$ denotes the number of singlet neutrinos (there is no a-priori reason to take $M = 3$). Note that the matrix σ_2 (τ_2) operates in Lorentz ($SU(2)_{\rm L}$) space. The N_m neutrinos are sometimes called "sterile" neutrinos, because of their lack of SM gauge interactions.

If (3.1) is the single source of neutrino masses then the Yukawa couplings have to be tiny,

$$Y_{\nu_i} \approx 10^{-11} \left(\frac{m_{\nu_i}}{eV}\right). \tag{3.2}$$

With the direct experimental bounds,

$$m_{\nu_e} \lesssim 10\ eV, \quad m_{\nu_\mu} \leq 0.17\ MeV, \quad m_{\nu_\tau} \leq 18.2\ MeV, \tag{3.3}$$

we have

$$Y_{\nu_e} \lesssim 10^{-10}, \quad Y_{\nu_\mu} \leq 10^{-6}, \quad Y_{\nu_\tau} \leq 10^{-3}. \tag{3.4}$$

But various cosmological constraints make (3.2) a more likely upper bound. If one is willing to accept the presence of such tiny couplings (after all, we already have $m_e/m_Z \sim 10^{-5}$!), this represents a viable extension of the SM. (It does, however, pose a problem for Big Bang Nucleosynthesis (BBN).)

For the SM with an extended lepton sector that includes three sterile neutrinos and only Dirac masses, the formalism of mixing and CP violation in the lepton sector is the same as in the quark sector. The global symmetry of this model is

$$U(1)_B \times U(1)_L. \tag{3.5}$$

The individual family lepton numbers are violated, but total lepton number is conserved. The mixing matrix for leptons can be parametrized with three mixing angles and a single CP violating phase.

3.2. An $SU(2)_{\rm L}$-Triplet Scalar

If there are no other neutral fermions added in, the only mass matrix for neutrinos is of the Majorana type. This means that some lepton number has to be violated. The only freedom therefore is to add scalar fields which, unlike the SM Higgs field, carry lepton number. For this to happen, the new scalar must have some Yukawa couplings to the leptons. There are three scalar representations that couple to lepton bilinears:

$$L^T_{(a}\sigma_2\tau_2\vec{\tau}L_{b)}(1,3)_{-1}, \quad L^T_{[a}\sigma_2\tau_2 L_{b]}(1,1)_{-1}, \quad \bar{\ell}_{(a}\sigma_2\bar{\ell}_{b)}(1,1)_{+2}. \tag{3.6}$$

Thus we can readily couple to the SM a new scalar with two units of lepton number. This could be (a) an $SU(2)_{\rm L}$-triplet with one doubly charged component, one singly charged component and one neutral, or (b) one singly charged $SU(2)_{\rm L}$-singlet, or (c) one doubly charged $SU(2)_{\rm L}$-singlet. Of these, only the triplet,

$$\Delta_L(1,3)_{+1} = \begin{pmatrix} \Delta_L^{++} \\ \Delta_L^{+} \\ \Delta_L^{0} \end{pmatrix}, \tag{3.7}$$

can give a tree-level mass to neutrinos. The other two cannot. It must be said that these couplings do not violate lepton number. But the new scalars are free to interact in such a way that lepton number is broken, either explicitly by some term in the potential, or spontaneously (which is allowed to happen only for the triplet).

The $\Delta_L(1,3)_{+1}$ triplet has, in general, the following Yukawa coupling that would lead to Majorana masses:

$$-\mathcal{L}_Y = f_{ij}L^T_{Li}\sigma_2\tau_2\vec{\tau}L_{Lj}\cdot\vec{\Delta}_L \qquad M_\nu^{\rm Majorana} = \frac{\langle\Delta_L^0\rangle}{2}f. \tag{3.8}$$

Here f is a symmetric 3×3 matrix. Of course, $\langle\Delta_L^0\rangle \neq 0$ breaks lepton number. If the potential preserves lepton number, lepton number is broken spontaneously. In the minimal extension, there results a Goldstone boson, called the *triplet Majoron* [6]. This triplet Majoron model is excluded by experiment through its coupling to the Z boson. It could also be that the potential breaks lepton number explicitly. In this case, there is no Goldstone boson, and the only puzzle is the smallness of neutrino masses. In other words, one has to devise a model where the VEV of the triplet is naturally much smaller than that of the doublet.

3.3. The seesaw mechanism

Suppose that we add to the SM singlet fermions, $N_m^I(1,1)_0$, but do not impose lepton number symmetry. Then, in addition to the Dirac mass terms of eq. (3.1), we could have bare mass terms of the Majorana type for the singlets:

$$-\mathcal{L}_M = (M_R)_{ij} N_i^T \sigma_2 N_j. \tag{3.9}$$

Denoting the Dirac mass matrix by M_D,

$$M_D = \frac{v}{\sqrt{2}} Y^\nu, \tag{3.10}$$

and a possible contribution from a triplet scalar to the Majorana masses of the active neutrinos by M_L,

$$M_L = \frac{\langle \Delta_L^0 \rangle}{2} f, \tag{3.11}$$

we can write the overall mass matrix for neutrinos in the basis $\{\nu_{Li}, N_m\}$ as

$$M^\nu = \begin{pmatrix} M_L & M_D \\ M_D^T & M_R \end{pmatrix}. \tag{3.12}$$

The three independent blocks in M^ν are expected have very different mass scales. Let us denote the three scales by v_L, v_D and v_R where we can roughly take (assuming no vanishing eigenvalues) $v_X = (\det M_X)^{1/3}$. Then we expect $v_L \ll v_D \ll v_R$. The triplet scalar VEV which sets the scale for M_L is required to be much smaller than the EW breaking scale because of the $\rho = 1$ relation. The doublet scalar VEV which sets the scale for M_D is, of course, at the the EW breaking scale. The M_R terms do not break the EW symmetry and could be at an arbitrarily high scale. Let us investigate the matrix (3.12) assuming that, for simplicity, $M_L = 0$ and that

$$\varepsilon_D \equiv \frac{v_D}{v_R} \ll 1. \tag{3.13}$$

We can now do the following basis transformation:

$$M^\nu \to \hat{M}^\nu = U M^\nu U^T \approx \begin{pmatrix} -M_D M_R^{-1} M_D^T & \\ & M_R \end{pmatrix}, \qquad U \approx \begin{pmatrix} 1 & -M_D M_R^{-1} \\ M_R^{-1} M_D^T & 1 \end{pmatrix}. \tag{3.14}$$

For each block in $\hat{M}^\nu$ and U we omitted subleading contributions of $\mathcal{O}(\varepsilon_D^2)$.

We learn that of the six neutrino mass eigenstates, three are heavy with masses of $\mathcal{O}(v_R)$, and three are light with masses of $\mathcal{O}(v_D^2/v_R)$. Moreover, the three heavy states are dominantly sterile neutrinos with an active component of $\mathcal{O}(\varepsilon_D)$, while the three light states are dominantly active neutrinos with a sterile component of $\mathcal{O}(\varepsilon_D)$. The states that are relevant to phenomenology are then the eigenstates of

$$M^\nu_{\rm light} = M_D M_R^{-1} M_D^T, \qquad m_{\rm light} = \mathcal{O}(v_D^2/v_R). \tag{3.15}$$

Eq. (3.15) not only describes how the (dominantly) left-handed neutrino acquire masses but also *explains* why they are so much lighter than the charged fermions. The ratio between the scale of neutrino masses and that of charged fermion masses is $\varepsilon_D = v_D/v_R$. Since $v_R \gg v_D$ (v_R signifies the high scale of new physics where the SM singlets acquire masses), this ratio is small. This explanation of the lightness of neutrinos is called *the seesaw mechanism* [7,8]. The reason for that is that the heavier the scale of sterile neutrino masses (v_R) is, the lighter the scale of active neutrinos masses (v_D^2/v_R) becomes.

Note that M^ν breaks the accidental symmetry (2.5). In particular, M_R breaks total lepton number. Both M_D and M_R break, in general, the family lepton numbers. Consequently, mixing and CP violation appear in the lepton sector.

3.4. Nonrenormalizable neutrino masses

In the previous section we described an extension of the SM with right-handed neutrinos that have heavy Majorana masses. The right-handed neutrinos are very heavy ($v_R \gg m_Z$) and do not take part in the SM gauge interactions (except for the small active component in the heavy mass eigenstates). It is useful then to integrate the right-handed neutrinos out and consider a low energy effective theory. This theory has the particle content and gauge symmetry structure of the SM. It can include, however, non-renormalizable interaction and/or mass terms for the left-handed neutrinos.

In particular, we can write the following dimension-five term:

$$\frac{Z_{ij}}{M} L_{Li}^T \sigma_2 \tau_2 \vec{\tau} L_{Lj} \cdot \phi^T \tau_2 \vec{\tau} \phi. \tag{3.16}$$

We would like to emphasize the following points:

(i) Eq. (3.16) has the same form as eq. (2.4). However, eq. (2.4) was presented in the context of the SM as a full theory. We argued that such terms are forbidden because of a global symmetry. Eq. (3.16) appears once we consider the SM as a low energy effective theory. Then nonrenormalizable terms are allowed. Accidental symmetries are likely to be broken because their validity crucially depends on the assumption of renormalizability.

(ii) Eq. (3.16) can be derived from eqs. (3.1) and (3.9) when the N_i fields are integrated out. However, it could also be generated by different full high energy theories. It is thus more general then the model discussed in the previous section. As long as the heavy particles are not directly probed, any theory that can produce (3.16) is equally valid from the phenomenological point of view. Hence the usefulness of using (3.16) for interpreting experimental results instead of considering a particular high energy theory.

(iii) Eq. (3.16) leads to Majorana masses for the active neutrinos,

$$M_\nu = \frac{\langle\phi\rangle^2}{M} Z. \tag{3.17}$$

Again, this is equivalent to eq. (3.15) in the specific model that we presented in the previous section. However, the seesaw mechanism is more general than that particular model. Its validity is best presented by (3.16).

The 3×3 matrix M_ν of eq. (3.17) (and similarly the matrix $M^\nu_{\rm light}$ of eq. (3.15)) is symmetric and complex. It can be diagonalized by a unitary transformation,

$$M_\nu^{\rm diag} = U_L^\nu M_\nu U_L^{\nu T}. \tag{3.18}$$

The charged lepton mass matrix, $M^\ell = \frac{v}{\sqrt{2}} Y^\ell$ can be diagonalized by a bi-unitary transformation,

$$M_\ell^{\rm diag} = U_L^\ell M_\nu U_R^{\ell\dagger}. \tag{3.19}$$

Then, the charged current weak interactions in the lepton mass basis will involve a mixing matrix,

$$V^\ell = U_L^{\ell\dagger} U_L^\nu. \tag{3.20}$$

The 3×3 unitary matrix V^ℓ can be parametrized by three real mixing angles and six phases. In contrast to the quark sector, one cannot remove five of these phases. Any phase rotation of ν_{Li} will modify $M_\nu^{\rm diag}$ and make its entries complex. We can only rotate the charged lepton fields with a phase matrix P_ℓ. Consequently, only three phases can be removed. We conclude that the number of lepton flavor parameters in the low energy effective theory is twelve: three charged lepton masses, three neutrino masses, three mixing angles and three CP violating phases.

In the next four subsections we briefly discuss five specific extensions of the Standard Model. Two of them, the LRS and the SO(10) models, predict see-saw masses for the active neutrinos. In SU(5) models and in the Supersymmetric Standard Model with R-parity, the situation is similar to the Standard Model, that is neutrinos are massless. In Supersymmetric models without R_p, neutrinos masses constitute a problem since their masses are, in general, not suppressed compared to the charged fermion masses.

3.5. Left-Right Symmetric Models

Left-Right Symmetric (LRS) models allow a situation where the discrete symmetries P, C and CP, are exact symmetries of the Lagrangian that are only spontaneously broken. The gauge group is extended to

$$G_{\rm LRS} = SU(3)_{\rm C} \times SU(2)_{\rm L} \times SU(2)_{\rm R} \times U(1)_{B-L} \times \mathcal{P}. \tag{3.21}$$

The last factor is the discrete Parity symmetry (but it could be replaced with charge conjugation). The electric charge is identified with

$$Q_{\rm EM} = T_{3L} + T_{3R} + \frac{1}{2}(B-L). \tag{3.22}$$

The fermionic matter fields are in three generations of the following representations:

$$Q_L(3,2,1)_{+1/3}, \quad Q_R(3,1,2)_{+1/3}, \quad L_L(1,2,1)_{-1}, \quad L_R(1,1,2)_{-1}. \tag{3.23}$$

Under $\mathcal{P}$,

$$Q_L \leftrightarrow Q_R, \quad L_L \leftrightarrow L_R, \quad W_L \leftrightarrow W_R. \tag{3.24}$$

This requires that the gauge couplings for the two $SU(2)$'s are equal, $g_L = g_R$. The Higgs sector consists of three multiplets:

$$\Phi(1,2,2)_0 = \begin{pmatrix} \phi^0 & \phi'^+ \\ \phi^- & \phi'^0 \end{pmatrix}, \quad \Delta_L(1,3,1)_2 = \begin{pmatrix} \Delta_L^{++} \\ \Delta_L^+ \\ \Delta_L^0 \end{pmatrix}, \quad \Delta_R(1,1,3)_2 = \begin{pmatrix} \Delta_R^{++} \\ \Delta_R^+ \\ \Delta_R^0 \end{pmatrix}. \tag{3.25}$$

The neutral members in these multiplets acquire VEVs, with a hierarchy of scales:

$$\langle \Delta_R^0 \rangle \gg \langle \phi^0 \rangle, \langle \phi'^0 \rangle \gg \langle \Delta_L^0 \rangle. \tag{3.26}$$

The Yukawa interactions are given by

$$
\begin{aligned}
-\mathcal{L}_Y = & \, h_{ab}\overline{L_{La}}\Phi L_{Rb} + \tilde{h}_{ab}\overline{L_{La}}\tilde{\Phi} L_{Rb} \\
& + f_{ab}\left[L_{La}^T C^{-1}\epsilon\vec{\tau}\cdot\vec{\Delta}_L L_{Lb} + L_{Ra}^T C^{-1}\epsilon\vec{\tau}\cdot\vec{\Delta}_R L_{Rb}\right] + \text{h.c.},
\end{aligned}
\tag{3.27}
$$

where $\tilde{\Phi} = \tau_2\Phi^*\tau_2$. The resulting mass matrix for neutrinos is then

$$
M_\nu = \begin{pmatrix} f\langle\Delta_L^0\rangle & h\langle\phi^0\rangle + \tilde{h}\langle\phi'^0\rangle \\ h^T\langle\phi^0\rangle + \tilde{h}^T\langle\phi'^0\rangle & f\langle\Delta_R^0\rangle \end{pmatrix}. \tag{3.28}
$$

Given the hierarchy in (3.26), this provides a realization of the see-saw mechanism [9].

3.6. $SU(5)$ GUT Models

$SU(5)$ is the smallest GUT group. The fermions of each generation are assigned to $\{10\}$ and $\{\bar{5}\}$, denoted respectively by T and F:

$$
F = \begin{pmatrix} \bar{d}_1 \\ \bar{d}_2 \\ \bar{d}_3 \\ \ell \\ \nu_\ell \end{pmatrix}, \quad T = \begin{pmatrix} 0 & \bar{u}_3 & -\bar{u}_2 & u_1 & d_1 \\ & 0 & \bar{u}_1 & u_2 & d_2 \\ & & 0 & u_3 & d_3 \\ & & & 0 & \bar{\ell} \\ & & & & 0 \end{pmatrix}. \tag{3.29}
$$

All fields here are left-handed and the indices are color indices. The matrix T is antisymmetric. The breaking down of $SU(5) \to G_{\rm SM}$ is achieved with a $\Phi \equiv \{\bar{2}4\}$ scalar:

$$
\langle\Phi\rangle = \text{diag}\left(V, V, V, -\frac{3}{2}V, -\frac{3}{2}V\right), \tag{3.30}
$$

while the breaking of $G_{\rm SM} \to SU(3)_{\rm C} \times U(1)_{\rm EM}$ is achieved with an $H \equiv \{\bar{5}\}$ scalar:

$$
\langle H\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v/\sqrt{2} \end{pmatrix}. \tag{3.31}
$$

The most general gauge invariant Yukawa couplings are given by

$$
\mathcal{L}_Y = h_1 T_{ij}^T C^{-1} F^i H^j + h_2 \epsilon^{ijklm} T_{ij}^T C^{-1} T_{kl} H_m + \text{h.c.}, \tag{3.32}
$$

where the displayed indices are $SU(5)$ indices. Once H acquires a VEV, eq. (3.32) leads to

$$
M^d = M^\ell = h_1 v/\sqrt{2}, \quad M^u = h_2 v/\sqrt{2}. \tag{3.33}
$$

The neutrinos remain massless. To see that, note that $\mathcal{L}_Y$ is invariant under the global $U(1)_{\mathcal{F}}$ symmetry, where

$$\mathcal{F}(H_i) = -2/3, \quad \mathcal{F}(T_{ij}) = +1/3, \quad \mathcal{F}(F_i) = 1. \tag{3.34}$$

This combines with the $U(1)_{\mathrm{Y}}$ symmetry to generate the $B-L$ symmetry:

$$B - L = \frac{3}{5}\mathcal{F} - \frac{2}{\sqrt{15}}\lambda_{24}, \tag{3.35}$$

where λ_{24} is one of the $SU(5)$ generators.

The simplest way to modify the minimal $SU(5)$ model to generate a massive neutrino is to include a $\{15\}$-dimensional scalar representation $S_{ij}(= S_{ji})$. It allows a Yukawa coupling of the form

$$\mathcal{L}'_Y = f F_i^T C^{-1} F_j S^{ij} + \text{h.c.}. \tag{3.36}$$

3.7. $SO(10)$ GUT Models

The next higher symmetry useful for grand unification is $SO(10)$. An interesting aspect of $SO(10)$ is that it contains the LRS gauge group (3.21) and, therefore, it automatically contains the right-handed neutrino. Each fermion generation is entirely contained in the spinor $\psi\{16\}$ representation, where the extra degree of freedom is a SM singlet, namely the right-handed neutrino. To see the Higgs bosons which lead to gauge invariant Yukawa couplings, we note that

$$\{16\} \times \{16\} = \{10\} \oplus \{120\} \oplus \{126\}. \tag{3.37}$$

We can write the Yukawa couplings as

$$\begin{aligned} \mathcal{L}_Y = & f^{(ab)}_{\{10\}} \psi_a^T \mathcal{B} C^{-1} \Gamma^i \psi_b \{10\}_i + f^{[ab]}_{\{120\}} \psi_a^T \mathcal{B} C^{-1} \Gamma^i \Gamma^j \Gamma^k \psi_b \{120\}_{ijk} \\ & + f^{(ab)}_{\{126\}} \psi_a^T \mathcal{B} C^{-1} \Gamma^i \Gamma^j \Gamma^k \Gamma^l \Gamma^m \psi_b \{126\}_{ijklm} + \text{h.c.}, \end{aligned} \tag{3.38}$$

where Γ^i are the analog of the Dirac gamma matrices for $SO(10)$ and $\mathcal{B}$ is the analog of the conjugation matrix for the spinor of $SO(10)$. The VEV $\langle\{126\}\rangle$ gives Majorana masses for the left-handed and the right-handed neutrinos, while the VEVs $\langle\{10\}\rangle$ and $\langle\{120\}\rangle$ provide them with Dirac masses. The heavy scale is that of $B-L$ breaking, M_{BL}. Once we assume that the $SU(5)$ singlet component of $\{126\}$ gets a VEV at this scale, we find that a four-scalar term of the form $\lambda\{126\}\{126\}\{10\}\{10\}$ induces a very small VEV, of order v_L^2/M_{BL} (v_L is the EW breaking scale), to the neutral member of the $SU(2)_{\mathrm{L}}$-triplet in the $\{126\}$ multiplet. We get then the following mass matrix for the neutrinos:

$$M_\nu = \begin{pmatrix} \lambda f_{\{126\}} \frac{v_L^2}{M_{BL}} & \frac{1}{3} f_{\{10\}} v_L \\ \frac{1}{3} f_{\{10\}} v_L & f_{\{126\}} M_{BL} \end{pmatrix}. \tag{3.39}$$

We find then that $SO(10)$ models can naturally give see-saw masses for the light, active neutrinos.

3.8. The Supersymmetric Standard Model

A supersymmetric extension of the Standard Model requires that there exist four fermions (in addition to the SM neutrinos) that are neutral under $SU(3)_{\rm C} \times U(1)_{\rm EM}$. These are the photino $\tilde{\gamma}$, the zino $\tilde{Z}$, and the two neutral higgsinos. Furthermore, in a generic Supersymmetric Standard Model (SSM), the neutrinos mix with the $Y = +1/2$ higgsino through a term $\mu_i L_i H_u$ in the superpotential and with the Zino through a gauge coupling $\frac{g}{2\cos\theta_W}\langle\tilde{\nu}_i\rangle\tilde{Z}\nu_i$. As a result, one combination of the three neutrinos acquires a mass. Since the neutralinos must have masses of the order of the electroweak breaking scale, there is no see-saw mechanism at work. The induced neutrino mass is expected to be at the electroweak breaking scale which is, of course, phenomenologically unacceptable. We conclude that neutrino masses constitute a problem for generic supersymmetric extensions of the Standard Model. (One can arrange that there is no tree level contribution to m_ν by aligning the Supersymmetry breaking B-terms with the μ-terms. In such a case neutrinos still acquire masses from loop corrections.)

Neutrino masses are only a part of a bigger problem. In a generic SSM, the accidental symmetry of the SM, that is $G_{\rm SM}^{\rm global} = U(1)_B \times U(1)_e \times U(1)_\mu \times U(1)_\tau$, is broken, leading to fast proton decay, neutrino masses, lepton flavor changing decays, etc. To prevent such a phenomenological disaster, one usually imposes an extra discrete symmetry on the SSM, that is R parity. As a result of R_p, $G_{\rm SM}^{\rm global}$ becomes an accidental symmetry of the SSM. Neutrino masses are then forbidden just as in the SM.

4. Kinematical Tests of Neutrino Masses

4.1. m_{ν_e}

The experimental manifestations of neutrino masses have been searched for in several different types of experiments. The first type involves kinematic searches. The idea is that, if neutrinos have masses, this will affect the electron energy distribution in nuclear beta decay. The process at the nuclear level is

$$(A, Z) \to (A, Z+1) + e^- + \nu_e,$$

with the best bound coming from

$${}^3H \to {}^3He + e^- + \nu_e.$$

At the hadronic level we have

$$n \to p + e^- + \nu_e.$$

The relevant amplitude is

$$T_{fi} = -i\frac{G_F \cos\theta_C}{\sqrt{2}} \int d^4x \left[\bar{\psi}_e(x)\gamma_\lambda(1-\gamma_5)\psi_{\nu_e}(x)\right] \left[\bar{\psi}_p(x)\gamma^\lambda(1-\gamma_5)\psi_n(x)\right]. \qquad (4.1)$$

Specializing to the case where the initial and final nucleus have the same parity, the axial-vactor part does not contribute. Furthermore, since the energy release is much smaller than the nucleus mass, the final nucleus is non-relativistic and the only non-negligible contribution will come from γ^0:

$$T_{fi} = -i\frac{G_F \cos\theta_C}{\sqrt{2}} \left[\bar{u}_e(p)\gamma_0(1-\gamma_5)v_{\nu_e}(k)\right] \int d^4x \Psi_p^\dagger(x)\Psi_n(x)e^{-i(p+k)\cdot x}. \qquad (4.2)$$

The de Broglie wavelengths of the electron and the antineutrino are much larger than the nuclear radius (their energies are $\lesssim 1\ MeV$), so we can safely set $e^{-i(p+k)\cdot x} = 1$. (This is called the allowed approximation.) Performing the spatial integration, we obtain

$$\mathcal{M} = \frac{G_F \cos\theta_C}{\sqrt{2}} \left[\bar{u}_e(p)\gamma_0(1-\gamma_5)v_{\nu_e}(k)\right] 2m_N I, \qquad (4.3)$$

where the factor $2m_N$ comes from the normalization of the hadronic wave-functions and I is a process-dependent isospin factor.

Now we can calculate the decay rate. Integrating over the 3-momentum of the final proton, we get

$$\begin{aligned} d\Gamma = & \frac{G_F^2 \cos^2\theta_C I^2}{2} \left| \sum_{\text{spin}} \bar{u}_e(p)\gamma_0(1-\gamma_5)v_{\nu_e}(k) \right|^2 \\ & \times \frac{d^3p}{(2\pi)^3 2E} \frac{d^3k}{(2\pi)^3 2\omega} 2\pi\delta(E_0 - E - \omega). \end{aligned} \qquad (4.4)$$

Here, E_0 denotes the total energy available to the lepton pair which, to a good approximation, is given by the mass difference between the initial and final nuclei.

The spin sum gives

$$\begin{aligned} \left| \sum_{\text{spin}} \bar{u}_e(p)\gamma_0(1-\gamma_5)v_{\nu_e}(k) \right|^2 & = \frac{1}{2}\text{tr}[\not{k}\gamma_0\ \not{p}\gamma_0(1-\gamma_5)] \\ & = 2[E\omega + \vec{p}\cdot\vec{k}] = 2E\omega[1 + v_e v_\nu \cos\theta], \end{aligned} \qquad (4.5)$$

where v_e, v_ν are the magnitudes of the velocities of the outgoing leptons and θ is the angle between their directions. Putting this in eq. (4.4) and integrating over the angular variables other than θ, we obtain

$$\begin{aligned} d\Gamma = & \frac{2G_F^2 \cos^2\theta_C I^2}{(2\pi)^3} |\vec{p}|\ E\ |\vec{k}|\ \omega\ \delta(E_0 - E - \omega) \\ & \times [1 + v_e v_\nu \cos\theta]\ dE\ d\omega\ d(\cos\theta). \end{aligned} \tag{4.6}$$

Integrating over $\cos\theta$, the term proportional to $v_e v_\nu$ drops out and the other term gives 2. Finally, integrating over ω, we obtain

$$\frac{d\Gamma}{dE} = \frac{G_F^2 \cos^2\theta_C I^2}{2\pi^3} |\vec{p}| E (E_0 - E) \sqrt{(E_0 - E)^2 - m_{\nu_e}^2}. \tag{4.7}$$

Define $Q \equiv E_0 - m_e$ and $T_e = E - m_e$ (the kinetic energy of the electron). The number of electrons emitted with energy between E and $E + dE$ is given by

$$n(E)dE = \frac{G_F^2 \cos^2\theta_C I^2}{2\pi^3} F(E,Z) |\vec{p}| E (Q - T_e) \sqrt{(Q - T_e)^2 - m_{\nu_e}^2}\, dE, \tag{4.8}$$

where $F(E,Z)$ is the Coulomb correction factor. If there is lepton mixing and ν_e has components in several mass eigenstates, then the square root expression is replaced by $\sum_{i=1}^{3} \sqrt{(Q-T_e)^2 - m_{\nu_i}^2}\ |U_{ei}|^2$. If $m_\nu = 0$, then

$$K(E) \equiv \left[\frac{n(E)}{F(E,Z)|\vec{p}|E} \right]^{1/2} \propto Q - T_e. \tag{4.9}$$

If we plot the quantity $K(E)$ vs T_e, we would obtain a straight line. Such a plot is called the Kurie plot. The maximum value of T_e would be Q. The linearity of the Kurie plot is lost if the neutrino has a mass. The effect becomes appreciable only near the end point, where $Q - T_e$ is comparable to m_{ν_e}. The maximum kinetic energy of the electron becomes $Q - m_{\nu_e}$. If several ν_i are involved, the different m_{ν_i} produce kinks in the Kurie-plot where the size of the kinks is a measure for the corresponding $|U_{ei}|^2$. The most recent experiments were performed in Mainz and in Troitzk, and the present bound is [10-12]:

$$m_{\nu_e} \lesssim 10\ eV. \tag{4.10}$$

4.2. m_{ν_μ}

The easiest way to obtain limits on m_{ν_μ} is given by the two-body decay of π^+:

$$\pi^+(p) \to \mu^+(k) + \nu_\mu(q).$$

Momentum conservation implies $q = p - k$. Then, for pion decay at rest, the neutrino mass is determined by

$$m_{\nu_\mu}^2 = m_{\pi^+}^2 + m_{\mu^+}^2 - 2m_{\pi^+}(|\vec{p}_{\mu^+}|^2 + m_{\mu^+}^2)^{1/2}. \tag{4.11}$$

Therefore, a precise measurement of the muon momentum p_μ and knowledge of m_μ and m_π is required. These measurements were done at the PSI, resulting in the limit [13]:

$$m_{\nu_\mu} \leq 170\ keV. \tag{4.12}$$

The largest uncertainty comes from the pion mass.

4.3. m_{ν_τ}

Before discussing m_{ν_τ} we should mention that ν_τ has not yet been directly observed via CC interactions and all evidence for its existence is indirect. The present bounds on m_{ν_τ} come from ARGUS [14], CLEO [15], OPAL [16], DELPHI [17], and ALEPH [18]. Practically all experiments use the τ-decay into five charged pions,

$$\tau \to \nu_\tau + 5\pi^\pm(\pi^0), \tag{4.13}$$

with a branching ratio of $\mathrm{BR}(\tau \to \nu_\tau + 5\pi^\pm(\pi^0)) = (9.7 \pm 0.7) \times 10^{-4}$. The best bounds are obtained in a two-dimensional analysis of the energy and the mass of the hadronic system. The strongest bound [18] is

$$m_{\nu_\tau} \leq 18.2\ MeV. \tag{4.14}$$

Plans for future charm- or B-factories might allow to explore m_{ν_τ} down to $1 - 5\ MeV$.

5. Neutrino Oscillations

Whatever the mechanism for generating neutrino masses and mixing angles, ingenious ways of measuring these parameters have to be devised. In this case, it is much harder than for charged quarks. If a strange quark is produced with an amplitude proportional to some mixing angle, that angle can be measured by actually catching the strange particle (or its decay products) in the detector. The fact is that these particles interact strongly and decay right away with decay products that are readily identifiable. Contrast this with the case of neutrinos. What if the wrong kind of neutrino is produced? There is no detector that can catch it right away, and as far as we know they do not decay. How can you catch a neutrino? In fact, it was not until 1959 that Cowan and Reines managed to catch one [19].

5.1. Theory of neutrino oscillations

Back in 1957, motivated by rumors of evidence that neutrinos were sometimes produced in beta decays, Pontecorvo [20] proposed that one should consider, in analogy with the $K^0 - \bar{K}^0$ system, that antineutrinos produced in normal beta decay might oscillate into neutrinos. The rumors turned out to be just that, but the idea of neutrino oscillations remained. This proposal was made before it was known that there were several kinds of neutrinos [21]. That year, the group at Nagoya [22] proposed the idea of oscillations between neutrinos, which we now call flavor oscillations. They also introduced an arbitrary mixing angle between the species, one year before this was done in the hadronic sector with spectacular success by Cabibbo [23].

In the following we will discuss only the case of flavor oscillations, which mix neutrinos with neutrinos, and preserves total lepton number. The case of neutrino-antineutrino oscillations which breaks lepton number is much more difficult to track since its effects would necessarily be suppressed by the ratio of the small Majorana mass to the large momentum. We will simplify the discussion even further by assuming two kinds of neutrinos only, and thus one mixing angle. In vacuum, the neutrino mass eigenstates satisfy the equation

$$i\frac{d}{dt}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} = \begin{pmatrix}E_1 & 0\\ 0 & E_2\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}, \tag{5.1}$$

where the energies are given by

$$\begin{aligned} E_1 &= \sqrt{p^2+m_1^2} \approx p + \frac{m_1^2}{2p},\\ E_2 &= \sqrt{p^2+m_2^2} \approx p + \frac{m_2^2}{2p}. \end{aligned} \tag{5.2}$$

We can rewrite the evolution equation in the form

$$i\frac{d}{dt}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} = \frac{1}{2}(E_1+E_2)\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} + \begin{pmatrix}2\pi/L_0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}. \tag{5.3}$$

The parameter L_0 with dimension of length is given by

$$L_0 = \frac{4\pi p}{\Delta m^2}, \qquad (\Delta m^2 \equiv m_1^2 - m_2^2). \tag{5.4}$$

Its physical meaning will emerge shortly. The unit diagonal component of the mass matrix will play no role in oscillations which depend only on differences between the two states.

The mass eigenstates $\nu_1(t)$ and $\nu_2(t)$ are related to the interaction eigenstates through

$$\begin{pmatrix} \nu_1(0) \\ \nu_2(0) \end{pmatrix} = \begin{pmatrix} \cos\theta_0 & \sin\theta_0 \\ -\sin\theta_0 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}, \tag{5.5}$$

where θ_0 is the mixing angle. Let us assume that at $t = 0$ an electron neutrino state is created in a weak interaction. It is given by a linear combination of mass eigenstates,

$$|\nu_e\rangle = \cos\theta_0|\nu_1\rangle + \sin\theta_0|\nu_2\rangle. \tag{5.6}$$

At a later time t, the neutrino state will evolve into

$$|\nu_e(t)\rangle = \cos\theta_0 e^{-iE_1t}|\nu_1\rangle + \sin\theta_0 e^{-iE_2t}|\nu_2\rangle. \tag{5.7}$$

Suppose that we use a detector which triggers on the weak eigenstate ν_μ,

$$|\nu_\mu\rangle = -\sin\theta_0|\nu_1\rangle + \cos\theta_0|\nu_2\rangle. \tag{5.8}$$

The (appearance) probability of triggering that detector at time t will be

$$\begin{aligned} P(\nu_e \to \nu_\mu) &= |\langle\nu_\mu|\nu_e(t)\rangle|^2 \\ &= (\cos\theta_0 \sin\theta_0)^2|1 - e^{i(E_1-E_2)t}|^2 \\ &= \sin^2 2\theta_0 \sin^2\frac{\pi t}{L_0}. \end{aligned} \tag{5.9}$$

Thus we see that the probability peaks whenever the neutrino has travelled a distance equal to $(n + 1/2)L_0$ $(n = 0, 1, \ldots)$. In this case we have computed the appearance probability; it is proportional to the square of the mixing angle. The distance between peaks is L_0, the oscillation length. A convenient way to remember the size of the oscillation length is

$$L_0 = 2.47\,\frac{E(GeV)}{\Delta m^2(eV^2)}\ \text{meters}. \tag{5.10}$$

The generalization to n neutrino flavors is straightforward:

$$P_{\nu_\alpha\to\nu_\beta} = \left|\delta_{\alpha\beta} + \sum_{k=2}^{n} U_{\beta k}U^*_{\alpha k}\left[\exp\left(-i\frac{\Delta m^2_{k1}L}{2E}\right) - 1\right]\right|^2. \tag{5.11}$$

5.2. Neutrino oscillation experiments

In the previous section, we found that

$$P(\nu_\mu \to \nu_e) = \sin^2 2\theta \sin^2\left(\frac{1.27\Delta m^2[eV^2]L[\text{km}]}{E[GeV]}\right). \tag{5.12}$$

We can see then that three important issues confront the designer of a neutrino oscillation experiment. First, if one is searching for oscillations in the very small Δm^2 region, then large L/E must be chosen in order to enhance the $\sin^2(1.27\Delta m^2 L/E)$ term. If, however, L/E is too large in comparison to $1/\Delta m^2$, then oscillation occur rapidly. Because experiments have finite resolution on L and E, and a spread in beam energies, the $\sin^2(1.27\Delta m^2 L/E)$ term averages to $1/2$ when $\Delta m^2 \gg E/L$ and one loses sensitivity to Δm^2. Finally, because the probability is directly proportional to $\sin^2 2\theta$, if the mixing angle is small, then high statistics is required to observe an oscillation signal.

There are two types of oscillation searches: *disappearance* and *appearance*. To be simplistic, consider a pure source of neutrinos of type x. In a disappearance experiment, one looks for a deficit in the expected flux of ν_x. Appearance experiments search for $\nu_x \to \nu_y$ by directly observing interactions of neutrinos of type y. The case for oscillations is most persuasive if the deficit or excess has the (L/E) dependence predicted by eq. (5.12).

Imagine that a hypothetical perfect experiment sees no oscillation signal, based on N events. The experimenters can exclude the probability for oscillations at some confidence level (CL). A typical choice of CL is 90%, so in this case $P \le 1.28\sqrt{N}/N$. There is only one measurement and there are two unknowns, so this translates into an excluded region in the $\sin^2 2\theta - \Delta m^2$ plane. At high Δm^2, the limit on $\sin^2 2\theta$ is driven by the experimental statistics:

$$\sin^2 2\theta \le 2P \quad (\Delta m^2 \gg E/L). \tag{5.13}$$

At $\sin^2 2\theta = 1$, the low Δm^2 limit is driven by L and E:

$$\Delta m^2 \le \frac{E}{1.27L}\sqrt{P} \quad (\sin^2 2\theta = 1). \tag{5.14}$$

The imperfections of a real experiment affect the limits which can be set. Systematic uncertainties in the efficiencies and backgrounds reduce the sensitivity of a given experiment. Background sources introduce multiple flavors of neutrinos in the beam. Misidentification of the interacting neutrino flavor in the detector can mimic oscillation signatures. In addition, systematic uncertainties in the relative acceptance versus distance and energy need to be understood and included in the analysis of the data.

There are several types of neutrino oscillation experiments that are held at present or will be held in the near future. In the table below [24] we give an order of magnitude estimate of the values of Δm^2 which can be probed in reactor short-baseline (SBL) and long-baseline (LBL), accelerator SBL and LBL, atmospheric and solar neutrino oscillation experiments.

Experiment	L [m]	E [MeV]	Δm^2 [eV^2]
Reactor SBL	10^2	1	10^{-2}
Reactor LBL	10^3	1	10^{-3}
Accelerator SBL	10^3	10^3	1
Accelerator LBL	10^6	10^3	10^{-3}
Atmospheric	10^7	10^3	10^{-4}
Solar	10^{11}	1	10^{-11}

5.3. Reactor Experiments

Reactor experiments are disappearance experiments looking for $\bar{\nu}_e \to \bar{\nu}_x$. Reactors are a source of MeV neutrinos due to the fission of nuclear fuel. The main isotopes involved are ^{235}U, ^{238}U, ^{239}Pu and ^{241}Pu. The detection reaction is

$$\bar{\nu}_e + p \to e^+ + n, \tag{5.15}$$

with an energy threshold of 1.804 MeV. Two new reactor experiments are CHOOZ in France and Palo Verde near Phoenix, AZ (USA). The former excludes values of $\Delta m^2 \gtrsim 10^{-3}\ eV^2$ for large $\sin^2 2\theta$ and $\sin^2 2\theta \gtrsim 0.2$ for large Δm^2 [25]. A future LBL reactor experiment, KamLAND, is approved and will be constructed at the Kamioka site. It will probe Δm^2 down to $10^{-5}\ eV^2$.

5.4. Accelerator Experiments

The second source of terrestrial neutrinos are high energy accelerators. Experiments can be of either appearance or disappearance type. Accelerators typically produce neutrino beams by shooting a proton beam on a fixed target. The produced secondary pions and kaons decay and create a neutrino beam dominantly consisting of ν_μ. The detection process is via charged weak currents,

$$\nu_i + N \to i + X, \qquad i = e, \mu, \tau, \tag{5.16}$$

where N is a nucleon and X is the final hadronic state. Depending on the intended goal, the search for oscillations therefore requires a detector which is capable of detecting electrons, muons and τ-leptons in the final state.

At present, there are two experiments, LSND and KARMEN, running with neutrinos at medium energies, $E_\nu \approx 30 - 50\ MeV$. The production process is $\pi^+ \to \mu^+ \nu_\mu$ followed by $\mu^+ \to e^+ \bar{\nu}_\mu \nu_e$. LSND observed a signal consistent with $\bar{\nu}_\mu - \bar{\nu}_e$ oscillations [26], and KARMEN does not [27]. Specifically, the $\nu_\mu - \nu_e$ $(\bar{\nu}_\mu - \bar{\nu}_e)$ interpretation of the LSND data favors

$$
\begin{aligned}
0.2\ eV^2 \lesssim \Delta m_{12}^2 \lesssim 10\ eV^2, \\
0.002 \lesssim \sin^2 2\theta_{12} \lesssim 0.03.
\end{aligned}
\tag{5.17}
$$

At high energies, there are two experiments, CHORUS and NOMAD, designed to improve the existing limits on $\nu_\mu - \nu_\tau$ oscillations. Both experiments use a high-intensity ν_μ beam produced at CERN. The neutrino energies range from $10 \sim 40\ GeV$. The $\bar{\nu}_\mu$ contamination in the beam is only $\sim 5\%$. The prompt ν_τ contamination is $(3 \pm 4) \times 10^{-6}$. For large $\Delta m_{23}^2 (\gtrsim 1\ eV^2)$, the bound on the mixing is [28,29]

$$
\sin^2 2\theta_{23} \leq 1.2 \times 10^{-3}. \tag{5.18}
$$

Among the future LBL accelerator experiments, we should mention K2K (KEK to Super-K, 235 km), MINOS (Fermilab to Soudan, 735 km), ICARUS, NOE, AQUA-RICH and OPERA (CERN to Gran Sasso, 732 km).

5.5. Atmospheric Neutrinos

The behavior of atmospheric neutrinos, produced in the earth's atmosphere by cosmic rays, now provides rather strong evidence that neutrinos have mass. The atmospheric neutrinos are studied by underground detectors. The Super-Kamiokande underground experiment (Super K) has found evidence for neutrino flavor oscillation which is particularly convincing because it is almost independent of theoretical input [30]. This persuasive evidence comes from the dependence of the detected atmospheric neutrino flux on the direction from which the neutrinos are coming.

Suppose that neutrinos do not oscillate, and consider the flux of neutrinos of a given flavor coming from polar angle θ with respect to the zenith – the point directly above the underground detector. Consider only neutrinos with energies above a few GeV, so that geomagnetic effects can be neglected. Then, given the fact that the flux of cosmic rays which produces the neutrinos in our atmosphere is isotropic, it follows from a simple geometric argument that, at the detector, the neutrino fluxes from θ and from $\pi - \theta$ will be equal to within a few percent. That is, the neutrino flux will be up-down symmetric.

By contrast, if neutrinos do oscillate, then in general the neutrino flux of a given flavor will not be up-down symmetric. In the usual approximation that only two neutrino flavors mix appreciably, we have a conversion probability as in eq. (5.12) ($f \neq f'$):

$$P(\nu_f \to \nu_{f'}) = \sin^2 2\alpha \sin^2 \left(\frac{1.27 \Delta m^2 [eV^2] L[\mathrm{km}]}{E[GeV]} \right). \tag{5.19}$$

Now, since the Super K detector is not far below the earth's surface, downward going neutrinos come from $\theta \approx 0$, and thus produced in the atmosphere directly above the detector, travel only $\sim$ 15 km to reach it. In contrast, upward going neutrinos coming from $\theta \approx \pi$, and thus produced in the atmosphere on the opposite side of earth from the detector, travel the entire diameter of the earth $\sim$ 13000 km, to reach the detector. Let us suppose that $\Delta m^2 \sim 10^{-3}\ eV^2$, and $E \sim\ GeV$. Then, for neutrinos from $\theta \approx 0$, $\Delta m^2[eV^2]L[\mathrm{km}]/E[GeV] \ll 1$, while for the ones from $\theta \approx \pi$, $\Delta m^2[eV^2]L[\mathrm{km}]/E[GeV] > 1$. Hence, from eq. (5.19), upward going neutrinos ν_f from $\theta \approx \pi$ can oscillate away into neutrinos of a different flavor $\nu_{f'}$, while downward going neutrinos from $\theta \approx 0$ cannot. This distinction will result in a difference between the detected upward- and downward-going ν_f-fluxes.

Now, what do the data show? Let us consider the super K multi-GeV events, for which the visible energy deposition in the detector exceeds 1.33 GeV, and geomagnetic effects can be neglected. For these events, the ν_e flux is consistent with being up-down symmetric, but the ν_μ flux has a large up-down asymmetry. To be more quantitative, let us call the flux corresponding to $-1.0 < \cos\theta < -0.2$ "Up", and the one corresponding to $+0.2 < \cos\theta < +1.0$ "Down". Then, for the ν_e events,

$$\nu_e : \quad \frac{\mathrm{Up}}{\mathrm{Down}} = 0.93^{+0.13}_{-0.12}, \tag{5.20}$$

but for the ν_μ events,

$$\nu_\mu : \quad \frac{\mathrm{Up}}{\mathrm{Down}} = 0.54^{+0.06}_{-0.05}. \tag{5.21}$$

We see that Up/Down for the ν_μ is very inconsistent with unity, the value it must have if there is no neutrino oscillation. Instead, Up/Down for these events suggests that the muon neutrinos which travel a significant fraction of the earth's diameter to reach the detector (the "Up" flux) are oscillating away into another flavor, while the ones which travel only a short distance to reach the detector (the "Down" flux) are not. Since for this data sample $\langle E\rangle$ = a few GeV, we see from our previous discussion that a $\Delta m^2 = \mathcal{O}(10^{-3}\ eV^2)$ would produce this behavior. Since Up/Down for the ν_e events is consistent with unity, it appears that (at least most of) the muon neutrinos which oscillate away do not become electron neutrinos, but tau or sterile neutrinos, to which the detector is insensitive.

The Super K detector cannot determine the L or the E of an event accurately. However, estimating these quantities for each event as best as possible, the Super K experiment finds that, interpreting the data with $\nu_\mu - \nu_\tau$ oscillations, $\Delta m^2 \approx 2.2 \times 10^{-3}\ eV^2$ and $\sin^2 2\alpha = 1$.

Together, the atmospheric neutrino data from Super K [30], Soudan 2, and MACRO imply that the parameters of the $\nu_\mu \rightarrow \nu_\tau$ or $\nu_\mu \rightarrow \nu_s$ oscillation of the atmospheric neutrinos are

$$6 \times 10^{-4}\ eV^2 \lesssim \Delta m^2 \lesssim 10^{-2}\ eV^2, \tag{5.22}$$

and

$$\sin^2 2\theta_{23} \approx 1. \tag{5.23}$$

To confirm that the atmospheric neutrinos are oscillating, one would like to observe the same oscillations in an accelerator-generated ν_μ beam. To do that, one will have to allow the beam to travel a distance L which is not negligible compared to the earth's diameter, since $\Delta m^2 \sim 10^{-3}\ eV^2$ and the beam energy will not be less than $\mathcal{O}(1\ GeV)$. During the coming years, several such *long baseline* accelerator neutrino oscillation searches will be performed, using beams generated at KEK, Fermilab and CERN. It seems that this program of experiments will be able to look for oscillation with Δm^2 down to $1 \times 10^{-3}\ eV^2$ but not lower. It is unlikely though not impossible that Δm^2 lies below this value.

6. Solar Neutrinos

6.1. Production and Detection

According to the theory of solar engine [31], electron neutrinos are produced during the so-called $p - p$ chain, which starts with protons and ends up with α-particles:

$$4p \ \rightarrow \ \alpha + 2e^+ + 2\nu_e + 28\ MeV. \tag{6.1}$$

This reaction represents in a compact form a series of successive reactions and is at the core of the field of neutrino astronomy. The released 28 MeV binding energy of the α-particle diffuses out of the core, getting degraded in frequency in the process to appear as sunlight. Our concern here is with the two neutrinos which have energies roughly in the range of a few MeVs, depending on the details of the nuclear reaction. Recalling that the density of a typical stellar core is roughly 100 g/cm^3 and typical $\nu_e e$ scattering cross section is of order 10^{-43} cm^2, the mean free path of a neutrino is of order 10^{17} cm, which is much larger than the radius of the sun and of typical stars. From the observed solar luminosity, one can determine that the flux of neutrinos from the sun is about 6×10^{10} cm^{-2}s^{-1} on the earth. These neutrinos carry away about 2-3% of the total energy emitted by the sun. Thus neutrinos from stars are another messenger of physics information in addition to the usual electromagnetic radiation. An advantage of neutrinos over electromagnetic radiation is that they carry information about the core.

Neutrinos are also produced during the CNO cycle, but in such reduced rates that we do not discuss them in the following. In fact, the $p-p$ chain is responsible for 98.5% of the energy generated by the sun.

The $p-p$ chain recations are as follows. 86% of the neutrions are produced in the ppI chain:

$$\begin{aligned} & p+p \ \rightarrow \ {}^2\mathrm{H} + e^+ + \nu_e, \qquad 0 < E_\nu < 0.420 \ MeV, \\ & p+e^- + p \ \rightarrow \ {}^2\mathrm{H} + \nu_e, \qquad E_\nu = 1.44 \ MeV, \\ & {}^2\mathrm{H} + p \ \rightarrow \ {}^3\mathrm{He} + \gamma, \\ & {}^3\mathrm{He} + \ {}^3\mathrm{He} \ \rightarrow \ {}^4\mathrm{He} + p + p. \end{aligned} \tag{6.2}$$

The ^{3}He produced in the above chain leads also to two other chains, called hep chain, which contributes only a small fraction ($\sim 10^{-7}$) of the neutrino flux, and ppII chain which contributes about 14% of the neutrino flux:

$${}^3\mathrm{He} + p \ \rightarrow \ {}^4\mathrm{He} + e^+ + \nu_e, \qquad E_\nu < 18.77 \ MeV, \tag{6.3}$$

$$\begin{aligned} & {}^3\mathrm{He} + \ {}^4\mathrm{He} \ \rightarrow \ {}^7\mathrm{Be} + \gamma, \\ & {}^7\mathrm{Be} + e^- \ \rightarrow \ {}^7\mathrm{Li}(+\gamma) + \nu_e, \qquad E_\nu = \begin{cases} 0.861 \ MeV & 90\% \\ 0.383 \ MeV & 10\% \end{cases}, \\ & {}^7\mathrm{Li} + p \ \rightarrow \ {}^4\mathrm{He} + \ {}^4\mathrm{He}. \end{aligned} \tag{6.4}$$

The reason for the two lines in ^{7}Be capture is that 10% of the time the ^{7}Li is produced in a metastable state which then radiatively decays. Finally, there is the ppIII chain which is responsible for a fraction 1.5×10^{-4} of the total neutrino flux. Despite the smallness of this fraction, it is of direct interest to us since these are the neutrinos detected in the water (Super-)Kamiokande experiment, and also constitute the major source in the experiment of Davis described below. It uses the ^{7}Be produced in the ppII chain:

$$\begin{aligned}
{}^7\mathrm{Be} + p &\rightarrow {}^8\mathrm{B} + \gamma, \\
{}^8\mathrm{B} &\rightarrow {}^8\mathrm{Be}^* + e^+ + \nu_e, \qquad 0.81 < E_\nu < 14.6\ MeV, \\
{}^8\mathrm{Be}^* &\rightarrow {}^4\mathrm{He} + {}^4\mathrm{He}.
\end{aligned} \tag{6.5}$$

Thus we see that we have three neutrino lines and two continuous bands, one at low energy coming from the original fusion reaction, and the second at higher energies coming from the Boron decay. The flux of the lower energy neutrinos depends on the proton density while the flux of the higher energy neutrinos depends on the abundance of ^8B. Clearly, the amount of Boron is much more environmentally sensitive than that of the primeval protons. This translates into the dependence of the neutrino flux on the sun's temperature. The flux of neutrinos from Boron decay depends on T^{18}, to be compared with the $T^{-1.2}$ dependence of the flux from the fusion reaction. There are three types of experiments that measure the solar neutrino flux.

The first experiment, performed deep underground in the Homestake mine, relies on the reaction

$$\nu_e + {}^{37}\mathrm{Cl} \rightarrow e^- + {}^{37}\mathrm{Ar}. \tag{6.6}$$

Energetics are such that only neutrinos with at least 0.814 MeV energy are capable of producing Argon. Thus, only neutrinos from the $p - e - p$ capture, Beryllium capture and Boron decay can trigger this detector.

The second type of experiment is also radiochemical, but uses cleverly chosen elements to be triggered by much lower energy neutrinos. The reaction in this case is

$$\nu_e + {}^{71}\mathrm{Ga} \rightarrow e^- + {}^{71}\mathrm{Ge}. \tag{6.7}$$

It can be triggered by neutrinos with as low energy as 0.23 MeV. Thus it is sensitive to the neutrinos that come from the primeval fusion reaction, as well as those that come from all other sources. However, the flux of $p - p$ neutrinos is many orders of magnitude higher than that of the ones emanating from Boron decay. Hence, this type of detector can really measure the number of the low energy solar neutrinos. This type of experiment is being conducted in SAGE near Baksan in the Caucasus mountain range, and in GALLEX in the Gran Sasso tunnel near Rome.

The third type of experiment is conducted in Japan, first at the Kamiokande detector and presently at the Super-Kamiokande detector. There, a neutrino entering the detector can scatter with an electron:

$$\nu_e + e^- \ \rightarrow \ \nu_e + e^-. \tag{6.8}$$

The electron recoils in the direction it was hit. Thus by detecting the recoil electron, one should find an excess of such electrons opposite the direction of the sun at the time of impact. This detector can only detect electrons if they have been hit sufficiently hard, limiting the detection to neutrinos with $E_\nu \gtrsim 6.5\ MeV$. Thus only neutrinos produced at the upper end of the Boron decay spectrum are detectable.

The most recent results from the five experiments are given in the table below.

Experiment	Result	Theory	Result/Theory
Homestake	$2.56 \pm 0.16 \pm 0.16$	$7.7^{+1.2}_{-1.0}$	0.33 ± 0.029
Kamiokande	$2.80 \pm 0.19 \pm 0.33$	$5.15^{+1.0}_{-0.7}$	0.54 ± 0.07
GALLEX	$77.5 \pm 6.2^{+4.3}_{-4.7}$	129^{+8}_{-6}	0.60 ± 0.06
SAGE	$66.6^{+7.8}_{-8.1}$	129^{+8}_{-6}	0.52 ± 0.06
Super K	$2.44 \pm 0.05^{+0.09}_{-0.07}$	$5.15^{+1.0}_{-0.7}$	0.474 ± 0.020

We learn that there is a deficit in all experiments. Furthermore, the deficits are such that there is no consistent explanation from modifications of the standard solar model.

6.2. Solar Neutrino Vacuum Oscillations

The most outrageous possibility is that the earth is at the right distance from the sun to be in a trough in the variation of the probability with distance. The earth-sun distance is 1.5×10^{11} meters. For $\langle E_\nu \rangle \sim 10\ MeV$, the difference between the masses-squared of ν_e and some other neutrino ν_x should be of $\mathcal{O}(10^{-11}\ eV^2)$. The best-fit value is actually [32]:

$$\Delta m^2_{1x} = 6.5 \times 10^{-11}\ eV^2, \quad \sin^2 2\theta_{1x} = 0.75. \tag{6.9}$$

Note that if we had $\Delta m^2_{1x} \gg 10^{-11}\ eV^2$, then in an effective two neutrino framework, we would have

$$P_{\nu_e \to \nu_e} \approx 1 - \frac{1}{2}\sin^2 2\theta \geq \frac{1}{2}, \tag{6.10}$$

which cannot explain the results of the Chlorine experiment. In a three generation framework, one has $P_{\nu_e \to \nu_e} \geq \frac{1}{3}$, which is barely consistent with the Chlorine experiment. However, since in such a case there is no sensitivity to E_ν, the suppression should have been the same for all experiments which is, again, inconsistent with the results. In the opposite case, $\Delta m^2_{1x} \ll 10^{-11}\ eV^2$, the oscillation probability vanishes. Thus we see that indeed a only a very fortunate combination of scales, $\Delta m^2_{1x} L/E_\nu = \mathcal{O}(1)$, can fit the experimental results.

6.3. Matter Oscillations

It is remarkable that, even if the fundamental parameters do not lend themselves to this "just so" scenario, the deficit of solar neutrinos could still be explained in terms of oscillations. Solar neutrinos are emitted at or near the core of the sun; in order to get to us they must traverse the sun. The neutrino mean free path is much longer than the radius of the sun, they traverse it without losing coherence. This is akin to a beam of light through water, which can be described in geometrical optics by means of an index of refraction: the electron neutrino is slowed by forward coherent scattering with electrons and other particles. The sun's interior acts as a refractive medium to neutrinos. Wolfenstein [33], who first realized it, also noted that the index of refraction would be different for another species of neutrinos, say ν_μ, since its interactions with matter would proceed differently (neutrinos with such low energy could not produce a muon). Hence, if an electron neutrino oscillates while in the sun, this effect produces a new twist on the situation since the indices of refraction of the two beams will be different. It was shown by Mikheyev and Smirnov [34] that the effect could have spectacular consequences: a resonance effect can take place so that in a certain region of the sun, the diagonal element of the effective mixing matrix can vanish, leading to maximum mixing for a large range of fundamental parameters. This is the MSW effect, which we now analyze in detail.

For a quantitative treatment of the above ideas, let us again consider the simplest case of two flavors, ν_e and ν_μ. We first recall the case of vacuum oscillations. If the mass eigenstates are ν_1 and ν_2, the evolution equation for these states can be written as

$$i\frac{d}{dt}\begin{pmatrix} \nu_1(t) \\ \nu_2(t) \end{pmatrix} = H \begin{pmatrix} \nu_1(t) \\ \nu_2(t) \end{pmatrix}, \tag{6.11}$$

where H is diagonal in this basis:

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \simeq E + \begin{pmatrix} m_1^2/2E & 0 \\ 0 & m_2^2/2E \end{pmatrix}. \tag{6.12}$$

In the interaction basis, (6.11) is rewritten as

$$i\frac{d}{dt}\begin{pmatrix} \nu_e(t) \\ \nu_\mu(t) \end{pmatrix} = H' \begin{pmatrix} \nu_e(t) \\ \nu_\mu(t) \end{pmatrix}, \tag{6.13}$$

where

$$H' = UHU^\dagger = E + \frac{m_1^2 + m_2^2}{4E} + \frac{\Delta m^2}{4E}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}. \tag{6.14}$$

From this, we can see that the diagonalizing angle θ is given by

$$\tan 2\theta = \frac{2H'_{12}}{H'_{22} - H'_{11}}. \tag{6.15}$$

We now consider the same problem when the neutrinos travel through matter. For simplicity, we assume that the density of matter is uniform, with n_e, n_p and n_n denoting the number density of electrons, protons and neutrons. Elastic scattering off these particles change the effective masses of neutrinos, whose magnitude we now estimate.

Consider first elastic scattering through charged current interactions. The only interaction of this sort is the $\nu_e e$ scattering, since there are no μ's or τ's in the background at normal temperatures. The contribution to the effective Lagrangian is given by

$$i\mathcal{L}_{\text{eff}} = \left(\frac{ig}{\sqrt{2}}\right)^2 \{\bar{e}(p_1)\gamma^\sigma \nu_{eL}(p_2)\} \frac{ig_{\sigma\rho}}{m_W^2} \{\overline{\nu_e}(p_3)\gamma^\rho e_L(p_4)\}, \tag{6.16}$$

where we have neglected the momentum dependence in the W-propagator since we are interested in the leading order contribution in G_F. Using $\frac{g^2}{8m_W^2} = \frac{G_F}{\sqrt{2}}$, we obtain

$$\begin{aligned} i\mathcal{L}_{\text{eff}} &= -\frac{4G_F}{\sqrt{2}} \{\bar{e}(p_1)\gamma_\sigma \nu_{eL}(p_2)\} \{\overline{\nu_e}(p_3)\gamma^\sigma e_L(p_4)\} \\ &= -\frac{4G_F}{\sqrt{2}} \{\overline{\nu_e}(p_3)\gamma_\sigma \nu_{eL}(p_2)\} \{\bar{e}(p_1)\gamma^\sigma e_L(p_4)\}, \end{aligned} \tag{6.17}$$

where the second term is obtained via Fierz transformation. For forward scattering where $p_2 = p_3 = p$, this gives the following contribution that affects the propagation of the ν_e:

$$-\sqrt{2}G_F \overline{\nu_{eL}}(p)\gamma_\sigma \nu_{eL}(p)\langle \bar{e}\gamma^\sigma(1-\gamma_5)e\rangle, \tag{6.18}$$

averaging the electron field bilinear over the background. To calculate that average, note that the axial current reduces to spin in the non-relativistic approximation, which is negligible for a non-relativistic collection of electrons. The spatial components of the vector current give the average velocity, which is negligible as well. So the only non-trivial average is

$$\langle \bar{e}\gamma^0 e\rangle = \langle e^\dagger e\rangle = n_e, \tag{6.19}$$

which gives the contribution to the effective Lagrangian

$$-\sqrt{2}G_F n_e \overline{\nu_{eL}}\gamma_0 \nu_{eL}. \tag{6.20}$$

One can interpret the density dependent term as a contribution to the potential energy:

$$V_{\rm CC} = \sqrt{2}G_F n_e. \tag{6.21}$$

(For a careful, field theoretic derivation, see [35-37].)

Turning now to the neutral current contributions, we find in an exactly similar way the following contributions to the effective energies of both ν_e and ν_μ:

$$V_{\rm NC} = \sqrt{2}G_F \sum_f n_f \left[I_{3L}^{(f)} - 2\sin^2\theta_W Q^{(f)}\right], \tag{6.22}$$

where $f = e, p, n$. For normal neutral matter $n_e = n_p$. Therefore the contributions of the electron and the proton cancel each other and we are left with the neutron contribution, which is

$$V_{\rm NC} = -\frac{\sqrt{2}}{2}G_F n_n. \tag{6.23}$$

We emphasize that this neutral current contribution is the same for all flavors of neutrinos whereas the charged current contribution affects ν_e only. Thus, the evolution equation of neutrino beams should now be given by eq. (6.13) with H' replaced by

$$\tilde{H} = E + \frac{m_1^2 + m_2^2}{4E} - \frac{1}{\sqrt{2}}G_F n_n + \begin{pmatrix} -\frac{\Delta m^2}{4E}\cos 2\theta + \sqrt{2}G_F n_e & \frac{\Delta m^2}{4E}\sin 2\theta \\ \frac{\Delta m^2}{4E}\sin 2\theta & \frac{\Delta m^2}{4E}\cos 2\theta \end{pmatrix}. \tag{6.24}$$

The effective mixing angle in matter, θ_m, would accordingly be given by

$$\tan 2\theta_m = \frac{2\tilde{H}_{12}}{\tilde{H}_{22} - \tilde{H}_{11}} = \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta - A}, \tag{6.25}$$

where for the sake of convenience we defined

$$A = 2\sqrt{2}G_F n_e E. \tag{6.26}$$

The effective mixing angle thus changes inside matter. The change is most dramatic if $A = \Delta m^2 \cos 2\theta$, i.e., if the electron number density is given by

$$n_e = \frac{\Delta m^2 \cos 2\theta}{2\sqrt{2}G_F E}. \tag{6.27}$$

Then, even if the vacuum mixing angle θ is small, we obtain $\theta_m = \pi/4$, which is to say that ν_e and ν_μ mix maximally.

To understand the exact nature of this maximal mixing, let us write down the eigenvalues of $\tilde{H}$. The results are

$$E_\alpha = E - \frac{G_F}{\sqrt{2}} n_n + \frac{\tilde{m}_\alpha^2}{2E}, \tag{6.28}$$

where

$$\tilde{m}_{1,2}^2 = \frac{1}{2}\left[(m_1^2 + m_2^2 + A) \mp \sqrt{(\Delta m^2 \cos 2\theta - A)^2 + (\Delta m^2 \sin 2\theta)^2}\right]. \tag{6.29}$$

Consider now the case when θ is very small. For $A = 0$, which corresponds to the vacuum, the lighter eigenstate ν_1 is almost purely ν_e, whereas the heavier one ν_2 is almost purely ν_μ. In the other extreme, where $A \gg \Delta m^2 \cos 2\theta$, $\theta_m \to \pi/2$, which means that the lighter eigenstate is almost purely ν_μ, whereas the heavier one is ν_e to a good approximation. Thus, if we follow the effective mass squared values of ν_e and ν_μ, we see that the effective mass squared value of ν_e starts lower than that of ν_μ but with the increase of A the difference diminishes until finally ν_e overtakes ν_μ. This phenomenon is known as *level-crossing.*

It was realized by Mikheyev and Smirnov that this phenomenon of level-crossing might be crucial in understanding the neutrino flux from the sun.

6.4. Matter Oscillations in the Sun

Let us rewrite the equation governing the neutrino propagation in matter in the form

$$i\frac{d}{dx}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} = \frac{1}{2E}M^2\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix}, \tag{6.30}$$

where

$$M^2 = \begin{pmatrix} -\frac{1}{2}\Delta m^2 \cos 2\theta + A & \frac{1}{2}\Delta m^2 \sin 2\theta \\ \frac{1}{2}\Delta m^2 \sin 2\theta & \frac{1}{2}\Delta m^2 \cos 2\theta \end{pmatrix}. \tag{6.31}$$

The effective mixing angle, which controls the oscillation probability, is given by

$$\sin^2 2\theta_m = \frac{(\Delta m^2)^2 \sin^2 2\theta}{(\Delta m^2 \cos 2\theta - A)^2 + (\Delta m^2)^2 \sin^2 2\theta}. \tag{6.32}$$

We see that when $A = \Delta m^2 \cos 2\theta$, the mixing angle is maximal and transition from ν_e to ν_μ occurs. This relation is equivalent to (6.25), but shows more readily that it is a resonance condition, with width $\Gamma \approx \Delta m^2 \sin 2\theta$.

To discuss this phenomenon in detail, let us write down the eigenvalues of the matrix M^2 in eq. (6.31):

$$\tilde{m}^2_{1,2} = \frac{1}{2}\left[A \mp \sqrt{(\Delta m^2 \cos 2\theta - A)^2 + (\Delta m^2)^2 \sin^2 2\theta}\right]. \tag{6.33}$$

Propagation through matter is determined by these two eigenvalues, with the corresponding eigenstates $\tilde{\nu}_\alpha$ related to the weak eigenstates as follows:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix} \begin{pmatrix} \tilde{\nu}_1 \\ \tilde{\nu}_2 \end{pmatrix}. \tag{6.34}$$

The equation of propagation for the $\tilde{\nu}_\alpha$'s can be obtained from eqs. (6.30), (6.31) and (6.34):

$$i\frac{d}{dx}\begin{pmatrix} \tilde{\nu}_1 \\ \tilde{\nu}_2 \end{pmatrix} = \begin{pmatrix} \frac{\tilde{m}_1^2}{2E} & i\frac{d\theta_m}{dx} \\ -i\frac{d\theta_m}{dx} & \frac{\tilde{m}_2^2}{2E} \end{pmatrix} \begin{pmatrix} \tilde{\nu}_1 \\ \tilde{\nu}_2 \end{pmatrix}. \tag{6.35}$$

Notice that if the density is constant, the off-diagonal terms in this matrix vanish, so that the states denoted by $\tilde{\nu}_\alpha$ are indeed the stationary eigenstates. For varying density as in the case of the sun, one can define the stationary eigenstates only for the Hamiltonian at a given point. However, as long as the off-diagonal elements are small, any matter eigenstate $\tilde{\nu}_\alpha$ traverses unchanged with their relative admixture of ν_e and ν_μ changing adiabatically according to the value of n_e at a given point. Since terms that are proportional to the unit matrix do not affect oscillation probabilities, the only physical parameter in the diagonal elements must be their difference. Thus, the *adiabacity condition* is

$$\left|\frac{d\theta_m}{dx}\right| \ll \left|\frac{\Delta m^2_{12}}{2E}\right|. \tag{6.36}$$

Using the resonant value of θ_m from (6.32) as well as the resonant density, we can rewrite it as

$$\gamma \gg 1, \tag{6.37}$$

where γ is called the *adiabaticity parameter*, defined to be

$$\gamma = \frac{\Delta m^2}{E} \frac{\sin^2 2\theta}{\cos 2\theta} \left| \frac{d \ln n_e}{dx} \right|_{\rm res}^{-1}. \tag{6.38}$$

The physical meaning of this condition is the following. The resonance condition $A = \Delta m^2 \cos 2\theta$ is approximately satisfied within a distance

$$\delta x = \frac{\delta n_e}{|dn_e/dx|_{\rm res}} \approx \frac{\Gamma/(2\sqrt{2}G_F E)}{|dn_e/dx|_{\rm res}} = \frac{\Delta m^2 \sin 2\theta}{|dn_e/dx|_{\rm res} 2\sqrt{2} G_F E}, \tag{6.39}$$

where we used $\Gamma = \Delta m^2 \sin 2\theta$. On the other hand, the oscillation length at the resonance is given by

$$L_{\rm res} = \frac{4\pi E}{\Delta m^2 \sin 2\theta}. \tag{6.40}$$

The ν_μ-fraction of the wave in ν_e has enough time to build if $\delta x \gg L_{\rm res}$, which is precisely the resonance condition of eq. (6.37).

To see the essence of resonant matter oscillation (or MSW effect), we recall [38] that when the adiabaticity condition is satisfied, $\tilde{\nu}_\alpha$ remains unchanged. Let us assume that ν_e starts at a point where the electron density is infinite. Then $\theta_m^{\rm p} = \pi/2$, where the superscript p denotes the production point. Eq. (6.34) then implies that $\tilde{\nu}_2$ corresponds to ν_e. On the way out of the solar core, the ν_e component in the beam decreases continuously, since at a general point $\tilde{\nu}_2 = \nu_e \sin\theta_m + \nu_\mu \cos\theta_m$, and θ_m decreases with decreasing density. When finally the neutrino beam comes out of the sun, $n_e \approx 0$ so that $\theta_m = \theta$ (the vacuum mixing angle). Thus $P_{\nu_e \to \nu_e}$ decreases from almost unity to the final value of $\sin^2\theta$. This is the *resonant conversion* phenomenon, which is most dramatic if θ is small, so that the ν_e beam is almost totally depleted. The survival probability of ν_e's is given, for the case of adiabatic propagation, by

$$P^{\rm (ad)}_{\nu_e \to \nu_e} = \frac{1}{2}\left[1 + \cos 2\theta_m^{\rm p} \cos 2\theta + \sin 2\theta_m^{\rm p} \sin 2\theta \cos\left(\int dy \frac{\tilde{m}_2^2 - \tilde{m}_1^2}{2E}\right)\right]. \tag{6.41}$$

In the more general case when the adiabaticity condition is not necessarily satisfied, the oscillation probability is given by [39,40]

$$P_{\nu_e \to \nu_e} = \frac{1}{2}\left[1 + (1 - 2P_{\rm LSZ}) \cos 2\theta_m^{\rm p} \cos 2\theta\right], \tag{6.42}$$

where

$$P_{\rm LSZ} = \exp\left(-\frac{\pi}{4}\gamma\right) \tag{6.43}$$

is the Landau-Zenner-Stuckelberg probability of jumping from one eigenstate to another, assuming the density to be linearly varying near the resonance and that γ is not very much smaller than unity.

To see how all this applies to the sun, we need to pay attention to two conditions. First, since the electron density decreases on the way out, the resonance is never reached unless the neutrino is produced at a region where the density is higher than the resonance density, $A^{\rm p} \geq \Delta m^2 \cos 2\theta$. Most neutrinos are produced at the solar core, where

$$n_0 \approx 6 \times 10^{25} \text{ cm}^{-3}. \tag{6.44}$$

This gives the following condition for the occurence of the resonance:

$$\frac{\Delta m^2 \cos 2\theta}{E} < 1.5 \times 10^{-11} \ eV \equiv m_{\rm res}. \tag{6.45}$$

The second condition is that of the onset of adiabaticity as given in (6.37), (6.38). To use it in the present context, note that n_e in the sun (except for the inner 10% of the radius) can be given by

$$n_e(x) = 2n_0 \exp(-10x/R), \tag{6.46}$$

where $R \approx 7 \times 10^{10}$ cm is the solar radius. Using (6.38), we thus find that the adiabatic condition is satisfied if

$$\frac{\Delta m^2}{E}\frac{\sin^2 2\theta}{\cos 2\theta} \gg \frac{10}{R} = 3 \times 10^{-15} \ eV \equiv m_{\rm ad}. \tag{6.47}$$

Let us now consider a beam of neutrinos of some average energy E. For the sake of simplicity, we consider the range of energies to be small enough that we can treat the beam as monochromatic, but large enough that the interference term of eq. (6.41) can be ignored. As an illustration, suppose that one has obtained experimentally some value P_0 for $P_{\nu_e \to \nu_e}$, $P_0 < 0.5$. We can seek for values of $\Delta m^2/E$ and θ which will give the required suppression in flux.

To find these solutions, consider first the region of parameter space where (6.47) is satisfied i.e. neutrino propagation is adiabatic. If $\theta \ll 1$, resonance occurs for $\Delta m^2/E < m_{\rm res}$. If $\Delta m^2/E \gg m_{\rm res}$, matter effects are negligible, so that the survival probability is given by the neutrino oscillation formulae in the vacuum, and is therefore always larger than 1/2. If $\Delta m^2/E = m_{\rm res}$, eq. (6.32) implies $\theta_m^{\rm p} = \pi/4$, so that $P_{\nu_e\to\nu_e} = 0.5$. And if $\Delta m^2/E \ll m_{\rm res}$, $P_{\nu_e\to\nu_e} = \sin^2\theta$, as argued earlier. Thus, $P_{\nu_e\to\nu_e} = P_0$ must be obtained for a value of $\Delta m^2/E$ which is somewhat smaller than $m_{\rm res}$. This solution is independent of the value of θ, as long as $\theta \ll 1$, and is valid unless non-adiabatic effects are important, which implies, from (6.47), $4\theta^2 \gg m_{\rm ad}/m_{\rm res} = 2\times 10^{-4}$. This solution appears as a horizontal line (for a given value of P_0) in the $\log(\Delta m^2/E) - \log(\sin^2 2\theta)$ plane.

There is a second way of satisfying $P_{\nu_e\to\nu_e} = P_0$ in the adiabatic region. This occurs for $\sin^2\theta = P_0$ if $\Delta m^2/E \ll m_{\rm res}$ so that $\theta_m^{\rm p} = \pi/2$. Since this branch has a constant value of θ, it appears as a vertical line in the plot.

Finally, if non-adiabatic effects are important, putting $\theta_m^{\rm p} \approx \pi/2$, we see that eq. (6.42) reduces to

$$P_{\nu_e\to\nu_e} = \sin^2\theta + P_{\rm LSZ}\cos 2\theta, \tag{6.48}$$

which reduces to just $P_{\rm LSZ}$ if $\theta \ll 1$. This leads to the non-adiabatic solution [41], given by the solution

$$\frac{\Delta m^2}{E}\sin^2 2\theta = -\frac{40\ln P_0}{\pi R}. \tag{6.49}$$

This is a diagonal line in the plot.

Taking all three branches together, we see that any equal probability line in the plot is roughly of the form of a triangle, with a horizontal, a vertical and a diagonal branch. Of course, near the vertices, the triangle is a little rounded since the extreme assumptions that we took do not apply here.

For each experiment, there corresponds an allowed band in the form of a triangle. As long as $\Delta m^2\cos 2\theta < E m_{\rm res} = A^{\rm p}$ is satisfied for some energy, the resonance condition will be reached somewhere in the sun. In the horizontal branch, for the most part $\cos 2\theta \approx 1$. Thus, for a fixed value of Δm^2, virtually all neutrinos having an energy greater than a certain value are converted to ν_μ and only the less energetic ν_e's survive. The lower the value of Δm^2, the smaller the ν_e flux because lower and lower energies can satisfy the resonance condition and are converted to ν_μ.

In the vertical branch, $\cos 2\theta$ is small enough that even very low energy neutrinos satisfy (6.45) and undergo resonant conversion. The survival probability $\sin^2\theta$ is practically the same for neutrinos with all energies. The range of values of θ in the vertical line is such that $\sin^2\theta$ gives the range of survival probabilities suggested by experiments.

Both these solutions are bounded, for small Δm^2 and/or small θ, by the validity of the adiabaticity condition given in eq. (6.37). Finally, the slanted branch denotes the range of values where non-adiabatic oscillations can produce the required depletion through (6.42). This can occur only if $P_{\rm LSZ}$ is large, i.e. the energies of the neutrinos are large. Thus, in this branch, the higher energy neutrinos survive whereas the low energy ones are depleted, in contrast to the horizontal branch.

Combining the results of all solar neutrino experiments (see Table in section 6.1), we get two small allowed regions. This are centered around

$$\Delta m^2_{1x} = 5 \times 10^{-6}\ eV^2, \quad \sin^2 2\theta_{1x} = 6 \times 10^{-3} \quad \text{(SMA)}, \tag{6.50}$$

$$\Delta m^2_{1x} = 2 \times 10^{-5}\ eV^2, \quad \sin^2 2\theta_{1x} = 0.8 \quad \text{(LMA)}. \tag{6.51}$$

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