9. Lie algebras

An *algebra* $\mathcal{A}$ is a linear vector space over a field, with a binary operation of multiplication — for every $X, Y \in \mathcal{A}$, there is an element $X \cdot Y \in \mathcal{A}$. If the field is the real numbers, the algebra is called a *real algebra*; if the field is the complex numbers, it is called a *complex algebra*. The multiplication is linear and distributive $[(aA+bB)\cdot(cC+dD) = acA\cdot C+adA\cdot D+bcB\cdot C+bdB\cdot D$, where A, B, C, D are vectors in the space and a, b, c, d are elements of the field], but need not be associative or commutative. The *dimension* of the algebra is its dimension as a vector space, the number of independent vectors required to form a basis for the algebra. The algebra contains a *null vector* 0, such that $0 \cdot A = 0$ for any element A of the algebra.

A subspace of the vector space which is closed under the multiplication operation constitutes a *subalgebra* of the algebra. Every algebra contains two *improper* subalgebras — the algebra itself and the *null algebra* consisting entirely of the null element, $\{0\}$. Any other subalgebra is a *proper* subalgebra.

If the multiplication rule

- is antisymmetric, $X \cdot Y = -Y \cdot X$ for every $X, Y \in \mathcal{A}$, and
- satisfies the *Jacobi identity*, $X \cdot (Y \cdot Z) + Y \cdot (Z \cdot X) + Z \cdot (X \cdot Y) = 0$, for any three elements $X, Y, Z \in \mathcal{A}$ (note that the cyclic order X, Y, Z is maintained in the Jacobi identity),

then it defines a *Lie product*. One concrete example of a Lie product is the usual vector product of three-dimensional vectors.

$$[\vec{a} \times \vec{b} = -\vec{b} \times \vec{a};\ \vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times a) + \vec{c} \times (\vec{a} \times \vec{b}) = 0].$$

Another concrete realization of the Lie product is the familiar commutator:

$$[[A, B] = -[B, A];\ [A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0].$$

as may be seen by simply writing out the commutators in full.

A *Lie algebra* is an algebra in which the multiplication is a *Lie product*. The commutator is so common as a realization of the Lie product, and its use within the framework of matrix representations of Lie algebras so widespread in physics, that the standard notation used for the Lie product of two elements A, B is $[A, B]$. This notation will be used from now on, but it should be noted that it may, in some contexts, cause confusion. As a rule, $[A, B]$ should be considered primarily as the Lie product of the elements A, B, though it may also represent their commutator.

[The infinitesimal generators of a Lie group span a vector space which is invariant under commutation, since the commutator of any two generators is a linear combination of the generators. Defining the Lie product of two elements of this vector space as their commutator makes the space a Lie algebra, as mentioned previously.]

A Lie algebra may contain proper Lie subalgebras. A subalgebra $\mathcal{B}$ is *Abelian* if the Lie product of any two elements vanishes, i.e. $[X, Y] = 0$ for all $X, Y \in \mathcal{B}$. A subalgebra $\mathcal{B} \subset \mathcal{A}$ is *invariant* if the Lie product of any of its elements with any element of the algebra is in the subalgebra, i.e. $[X, Y] \in \mathcal{B}$ for any $X \in \mathcal{B},\ Y \in \mathcal{A}$. An invariant subalgebra is also called an *ideal*.

The *centre* of a Lie algebra is the set of elements whose Lie product with all elements of the algebra vanishes. (In more familiar terms, the set of elements which commutes with all elements of the algebra is the centre of the algebra.) The centre is an Abelian ideal of the algebra.

A Lie algebra is *simple* if it is non-Abelian and contains no proper ideals, *semi-simple* if it contains no Abelian ideals except the null subalgebra $\{0\}$. A simple algebra is semi-simple. It can be proved that a semi-simple Lie algebra is a direct sum of simple Lie algebras.

[The *direct sum* of two algebras is analogous to the direct product of two groups. If $\mathcal{A}$ and $\mathcal{B}$ are Lie algebras of dimensions d_a and d_b respectively, with bases $\{A_i\}$ and $\{B_i\}$ respectively, and the Lie product of any basis element of $\mathcal{A}$ with any basis element of $\mathcal{B}$ vanishes, $[A_i, B_j] = 0$, then the vector space spanned by the basis $\{A_i, B_j; i = 1, \ldots, d_a; j = 1, \ldots, d_b\}$ constitutes the algebra $\mathcal{A} \oplus \mathcal{B}$, of dimension $d_a + d_b$.]

Consider the Lie algebra $\mathcal{A}$ of dimension d and let it have a basis $\{X_i\}$, with $i = 1, \ldots, d$. Since the algebra is closed under the Lie product,

$$[X_i, X_j] = \sum_{k=1}^{d} c_{ij}^k X_k, \tag{1}$$

where the coefficients c_{ij}^k are called the *structure constants* of the algebra and determine its structure completely.

[Note carefully the positions of the indices on c_{ij}^k — the subscripts are the indices of the factors of the Lie product, in the same order in which the factors appear, and the superscript is the index of the appropriate basis vector in the expansion of the product.]

From the antisymmetry of the Lie product and from the Jacobi identity it satisfies, the structure constants satisfy the conditions

$$c_{ij}^k = -c_{ji}^k \tag{2}$$

$$\sum_{l=1}^{d} (c_{il}^m c_{jk}^l + c_{jl}^m c_{ki}^l + c_{kl}^m c_{ij}^l) = 0, \tag{3}$$

since the basis vectors are linearly independent.

To every element A of the Lie algebra $\mathcal{A}$ there corresponds a linear operator on the algebra, denoted $ad(A)$ and defined by $ad(A)X = [A, X]$ for every $X \in \mathcal{A}$. The set of all such operators constitutes a Lie algebra of the same dimension d as the original algebra $\mathcal{A}$, in which the Lie product is the commutator.

[The notation $ad(A)ad(B)$ implies consecutive action of the ad operators, from right to left, and the Lie product is defined as $[ad(A), ad(B)] = ad(A)ad(B) - ad(B)ad(A)$.]

There is a natural mapping $A \to ad(A)$ which preserves the Lie product.

[$ad([A, B])X = [[A, B], X] = [A, [B, X]]+[B, [X, A]] = ad(A)ad(B)X - ad(B)ad(A)X = [ad(A), ad(B)]X$ for all $X \in \mathcal{A}$, i.e. $ad([A, B]) = [ad(A), ad(B)]$, where the Jacobi identity has been used in the second step.]

The algebra of operators $ad(A)$ is therefore a representation of the algebra $\mathcal{A}$, known as the *adjoint representation.* If $\{X_i\}$ is a basis for $\mathcal{A}$, then $\{ad(X_i)\}$ is a basis for the adjoint representation and the matrix elements of the adjoint representation are the structure constants, $(ad(X_k))_{ij} = c_{kj}^i$.

[$ad(X_i)X_j = \sum_k (ad(X_i))_{kj} X_k$, but $ad(X_i)X_j = [X_i, X_j] = \sum_k c_{ij}^k X_k$, so $(ad(X_i))_{kj} = c_{ij}^k$.]

The representation matrices of the adjoint representation constitute a faithful representation of the algebra, with the same structure constants.

[Consider $([ad(X_k), ad(X_l)])_{ij} = \sum_m (ad(X_k))_{im}(ad(X_l))_{mj}$ $- \sum_m (ad(X_l))_{im}(ad(X_K))_{mj}$ and insert the structure constants, so $([ad(X_k), ad(X_l)])_{ij} = \sum_m (c^i_{km}c^m_{lj} - c^i_{lm}c^m_{kj}) = \sum_m c^i_{mj}c^m_{kl}$, using eqs.(2) and (3). This implies $([ad(X_k), ad(X_l)])_{ij} = \sum_m c^m_{kl}(ad(X_m)_{ij}$.]

With the aid of the adjoint representation, a very useful invariant bilinear form can be defined on any Lie algebra. The *Killing form* is defined by

$$g_{AB} = \mathrm{tr}(ad(A)ad(B)), \text{ for any two elements } A, B \text{ of the algebra,} \tag{4}$$

involving the trace of the simple product (in terms of consecutive action) of two operators of the adjoint representation. Because of the cyclic invariance of the trace, the Killing form is symmetric, $g_{AB} = g_{BA}$, and invariant in the sense that $g_{[A,B]C} = g_{A[B,C]}$.

[$\mathrm{tr}(ad([A, B])ad(C)) = \mathrm{tr}(ad(A)ad(B)ad(C) - ad(B)ad(A)ad(C)) = \mathrm{tr}(ad(A)ad(B)ad(C) - ad(A)ad(C)ad(B)) = \mathrm{tr}(ad(A)ad([B, C]))$, where the cyclic invariance of the trace has been used in the second step.]

It can be shown to be the unique such invariant bilinear form, up to a multiplicative constant.

The Killing form for the basis vectors $\{X_i\}$ may be rewritten in terms of the structure constants,

$$g_{ij} = tr(ad(X_i)ad(X_j)) = \sum_{k,l=1}^{d} c^k_{il}c^l_{jk}. \tag{5}$$

A Lie algebra is semi-simple if and only if its Killing form is non-degenerate, i.e. has a non-vanishing determinant. This is called the *Cartan criterion.*

[Suppose the algebra $\mathcal{A}$ is not semi-simple, i.e. contains an Abelian ideal $\mathcal{I}$. Select a basis for $\mathcal{I}$ and supplement it with additional basis vectors to complete a basis for $\mathcal{A}$. The basis is denoted $\{X_i\}$ and the basis vectors of $\mathcal{I}$ are distinguished by primes. Since $\mathcal{I}$ is an ideal, $[X_{i'}, X_j] \in \mathcal{I}$, which implies the structure constants $c^k_{i'j} = 0$ if $X_k \notin \mathcal{I}$. The index k must carry a prime, k', for the structure constant to be non-zero. Since $\mathcal{I}$ is Abelian, $[X_{i'}, X_{j'}] = 0$, which implies the structure constants $c^k_{i'j'} = 0$ for any k. Now consider the Killing form $g_{ij'}$for an arbitrary basis vector X_i and a basis vector $X_{j'} \in \mathcal{I}$ and apply these two results systematically to $g_{ij'} = \sum_{kl} c^k_{il} c^l_{j'k} = \sum_{kl'} c^k_{il'} c^{l'}_{j'k} = \sum_{k'l'} c^{k'}_{il'} c^{l'}_{j'k'} = 0$. This holds for any i, for given j', so a whole column of g_{ij} vanishes and $\det g$=0. If $\mathcal{A}$ is not semi-simple, its Killing form is degenerate. The converse was proved by Cartan.]

For semi-simple Lie algebras, the Killing form g_{ij} has an inverse, which is denoted g^{ij}, with raised indices.

A related structure constant may be defined by

$$c_{ijk} = \sum_l g_{il} c^l_{jk}. \tag{6}$$

Upon insertion of the expression (5), and by use of the Jacobi identity, this becomes $c_{ijk} = \sum_{lmn} c^m_{in} c^n_{lm} c^l_{jk} = \sum_{lmn} (c^m_{in} c^n_{jl} c^l_{km} + c^m_{ni} c^l_{mj} c^n_{lk})$. The right hand side of this expression is manifestly invariant under cyclic permutations of i, j, k. But the left hand side is antisymmetric under interchange of j and k, by (2). So the new structure constant c_{ijk} is totally antisymmetric in its three indices (i.e. it changes sign under interchange of any pair of indices). For semi-simple Lie algebras, the defining relation (6) can be inverted,

$$c^i_{jk} = \sum_l g^{il} c_{ljk}. \tag{7}$$

[Note how g_{ij} and g^{ij} play the familiar formal role of raising and lowering indices here.]

Given a Lie algebra, only the Lie product of two elements of the algebra is defined — the simple product of two elements is not defined within the algebra. However, in any representation of the algebra in terms of operators on a vector space or in terms of matrices, the simple product of two representative elements is defined, either as the consecutive operation of two operators or as the matrix product of two matrices. So in any representation of a semi-simple Lie algebra it is possible to define the *Casimir operator*

$$\mathsf{C} = \sum_{i,j=1}^{d} g^{ij} X_i X_j, \tag{8}$$

where d is the dimension of the algebra, $\{X_i\}$ is a basis for the algebra and g^{ij} is the inverse of the Killing form for the basis vectors. This operator commutes with all the elements of the algebra.

[The commutator $[\mathsf{C}, X_k] = \sum_{ij} g^{ij}[X_iX_j, X_k] = \sum_{ij} g^{ij}(X_i[X_j, X_k] + [X_i, X_k]X_j) = \sum_{ijl} g^{ij}(c^l_{jk}X_iX_l + c^l_{ik}X_lX_j) = \sum_{ijl} g^{ij}c^l_{jk}(X_iX_l + X_lX_i) = \sum_{ijlm} g^{ij}g^{lm}c_{mjk}(X_iX_l + X_lX_i)$, where use has been made of the fact that i, j are dummy indices and of eq. (7). If, in the final expression, the dummy indices i, l are interchanged and so are the dummy indices j, m, the product $g^{ij}g^{lm}$ remains unchanged, as does the sum $X_iX_l + X_lX_i$, while the antisymmetric structure constant c_{mjk} changes sign. The commutator is equal to minus the commutator, i.e. it vanishes.]

Racah generalized the notion of Casimir operator to operators of higher order, defining n^{th} order invariants by

$$\mathsf{C}_n = \sum c^{j_2}_{i_1j_1}c^{j_3}_{i_2j_2}\dots c^{j_1}_{i_nj_n}g^{i_1l_1}g^{i_2l_2}\dots g^{i_nl_n}X_{l_1}X_{l_2}\dots X_{l_n}, \tag{9}$$

where the summation is over all repeated indices. These operators commute with all elements of the semi-simple Lie algebra, $[\mathsf{C}_n, X_m] = 0$. Note that the Casimir operator defined in eq.(8) is $\mathsf{C} = \mathsf{C}_2$.

A real Lie algebra is said to be *compact* if its Killing form is negative definite. A compact algebra is necessarily semi-simple. Complex Lie algebras are non-compact.