6. Irreducible representations

It has been shown that no irrep of $\mathcal{G}$ can have dimension larger than $|\mathcal{G}|$. Even more stringent restrictions may be placed on the properties of irreps with the aid of two very powerful results known as *Schur's lemmas.*

Schur's first lemma states that if $\mathsf{D}^{(1)}$ and $\mathsf{D}^{(2)}$ are irreps of a group $\mathcal{G}$ and if there exists a matrix $\mathcal{A}$ such that $\mathcal{D}_R^{(1)}\mathcal{A} = \mathcal{A}\mathcal{D}_R^{(2)}$ for all $R \in \mathcal{G}$ then $\mathsf{D}^{(1)}$ and $\mathsf{D}^{(2)}$ are equivalent (i.e. $\mathcal{A}$ is invertible) or $\mathcal{A} = 0$.

> [Suppose that $\mathsf{D}^{(1)}$ and $\mathsf{D}^{(2)}$ have different dimensions. Then by the reducibility criterion one of them is reducible — a contradiction. So $\mathcal{A}$ must vanish. If they both have the same dimension d and $\{\psi_i\}$ form a basis for $\mathsf{D}^{(1)}$, then the set $\{\sum_{j=1}^{d} \mathcal{A}_{ji}\psi_j\}$ spans an invariant subspace of the carrier space of $\mathsf{D}^{(2)}$. If this set is linearly independent, then $\mathcal{A}$ is non-singular and $\mathcal{D}_R^{(1)} = \mathcal{A}\mathcal{D}_R^{(2)}\mathcal{A}^{-1}$ for all $R \in \mathcal{G}$ — the two irreps are equivalent. If the set is not linearly independent, then $\mathsf{D}^{(2)}$ is reducible, a contradiction which can only be avoided if $\mathcal{A} = 0$.]

Schur's second lemma states that if D is an irrep of the group $\mathcal{G}$ and there exists a matrix $\mathcal{A}$ which commutes with all the $\mathcal{D}_R$, i.e. $\mathcal{A}\mathcal{D}_R = \mathcal{D}_R\mathcal{A}$ for all $R \in \mathcal{G}$, then $\mathcal{A}$ is a multiple of the unit matrix, $\mathcal{A} = \lambda\mathbf{1}$.

> [$\mathcal{A}\mathcal{D}_R = \mathcal{D}_R\mathcal{A}$ for all $R \in \mathcal{G} \implies (\mathcal{A} - \lambda\mathbf{1})\mathcal{D}_{\mathbf{R}} = \mathcal{D}_{\mathbf{R}}(\mathcal{A} - \lambda\mathbf{1})$ for all $R \in \mathcal{G}$. This can be regarded as a special case of Schur's first lemma, for the case of equal dimensions, and implies that $\mathcal{A} - \lambda\mathbf{1}$ is either invertible or equal to zero. But $\det(\mathcal{A} - \lambda\mathbf{1}) = 0$, regarded as an equation for λ, has at least one solution (possibly complex). For such a λ, the matrix $\mathcal{A} - \lambda\mathbf{1}$ is not invertible, so it must vanish, implying $\mathcal{A} = \lambda\mathbf{1}$.]

Using Schur's lemmas, some remarkable orthogonality relations can be derived. Let D be an irrep of dimension d of the group $\mathcal{G}$ of order g. Construct the matrix $\mathcal{A} = \sum_{S\in\mathcal{G}} \mathcal{D}_S\mathcal{X}\mathcal{D}_{S^{-1}}$, where $\mathcal{X}$ is an arbitrary matrix. $\mathcal{A}$ commutes with $\mathcal{D}_R$ for every $R \in \mathcal{G}$.

> [$\mathcal{A}\mathcal{D}_R = \sum_{S\in\mathcal{G}} \mathcal{D}_S\mathcal{X}\mathcal{D}_{S^{-1}}\mathcal{D}_R = \sum_{S\in\mathcal{G}} \mathcal{D}_S\mathcal{X}\mathcal{D}_{S^{-1}R}$
> $= \sum_{S\in\mathcal{G}} \mathcal{D}_{RR^{-1}S}\mathcal{X}\mathcal{D}_{S^{-1}R} = \sum_{S\in\mathcal{G}} \mathcal{D}_R\mathcal{D}_{R^{-1}S}\mathcal{X}\mathcal{D}_{S^{-1}R}$
> $= \mathcal{D}_R\mathcal{A}$, using the rearrangement theorem.]

Therefore $\mathcal{A} = \lambda\mathbf{1}$. Choose the special matrix $\mathcal{X}_{ij} = \delta_{il}\delta_{jm}$, for certain fixed values of l, m (between 1 and d), and denote the corresponding λ by λ_{lm}.

Then $\sum_{S\in\mathcal{G}}(\mathcal{D}_S)_{il}(\mathcal{D}_{S^{-1}})_{mj} = \lambda_{lm}\delta_{ij}$. To evaluate λ_{lm}, set $i = j$ and sum over i, namely $\lambda_{lm}d = \sum_{S\in\mathcal{G}}(\mathcal{D}_{S^{-1}S})_{ml} = g\delta_{lm} \Longrightarrow \lambda_{lm} = g\delta_{lm}/d$. Finally, $\sum_{S\in\mathcal{G}}(\mathcal{D}_S)_{il}(\mathcal{D}_{S^{-1}})_{mj} = g\delta_{lm}\delta_{ij}/d$. For a unitary irrep, this can be rewritten $\sum_{S\in\mathcal{G}}(\mathcal{D}_S)_{il}(\mathcal{D}_S)^*_{jm} = g\delta_{lm}\delta_{ij}/d$.

To complete the result, consider two inequivalent irreps $\mathsf{D}^{(1)}, \mathsf{D}^{(2)}$. Then the matrix $\mathcal{A} = \sum_{S\in\mathcal{G}}\mathcal{D}^{(2)}_S\mathcal{X}\mathcal{D}^{(1)}_{S^{-1}}$, for any $\mathcal{X}$, satisfies $\mathcal{D}^{(2)}_R\mathcal{A} = \mathcal{A}\mathcal{D}^{(1)}_R$ for all $R \in \mathcal{G}$, which implies $\mathcal{A} = 0$. Choosing the same special $\mathcal{X}$ as above, $\sum_{S\in\mathcal{G}}(\mathcal{D}^{(2)}_S)_{il}(\mathcal{D}^{(1)}_{S^{-1}})_{mj} = 0$, which may be rewritten $\sum_{S\in\mathcal{G}}(D^{(2)}_S)_{il}(\mathcal{D}^{(1)}_S)^*_{jm} = 0$ for a unitary irrep. The two results may be combined to conclude that, for all inequivalent irreps $\mathsf{D}^{(\mu)}$, $\sum_{R\in\mathcal{G}}(\mathcal{D}^{(\mu)}_R)_{il}(\mathcal{D}^{(\nu)}_{R^{-1}})_{mj} = g\delta_{\mu\nu}\delta_{ij}\delta_{lm}/d_\mu$. Once again, for unitary irreps, this can be rewritten $\sum_{R\in\mathcal{G}}(\mathcal{D}^{(\mu)}_R)_{il}(\mathcal{D}^{(\nu)}_R)^*_{jm} = g\delta_{\mu\nu}\delta_{ij}\delta_{lm}/d_\mu$.

> [If μ is not the unit irrep, while ν is the unit irrep, this result implies $\sum_{R\in\mathcal{G}}(\mathcal{D}^{(\mu)}_R)_{ij} = 0$, a useful sum rule for non-unit irreps.]

The last result can be understood as an orthogonality relation for the g-dimensional vectors $(\mathcal{D}^{(\mu)}_R)_{ij}$, one vector for each choice of indices i, j, μ. There are $\sum_\mu d^2_\mu$ such mutually orthogonal g-dimensional vectors with non-zero norms, so $\sum_\mu d^2_\mu \le g$. There can be only a finite number of inequivalent irreps of a finite group and their dimensions are strongly limited.

In the orthogonality relation, it is possible to set $i = l, j = m$ and to sum over i, j to obtain an orthogonality relation for characters, $\sum_{R\in\mathcal{G}}\chi^{(\mu)}_R\chi^{(\nu)*}_R = g\delta_{\mu\nu}$. (This is the form appropriate for unitary irreps.) If $\mathcal{G}$ has n_c classes, with the k^{th} class containing g_k elements, this can be rewritten $\sum_{k=1}^{n_c} g_k\chi^{(\mu)}_k\chi^{(\nu)*}_k = g\delta_{\mu\nu}$.

> [As before, this result specialises to $\sum_k g_k\chi^{(\mu)}_k = 0$ if μ is not the unit irrep.]

Then, for given μ, $\{\sqrt{g_k}\chi^{(\mu)}_k\}$ form a non-zero n_c-dimensional vector. Inequivalent irreps provide mutually orthogonal vectors, so the number of inequivalent irreps is less than or equal to the number of classes.

Suppose a reducible representation D is decomposed into irreps as $\mathsf{D} = \sum_\mu a_\mu\mathsf{D}^{(\mu)}$, where the a_μ are non-negative integers. Taking the trace of this equation, $\chi_k = \sum_\mu a_\mu\chi^{(\mu)}_k \Longrightarrow \sum_k g_k\chi^{(\mu)*}_k\chi_k = ga_\mu \Longrightarrow a_\mu = \sum_k g_k\chi_k\chi^{(\mu)*}_k/g$. It follows that two representations with the same set of characters are equivalent, so the set of characters completely defines a representation. By a similar argument, $\sum_k g_k\chi_k\chi^*_k = g\sum_\mu a^2_\mu \ge g$ and $\sum_k g_k|\chi_k|^2 = g$ if and only if D is irreducible (since then only one a_μ can be non-zero and must equal 1). This is a practical and very useful criterion of reducibility.

It was shown earlier that the class operator $\mathcal{C}$, defined in the group algebra by the sum $\mathcal{C} = \sum R$ over all elements of the k^{th} class K_k of a group $\mathcal{G}$, commutes with all elements of the group. Given a representation $\mathsf{D}^{(\mu)}(\mathcal{G})$, the corresponding matrix $\mathcal{C} = \sum_{R\in\mathsf{K}_k} \mathcal{D}_R^{(\mu)}$ commutes with all the representation matrices $\mathcal{D}_R^{(\mu)}$. In the case of an irrep, by Schur's second lemma, this matrix must therefore be a multiple of the unit matrix, $\lambda_k^{(\mu)}\mathbf{1}$. Taking the trace of $\mathcal{C}$, it follows that $\lambda_k^{(\mu)} = g_k\chi_k^{(\mu)}/d_\mu$, where d_μ is the dimension of the irrep and g_k is the number of elements in K_k. The eigenvalues of the class operators can also serve to characterise an irrep.

An important representation, which has been previously mentioned, is the *regular representation*, where the elements of the group are looked at as basis vectors of the group algebra and the effect of a group element on a given basis vector is to transform it into a different basis vector (except for the identity element, which leaves every basis vector unchanged). The corresponding representation matrices have only 1 and 0 as entries, with every row and every column containing only a single 1. The identity element E is represented by the unit matrix, while all other representation matrices have only zeroes on the main diagonal. This is a g-dimensional representation, with characters $\chi_E = g, \chi_{R\neq E} = 0$. It satisfies $\sum_k g_k|\chi_k|^2 = g^2 > g$, so it is reducible. If it is decomposed into irreps, labelled μ, then $\chi_k = \sum_\mu a_\mu\chi_k^{(\mu)}$ for the k^{th} class. For the class of the identity, this reads $g = \sum_\mu a_\mu d_\mu$, where d_μ is the dimension of the irrep μ. But by the general result above, $a_\mu = \sum_k g_k\chi_k\chi_k^{(\mu)*}/g = d_\mu$, so that $g = \sum_\mu d_\mu^2$, turning the previously dervied inequality into an equality. This is a significant limitation on the number and dimensions of irreps of a group $\mathcal{G}$.

Suppose there are r inequivalent irreps of a group $\mathcal{G}$ having n_c classes. It was shown above, on the basis of the orthogonality relation for characters, that $r \leq n_c$. It is also possible to prove a second orthogonality relation for characters, namely $\sum_{\mu=1}^r \chi_i^{(\mu)}\chi_j^{(\mu)*} = g\delta_{ij}/g_i$, where g_k is the number of elements in the k^{th} class. For a given i, the $\{\chi_i^{(\mu)}\}$ form an r-dimensional vector with non-zero norm. There are n_c mutually orthogonal such vectors, so $n_c \leq r$. Hence $r = n_c$ — the number of inequivalent irreps of a group is equal to the number of classes it has. As a corollary to this result, it follows that all irreps of an Abelian group are one-dimensional.

[In an Abelian group, each element is a class in itself, so $n_c = g$.

Since $\sum_{\mu=1}^{n_c} d_\mu^2 = g$, $d_\mu = 1$ for all μ.]

Let $\mathcal{H}$ be an invariant subgroup of $\mathcal{G}$. There is a natural homomorphism of $\mathcal{G}$ to the factor group $\mathcal{G}/\mathcal{H}$ — each element $R \in \mathcal{G}$ is mapped to its coset $R\mathcal{H} \in \mathcal{G}/\mathcal{H}$. Any irrep of the factor group $\mathcal{G}/\mathcal{H}$ is an irrep of the group $\mathcal{G}$ via this homomorphism.

If a group $\mathcal{G}$ is a direct product of two groups $\mathcal{G}_1, \mathcal{G}_2$, so that $R \in \mathcal{G}$ can be written $R_1R_2, R_i \in \mathcal{G}_i$, then the irreps of $\mathcal{G}$ are given by the irreps of $\mathcal{G}_1, \mathcal{G}_2$, the basis of the former being the direct product of the bases of the latter. The representation matrices are $(\mathcal{D}^{(\mu,\nu)}_{R_1R_2})_{ij,kl} = (\mathcal{D}^{(\mu)}_{R_1})_{ik}(\mathcal{D}^{(\nu)}_{R_2})_{jl}$ and the characters are $\chi^{(\mu,\nu)}_{R_1R_2} = \chi^{(\mu)}_{R_1}\chi^{(\nu)}_{R_2}$. The classes of the direct product group are the products of classes of the component groups.

It has already been seen that any vector ψ on which the elements of a group $\mathcal{G}$ of order g can act will generate an invariant subspace of dimension at most g through $\{\mathsf{D}_R\psi, R \in \mathcal{G}\}$. This subspace will carry a representation of $\mathcal{G}$, which is generally reducible. By an appropriate sequence of basis changes, the representation can be fully reduced to block diagonal form, in which each submatrix along the diagonal is part of an irrep. The original vector ψ can be expressed in terms of the new basis as $\psi = \sum_{\mu,i} \phi_i^{(\mu)}$, where $\phi_i^{(\mu)}$ is a function belonging to the i^{th} row of the irrep μ, i.e. $\mathsf{D}_R\phi_i^{(\mu)} = \sum_j \mathcal{D}^{(\mu)}_{ji}\phi_j^{(\mu)}$. So the set of basis vectors for the irreps of $\mathcal{G}$ is complete — any vector can be expressed in terms of it. The operator $P_i^{(\mu)} = d_\mu \sum_{R\in\mathcal{G}} (\mathcal{D}^{(\mu)}_R)^*_{ii}\mathsf{D}_R/g$ is a projection operator on functions belonging to the i^{th} row of the irrep $\mathsf{D}^{(\mu)}$, while $P^{(\mu)} = d_\mu \sum_{R\in\mathcal{G}} \chi_R^{(\mu)*}\mathsf{D}_R/g$ is a projection operator on functions belonging to the irrep $\mathsf{D}^{(\mu)}$.

If D is an irrep of $\mathcal{G}$, then so are $\overline{\mathsf{D}}$ (the *conjugate* or *adjoint* irrep) and D^* (the *complex conjugate* irrep), in which the representation matrices are respectively the transpose of the inverse or the complex conjugate of the original representation matrices of D. For a unitary irrep D, $\overline{\mathsf{D}} = \mathsf{D}^*$. Note that $\overline{\chi}_R = \chi_{R^{-1}}$. There are three possibilities — D can be made real, or D is equivalent to D^* but cannot be made real, or D is inequivalent to D^*. The first kind of representation is called an *integer* representation and has real characters; the second is called a *half-integer* representation and has real characters; while the third kind of representation has complex characters (not real). It can be shown that the number of irreps with real characters is equal to the number of classes which contain the inverses of all their elements. (These are known as *ambivalent* classes.) Since permutations and their inverses have the same cycle structure, all classes of S_n are ambivalent, so all irreps of S_n have only real characters. It can be proved that $\sum_{R\in\mathcal{G}} \chi_{R^2} = cg$, where $c = +1, -1, 0$ respectively for the three kinds of representations listed above. The trivial unit irrep is always an integer representation.

Examples

1. Consider the cyclic group $C_2 = \{E, A\}$ of order $g = 2$. It is Abelian, so it has 2 irreps, each of dimension one. One is the trivial unit irrep, $\chi_E^{(1)} = 1, \chi_A^{(1)} = 1$. The other must be orthogonal to this irrep, so it must be $\chi_E^{(2)} = 1, \chi_A^{(2)} = -1$. Both are integer irreps.

2. Consider the cyclic group $C_3 = \{E, A, A^2\}$ of order $g = 3$, also Abelian, with three one-dimensional irreps. The unit irrep is $\chi_E^{(1)} = 1, \chi_A^{(1)} = 1, \chi_{A^2}^{(1)} = 1$. The other two must be orthogonal to this irrep and to one another. Since the irreps are one-dimensional and multiplication is preserved in a representation, $\chi_{A^2} = (\chi_A)^2$ and $\chi_A = \chi_{A^4} = (\chi_{A^2})^2$ for all irreps. So $(\chi_A)^3 = 1$. Denote $\epsilon = e^{2\pi i/3}$. Then the remaining irreps are $\chi_E^{(2)} = 1, \chi_A^{(2)} = \epsilon, \chi_{A^2}^{(2)} = \epsilon^2$ and $\chi_E^{(3)} = 1, \chi_A^{(3)} = \epsilon^2, \chi_{A^2}^{(3)} = \epsilon$. They are both complex representations. (Note that $\epsilon^* = \epsilon^2$.)

3. Consider the group of symmetries of the equilateral triangle, to be denoted $\mathcal{T} = \{1, r_1, r_2, m_1, m_2, m_3\}$, of order $g = 6$. It has three classes, $\{1\}$,$\{r_1, r_2\}$ and $\{m_1, m_2, m_3\}$, hence three irreps. One is the unit irrep, $\chi(1)_R = 1$ for all $R \in \mathcal{T}$, so the remaining two satisfy $d_2^2 + d_3^2 = 5$. The only possible solution is $d_2 = 1, d_3 = 2$, so $\chi_1^{(2)} = 1, \chi_1^{(3)} = 2$. The subgroup $\mathcal{R} = \{1, r_1, r_2\}$ is invariant, with cosets $\{1, r_1, r_2\}, \{m_1, m_2, m_3\}$, and the factor group is C_2, whose irreps are known. By the natural homomorphism, the second irrep of $\mathcal{T}$ is seen to be $\chi_1^{(2)} = 1, \chi_2^{(2)} = 1, \chi_3^{(2)} = -1$. By orthogonality, the last irrep is deduced to be $\chi_1^{(3)} = 2, \chi_2^{(3)} = -1, \chi_3^{(3)} = 0$. All three irreps are integer representations.

> [Recall that $\mathcal{T}$ is isomorphic to the symmetric group on 3 objects, $\mathcal{S}_3$. The second one-dimensional irrep can be recognised as the alternating irrep of $\mathcal{S}_3$. Vectors or functions transforming under $\mathcal{S}_n$ as the unit irrep are called *totally symmetric*; those transforming as the alternating irrep are called *totally antisymmetric*; those transforming as any other (higher-dimensional) irrep are said to be of *mixed symmetry*.]

Irreps are frequently displayed in *character tables*, where the columns are labelled by the classes of the group (and the number of elements in each class is often noted) and each row corresponds to a different irrep. The entries in the resulting matrix are the characters of the classes in each irrep. The above three examples have the character tables:

C_2 class	$\{E\}$	$\{A\}$
no. of elements	1	1
	1	1
	1	-1

C_3 class	$\{E\}$	$\{A\}$	$\{A^2\}$
no. of elements	1	1	1
	1	1	1
	1	$\epsilon = e^{2\pi i/3}$	ϵ^2
	1	ϵ^2	ϵ

$\mathcal{S}_3$ class	$\{1\}$	$\{r_1, r_2\}$	$\{m_1, m_2, m_3\}$
no. of elements	1	2	3
	1	1	1
	1	1	-1
	2	-1	0

In checking the orthogonality of characters, it is important to remember the g_k weighting of the columns and the complex conjugation of complex characters.