5a. Representations of groups – supplement

(This is a slightly expanded version of a paragraph on p.5 of chapter 5, dealing with carrier spaces of vectors dependent on arguments acted on by the group elements $R_i \in \mathcal{G}$.)

The vectors $\{f\}$ carrying a representation of a group $\mathcal{G}$ with elements $\{R_i\}$ may be functions of arguments $\{\boldsymbol{r}\}$ on which the group elements act. The transformed function $[\mathsf{D}_R f](\boldsymbol{r})$ is defined to have the same value for argument $\boldsymbol{r}$ as the original function $f(\boldsymbol{r})$ had for the argument that was transformed into $\boldsymbol{r}$, i.e. $[\mathsf{D}_R f](\boldsymbol{r}) = f(R^{-1}\boldsymbol{r})$.

Consider the action of the product of group elements R_1R_2 on the carrier space $\{f(\boldsymbol{r})\}$. A given vector $f(\boldsymbol{r})$, acted upon by R_2, is transformed into the vector $f'(\boldsymbol{r}) = [\mathsf{D}R_2 f](\boldsymbol{r}) = f(R_2^{-1}\boldsymbol{r})$. Action by R_1 transforms the resulting vector into $[\mathsf{D}R_1 f'](\boldsymbol{r}) = f'(R_1^{-1}\boldsymbol{r})$.

But $f'(\boldsymbol{s}) = f(R_2^{-1}\boldsymbol{s})$ for any argument $\boldsymbol{s}$, so $f'(R_1^{-1}\boldsymbol{r}) = f(R_2^{-1}R_1^{-1}\boldsymbol{r})$. Recall $R_2^{-1}R_1^{-1} = (R_1R_2)^{-1}$ and, because D is a homomorphism, $\mathsf{D}_{R_1}\mathsf{D}_{R_2} = \mathsf{D}_{R_1R_2}$.

Finally, $[\mathsf{D}_{R_1R_2} f](\boldsymbol{r}) = [\mathsf{D}_{R_1}[\mathsf{D}_{R_2} f]](\boldsymbol{r})$ and multiplication is preserved.