13. Representations of Lie algebras

As in the case of finite groups, there is considerable interest and importance in the subject of representations of continuous groups and their associated algebras. It is again necessary to define the action of the group or the algebra on an appropriate vector space, the *carrier space*. This may be a space of vectors (or states) $\{\phi\}$, in which case the action is of the form $\mathsf{D}\phi$, or a space of operators $\{\mathcal{O}\}$, in which case the action is of the form $[\mathsf{D}, \mathcal{O}]$, where D is the operator corresponding to a generator of the algebra.

> [It was shown that group elements take the form $e^{\sum_k \gamma_k \mathsf{M}_k}$, in terms of the parameters $\{\gamma\}$ and generators $\{\mathsf{M}\}$, and that operators transform as $R\mathcal{O}R^{-1}$ under the action of the group element R. For an infinitesimal transformation,$e^{\epsilon \mathsf{M}}\phi = (\mathbf{1} + \epsilon\mathsf{M})\phi = \phi + \epsilon\mathsf{M}\phi$, to lowest order in ϵ. The change in ϕ is given by $\mathsf{M}\phi$. Similarly, $e^{\epsilon\mathsf{M}}\mathcal{O}e^{-\epsilon\mathsf{M}} = (\mathbf{1} + \epsilon\mathsf{M})\mathcal{O}(\mathbf{1} - \epsilon\mathsf{M}) = \mathcal{O} + \epsilon[\mathsf{M}, \mathcal{O}]$, to lowest order in ϵ. The change in $\mathcal{O}$ is given by $[\mathsf{M}, \mathcal{O}]$.]

Recall that a representation is a homomorphism from the group or algebra to a group or algebra of operators on the carrier space of the representation, which preserves the multiplication or Lie product. For a d-dimensional carrier space, there will be a matrix representation in terms of $d \times d$ matrices. The representation is *irreducible* if the carrier space has no subspace invariant under the action of the elements of the group or the generators of the algebra. When it is possible to realise the group elements by exponentiation of the generators of the algebra, then irreducible representations of the group are obtained by exponentiation of the irreducible representations of the algebra. It is therefore reasonable to concentrate on representations of Lie algebras.

Within the Cartan-Weyl basis, any semi-simple Lie algebra is spanned by the generators of the Cartan subalgebra, denoted $\{H_i\}$, and pairs of root vectors $\{E_\alpha, E_{-\alpha}\}$, one pair for each positive root α of the algebra. The different H_i's commute with one another. (The same notation will be used for these generators of the algebra and for the operators on the carrier space which represent them.)

It is thus possible to find in the carrier space vectors which are simultaneous eigenvectors of all the H_i. These vectors are called *weight vectors* and the set of eigenvalues of the H_i associated with a weight vector is called a *weight*. It is clear that the weights, being linear functionals on the Cartan sub-algebra, belong to the root space of the algebra and can be expressed in terms of the simple roots, which form a basis for the root space.

> [Although the Cartan subalgebra $\mathcal{H}$ is uniquely defined, there is great flexibility in the choice of the basis $\{H_i\}$ of $\mathcal{H}$. The numerical values of the components of weights (or of roots) depend on the specific choice of basis for $\mathcal{H}$, but any quantities defined through scalar products, such as Dynkin indices, are independent of the choice of basis. Any convenient set of ℓ linearly independent ℓ-tuples can serve as the simple roots of the algebra, since the basis of $\mathcal{H}$ can always be transformed to produce those ℓ-tuples as eigenvalues of $\{H_i\}$. The choice is often made to accord with a particular physical interpretation. This is referred to as *standardising* the Cartan subalgebra.]

Let M be a weight associated with the weight vector $\phi \in \mathcal{V}$, where $\mathcal{V}$ is the carrier space. [Do not confuse the weights M with the infinitesimal generators M.] It is defined by the set of eigenvalues $\{M_i\}$ of the basis $\{H_i\}$ of the Cartan subalgebra $\mathcal{H}$. Then $E_\alpha\phi$ is a weight vector with weight $M+\alpha$, unless $E_\alpha\phi = 0$.

> [$H_i E_\alpha\phi = [H_i, E_\alpha]\phi + E_\alpha H_i\phi = \alpha_i E_\alpha\phi + E_\alpha M_i\phi = (M_i+\alpha_i)E_\alpha\phi$.]

For a carrier space of operators, the weight M is defined by eigenvalues determined via $[H_i, \mathcal{O}] = M_i\mathcal{O}$ for the weight vector $\mathcal{O}$. Then $[E_\alpha, \mathcal{O}]$ is a weight vector with weight $M + \alpha$, unless it vanishes.

> [$[H_i, [E_\alpha, \mathcal{O}]] = [[H_i, E_\alpha], \mathcal{O}] + [E_\alpha, [H_i, \mathcal{O}]] = \alpha_i[E_\alpha, \mathcal{O}] + M_i[E_\alpha, \mathcal{O}]$, where the Jacobi identity has been used.]

Repeated applications of E_α will generate an α-string of weights $M + k\alpha$. Similarly, repeated applications of $E_{-\alpha}$ will generate further weights $M - k\alpha$ in the string. For a finite-dimensional representation, such a string must terminate at both ends, at $M + p\alpha$ and at $M - m\alpha$. It is easy to see that the same arguments as were applied to the discussion of strings of roots will lead to the conclusion that the "magic formula" applies equally well to strings of weights. It may be concluded that $m - p = 2M \cdot \alpha/\alpha \cdot \alpha$. It also follows, again, that if a weight M is expanded in terms of simple roots, the expansion coefficients will be rational.

Starting from any weight M, additional weights may be produced by generating α-strings of weights, for any α. Further weights can be produced by generating α-strings from each of these weights, and so on. For a finite-dimensional representation, this process must terminate at some stage. The resulting set of weight vectors spans a minimal invariant space, which generates an irreducible representation. This space must contain a *highest weight*, one which satisfies $E_\alpha \phi_M = 0$ for all $\alpha > 0$. In fact, it is clearly sufficient to restrict the process to the simple roots.

For the state of highest weight, $p = 0$ in the magic formula, so all the Dynkin indices of this weight are non-negative. In fact, Dynkin showed that every weight with non-negative integer Dynkin indices is the highest weight of an irreducible representation. Both the highest weight and the irrep it determines will be denoted by Λ.

The Cartan matrix again provides a tool for generating all the weights in an irrep from the highest weight. Additional weights can be generated by subtracting simple roots. This is permitted provided the value of the corresponding m in the magic formula is positive. At the initial step, starting with the highest weight, $p = 0$ for all the simple roots, so m is given by the Dynkin indices of the highest weight. Those simple roots for which the corresponding coefficient is positive can be subtracted from the highest weight to get new weights. The Dynkin indices of the new weights are obtained by subtracting the appropriate row of the Cartan matrix from the indices of the previous weight. At each stage, the value of p for each simple root is known by inspection of the previous steps, so m is given by the sum of p and the Dynkin indices of the current weight. The process terminates when $m = 0$ for every simple root.

Any weight in an irrep labeled by the highest weight Λ can be written $\Lambda - \sum_i k_i \alpha^{(i)}$ in terms of the simple roots $\{\alpha^{(i)}\}$, where the $\{k_i\}$ are non-negative integers. The quantity $\sum_i k_i$ for a given weight is called the weight *level*. The highest weight has level 0.

[As an **illustrative example**, consider the $\Lambda = (0, 1)$ representation of G_2, which has the Cartan matrix $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$. The starting weight is $\Lambda = (0, 1)$, for which $p = (0, 0)$, so that $m = (0, 1)$. Since $m_2 > 0$, it is possible to generate another weight by subtracting $\alpha^{(2)}$. This gives $\Lambda - \alpha^{(2)}$, with Dynkin indices $(1, -1)$, obtained by subtracting the second row of the Cartan matrix from the Dynkin indices of Λ. Since a weight can be obtained by adding $\alpha^{(2)}$ but not by adding $\alpha^{(1)}$, this weight has $p = (0, 1)$ and hence, adding to this its Dynkin indices, $m = (1, 0)$. Another weight is now obtained by subtracting $\alpha^{(1)}$ (since $m_1 > 0$)

to get $\Lambda - \alpha^{(1)} - \alpha^{(2)}$, with Dynkin indices $(-1, 2)$. This weight has $p = (1, 0)$, so $m = (0, 2)$ and $\alpha^{(2)}$ can again be subtracted, producing $\Lambda - \alpha^{(1)} - 2\alpha^{(2)}$ with Dynkin indices $(0, 0)$. Here $p = (0, 1) \Longrightarrow m = (0, 1)$, so the next weight is $\Lambda - \alpha^{(1)} - 3\alpha^{(2)}$, with Dynkin indices $(1, -2)$. Now $p = (0, 2) \Longrightarrow m = (1, 0)$ and the next weight is $\Lambda - 2\alpha^{(1)} - 3\alpha^{(2)}$ with Dynkin indices $(-1, 1)$. At this stage, $p = (1, 0) \Longrightarrow m = (0, 1)$, so the next weight is $\Lambda - 2\alpha^{(1)} - 4\alpha^{(2)}$, Dynkin indices $(0, -1)$. Finally, $p = (0, 1) \Longrightarrow m = (0, 0)$ and the process terminates. This irrep has seven distinct weights, at levels 0,1,. . . ,6.]

The process may again be structured as outlined below.

$$\mathsf{A} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

<u>level 0</u>

$$\Lambda$$
$$\Delta = (0,1)$$
$$p = (0,0)$$
$$m = (0,1)$$

↘ (2)

<u>level 1</u>

$$\Lambda - \alpha^{(2)}$$
$$\Delta = (1,-1)$$
$$p = (0,1)$$
$$m = (1,0)$$

(1) ↙

<u>level 2</u>

$$\Lambda - \alpha^{(1)} - \alpha^{(2)}$$
$$\Delta = (-1,2)$$
$$p = (1,0)$$
$$m = (0,2)$$

↘ (2)

<u>level 3</u>

$$\Lambda - \alpha^{(1)} - 2\alpha^{(2)}$$
$$\Delta = (0,0)$$
$$p = (0,1)$$
$$m = (0,1)$$

↘ (2)

<u>level 4</u>

$$\Lambda - \alpha^{(1)} - 3\alpha^{(2)}$$
$$\Delta = (1,-2)$$
$$p = (0,2)$$
$$m = (1,0)$$

(1) ↙

<u>level 5</u>

$$\Lambda - 2\alpha^{(1)} - 3\alpha^{(2)}$$
$$\Delta = (-1,1)$$
$$p = (1,0)$$
$$m = (0,1)$$

↘ (2)

<u>level 6</u>

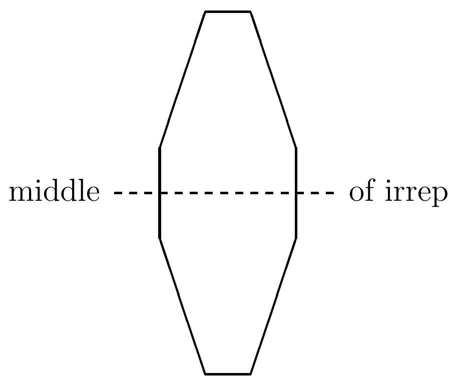
$$\Lambda - 2\alpha^{(1)} - 4\alpha^{(2)}$$
$$\Delta = (0,-1)$$
$$p = (0,1)$$
$$\underline{m = (0,0)}$$

The use of the Cartan matrix and the "magic formula" to determine the weights of an irrep by stepping down from the highest weight is essentially the inverse of the use of the same ingredients to establish the roots of an algebra by building up from the simple roots. In fact, the roots of the algebra are the weights of the adjoint representation. The last positive root found by the building-up process is the highest weight of the adjoint representation and its Dynkin indices label the adjoint representation.

However, if the process is reversed and the weights of the adjoint representation are generated from the highest weight, all the roots are produced — positive, negative and zero. It will be found that the zero root is produced only once, regardless of the rank of the algebra, i.e. the degeneracy of the zero root. This is a symptom of a general problem with the "step-down" method for finding the weights. While it does produce all the weights in an irrep, it gives no information about their degeneracy. Though the non-zero weights in the adjoint representation are non-degenerate, this is not generally true of the weights in other irreps.

Each weight belongs to a *weight space* whose dimension is the degeneracy of the weight and must be separately determined. Two useful rules are that the highest weight in an irrep is always non-degenerate, and that any weight which can be generated in only one way by the "step-down" procedure (i.e. there is only one path from the highest weight to the given weight) is also non-degenerate.

A helpful property of the irreps is that they are "spindle-shaped" —



— when built up step by step from the highest weight down, they cannot shrink in width until the middle of the irrep is reached (i.e. each level must have at least as many weights, including degeneracy, as the previous level) and they must be symmetric about the middle of the irrep, in terms of number of weights (including degeneracy) at each level. This can be a check on the determination of degeneracies.

Any weight can be expressed in terms of the simple roots, which form a basis for the root space in which the weights reside. For a given weight M, its expansion in terms of simple roots can be written $M = \sum_i a_i \alpha^{(i)}$, where the coefficients $\{a_i\}$ are to be determined. The Dynkin indices of the weight are $\Delta_i^M = 2M \cdot \alpha^{(i)}/\alpha^{(i)} \cdot \alpha^{(i)} = \sum_j a_j \mathsf{A}_{ji}$, from which $a_i = \sum_j \Delta_j^M (\mathsf{A}^{-1})_{ji}$. The inverse of the Cartan matrix may be used to obtain the expression for any weight in terms of simple roots, if desired. Since $M = \sum_{ij} \Delta_i^M (\mathsf{A}^{-1})_{ij} \alpha^{(j)}$, scalar products can be evaluated directly in terms of Dynkin indices, with the aid of the inverse A^{-1} of the Cartan matrix.

[$M \cdot N = \sum_{ijkl} \Delta_i^M (\mathsf{A}^{-1})_{ij} \Delta_k^N (\mathsf{A}^{-1})_{kl} \alpha^{(j)} \cdot \alpha^{(l)}$. But $\alpha^{(j)} \cdot \alpha^{(l)} = \mathsf{A}_{lj} \alpha^{(j)} \cdot \alpha^{(j)}/2$, so $M \cdot N = \sum_{ij} \Delta_i^M (\mathsf{A}^{-1})_{ij} \Delta_j^N \alpha^{(j)} \cdot \alpha^{(j)}/2$.]

It was shown that each Lie algebra has a Casimir operator which commutes with all the generators of the algebra. Though the definition of this operator was purely formal in the context of the algebra (since it involves the simple product of two generators, which is not generally defined), it becomes well-defined in the context of representations, where the simple product of two operators or of two matrices is clearly defined. The resulting operator or matrix commutes with all the representatives of the generators and hence, by Schur's lemma, is a multiple of the unit matrix in any irrep. Its eigenvalue characterizes the irrep and can be expressed directly in terms of the highest weight.

The Casimir operator is given by $\mathsf{C} = \sum_{\rho,\sigma=1}^{d} g^{\rho\sigma} X_\rho X_\sigma$, where d is the dimension of the algebra and X_ρ are its generators. In the Cartan-Weyl basis, this becomes $\mathsf{C} = \sum_{i,j=1}^{\ell} g^{ij} H_i H_j + \sum_{\alpha>0} (E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha)/g_{\alpha,-\alpha}$. But $[E_\alpha, E_{-\alpha}] = g_{\alpha,-\alpha} H_\alpha$, so $\mathsf{C} = \sum_{i,j=1}^{\ell} g^{ij} H_i H_j + \sum_{\alpha>0} H_\alpha + 2 \sum_{\alpha>0} E_{-\alpha} E_\alpha / g_{\alpha,-\alpha}$. When acting on the state of highest weight Λ, each H_i produces a factor Λ_i, while each $E_{\alpha>0}$ produces zero, by the definition of highest weight. Recalling that $H_\alpha = \sum_{i,j} g^{ij} \alpha_i H_j$, it is seen that the eigenvalue of the Casimir operator C for the state of highest weight Λ, and hence for every state in the irrep, is $\mathsf{c} = (\Lambda + 2\delta) \cdot \Lambda$, where the quantity δ is half the sum of the positive roots of the algebra, $\delta = \sum_{\alpha>0} \alpha/2$.

The eigenvalue of the Casimir operator is easily evaluated when the Cartan matrix and the Dynkin indices of the highest weight Λ are known, provided the Dynkin indices of the quantity $\delta = \sum_{\alpha>0} \alpha$ are also known. But the Dynkin indices of δ are $(1, 1, \ldots, 1)$ for every semi-simple Lie algebra.

[The Weyl reflection S_α, where α is a root, maps roots into roots. For any root β, it is defined by $S_\alpha\beta = \beta - 2(\beta\cdot\alpha)\alpha/(\alpha\cdot\alpha)$. By inspection, $S_{-\alpha} = S_\alpha$ and $S_\alpha\alpha = -\alpha$. Also, $S_\alpha^2\beta = S_\alpha\beta - 2(\beta\cdot\alpha)S_\alpha\alpha/(\alpha\cdot\alpha) = \beta \Longrightarrow S_\alpha^2 = \mathbf{1}$. It can be seen easily by direct evaluation that $S_\alpha\beta\cdot S_\alpha\gamma = \beta\cdot\gamma$, i.e. the Weyl reflection is unitary.

For any positive root $\alpha > 0$ and any simple root $\alpha^{(i)}$, the root $S_{\alpha^{(i)}}\alpha > 0$, for $\alpha \neq \alpha^{(i)}$.

[$\alpha = \sum_i k_i\alpha^{(i)}$, where the $\{k_i\}$ are non-negative integers. Then $S_{\alpha^{(i)}}\alpha = \sum_j k_j(\alpha^{(j)} - 2(\alpha^{(j)}\cdot\alpha^{(i)})\alpha^{(i)}/(\alpha^{(i)}\cdot\alpha^{(i)})) = \alpha^{(i)}(k_i - \sum_j 2(\alpha^{(j)}\cdot\alpha^{(i)})/(\alpha^{(i)}\cdot\alpha^{(i)})k_j) + \sum_{j\neq i} k_j\alpha^{(j)}$. But, for $\alpha \neq \alpha^{(i)}$, there must be some $j \neq i$ for which $k_j \neq 0$, i.e. $k_j > 0$. However, the expansion coefficients of a root in terms of simple roots are either all positive, for a positive root, or all negative, for a negative root. Therefore, $S_{\alpha^{(i)}}\alpha$ is a positive root.]

The images under $S_{\alpha^{(i)}}$ of different positive roots α are different.

[Since $S_\alpha^2 = \mathbf{1}$, it follows that $S_\alpha\beta = S_\alpha\gamma \Longrightarrow \beta = \gamma$.]

So $S_{\alpha^{(i)}}\delta = -\alpha^{(i)}/2 + \sum_{0<\alpha\neq\alpha^{(i)}}\alpha/2 = -\alpha^{(i)}/2 + \delta - \alpha^{(i)}/2 = \delta - \alpha^{(i)}$. Finally, $\delta\cdot\alpha^{(i)} = S_{\alpha^{(i)}}\delta\cdot S_{\alpha^{(i)}}\alpha^{(i)} = (\delta - \alpha^{(i)})\cdot(-\alpha^{(i)}) \Longrightarrow 2\delta\cdot\alpha^{(i)} = \alpha^{(i)}\cdot\alpha^{(i)} \Longrightarrow \Delta_i^\delta = 1$ for all i.]

In terms of the Dynkin indices which label the irrep Λ, the eigenvalue of the Casimir operator is thus $\mathsf{c}_\Lambda = (\Lambda + 2\delta)\cdot\Lambda = \sum_{ij}(\Delta_i^\Lambda + 2)(\mathsf{A}^{-1})_{ij}\Delta_j^\Lambda(\alpha_j\cdot\alpha_j)/2$.

Knowledge of the eigenvalue of the Casimir operator permits the derivation of a closed formula for the dimension d_M of the weight space associated with any weight M, i.e. for the degeneracy of the weight, in terms of the dimensions of weight spaces of higher weight. This *Freudenthal recursion formula* reads $d_M = \sum_{\alpha>0}\sum_{k=1}^{\infty} 2d_{M+k\alpha}(M + k\alpha)\cdot\alpha/(\Lambda + M + 2\delta)\cdot(\Lambda - M)$, where the inner sum is over all weights which can be reached from the weight M by adding multiples of the root α. In order to apply this formula, it is clearly necessary to know all the weights higher than M (which are automatically supplied by the "step-down" procedure) and all the positive roots, both in terms of simple roots.

Using these results, Weyl derived the *Weyl dimension formula* for the dimension d_Λ of an irrep Λ of a semi-simple Lie algebra, $d_\Lambda = \prod_{\alpha>0}(1 + \alpha\cdot$

$\Lambda/\alpha\cdot\delta)$. If the positive root $\alpha=\sum_i k_i^{(\alpha)}\alpha^{(i)}$, where the $k_i^{(\alpha)}$ are non-negative integers and the $\alpha^{(i)}$ are the simple roots, and if the Dynkin indices Δ_i^Λ of the irrep are known, then $\alpha\cdot\Lambda=\sum_i k_i^{(\alpha)}\Delta_i^\Lambda(\alpha^{(i)}\cdot\alpha^{(i)})/2$ and the formula becomes $d_\Lambda=\prod_{\alpha>0}(1+\sum_i k_i^{(\alpha)}\Delta_i^\Lambda\alpha^{(i)}\cdot\alpha^{(i)}/\sum_i k_i^{(\alpha)}\alpha^{(i)}\cdot\alpha^{(i)})$.

Examples

1. Consider the Lie algebra $\mathcal{A}_1$, corresponding to the matrix group $\mathcal{SU}(2)$. This is a rank-1 algebra, so the Cartan matrix is a single number, $\mathsf{A}=2$, as is its inverse, $\mathsf{A}^{-1}=1/2$.

Suppose an irrep is specified by the single Dynkin coefficient λ. In terms of the single simple root α, the highest weight is $\lambda\alpha/2$. Since $p=0$ for the highest weight, $m=p+\Delta=\lambda>0$, in general, so a lower weight can be obtained by subtracting α. This produces the weight $(\lambda/2-1)\alpha$ with Dynkin coefficient $\lambda-2$ (obtained by subtracting A from the previous Dynkin coefficient). For this weight, $p=1$, so $m=\lambda-1>0$, for large enough λ.

Again subtracting α produces the weight $(\lambda/2-2)\alpha$, with Dynkin coefficient $\lambda-4$. The process proceeds through the weights $(\lambda/2-k)\alpha$ with Dynkin indices $\lambda-2k$, for which $p=k, m=\lambda-k$, and terminates at the lowest weight, $-\lambda\alpha/2$, with Dynkin coefficient $-\lambda$, for which $p=\lambda, m=0$.

All the weights are non-degenerate, since each can be reached from the highest weight by only one path, and there are clearly $\lambda+1$ of them, which is the dimension of the irrep. If the representation is standardised by setting $\alpha=1$ (i.e. the Cartan subalgebra is scaled so that $\alpha=1$), then the $\lambda+1$ weights range from $\lambda/2$ to $-\lambda/2$ in unit steps. The usual notation is obtained by writing $\lambda=2j$.

2. Consider the Lie algebra $\mathcal{A}_2$, corresponding to the matrix group $\mathcal{SU}(3)$. This rank-2 algebra has two simple roots, denoted $\alpha^{(1)},\alpha^{(2)}$, and a third positive root, $\alpha^{(1)}+\alpha^{(2)}$. The simple roots are of equal length.

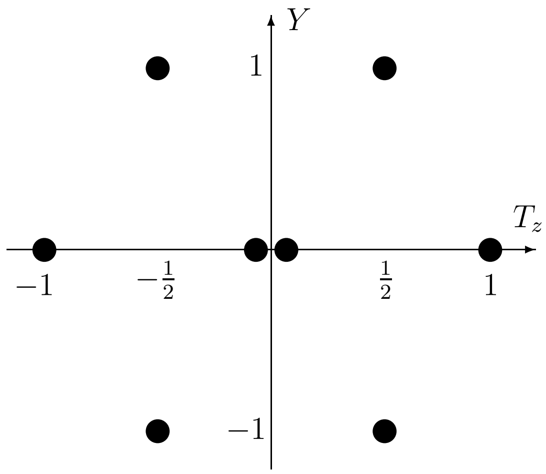
Let the irreps be denoted (λ,μ), the Dynkin indices of the corresponding highest weight. From the Weyl dimension formula, the corresponding dimension is $d_{\lambda,\mu}=(1+\lambda)(1+\mu)(1+(\lambda+\mu)/2)=(\lambda+1)(\mu+1)(\lambda+\mu+2)/2$. The Cartan matrix is $\begin{pmatrix}2&-1\\-1&2\end{pmatrix}$, its inverse $\frac{1}{3}\begin{pmatrix}2&1\\1&2\end{pmatrix}$.

The irrep $(1,0)$, of dimension 3, has the highest weight $(2\alpha^{(1)}+\alpha^{(2)})/3$, for which $p=(0,0)$ and $m=(1,0)$. Subtracting $\alpha^{(1)}$ produces the weight

$(-\alpha^{(1)} + \alpha^{(2)})/3$ with Dynkin indices $(-1, 1)$, for which $p = (1, 0)$ and $m = (0, 1)$. Subtracting $\alpha^{(2)}$ produces the weight $-(\alpha^{(1)} + 2\alpha^{(2)})/3$ with Dynkin indices $(0, -1)$, for which $p = (0, 1)$ and $m = (0, 0)$, so the process terminates.

One well-known application of $\mathcal{SU}(3)$ is to light-flavour quarks, in which the quarks u, d, s form a fundamental triplet. They have the quantum numbers isospin T_z and hypercharge Y, with the values $u\,[1/2, 1/3]$, $d\,[-1/2, 1/3]$ and $s\,[0, -2/3]$, where isospin refers to an $\mathcal{SU}(2)$ subgroup. If isospin is identified with the $\mathcal{SU}(2)$ associated with the simple root $\alpha^{(1)}$, then the weight $(1, 0)$ is identified with u and the weight $(-1, 1)$ with d. Thus, $(2\alpha^{(1)} + \alpha^{(2)})/3 = [1/2, 1/3]$, $(-\alpha^{(1)} + \alpha^{(2)})/3 = [-1/2, 1/3] \Longrightarrow \alpha^{(1)} = [1, 0]$, $\alpha^{(2)} = [-1/2, 1]$, which standardises the Cartan subalgebra.

The weights of the adjoint representation $(1, 1)$, which are also the roots of the algebra, have Dynkin indices (obtained by the “step-down” procedure) $(1, 1)$, $(-1, 2)$, $(2, -1)$, $(0, 0)$(twice), $(-2, 1)$, $(1, -2)$, $(-1, -1)$. Expressed in terms of the standardised simple roots, these become the well-known octuplet $[1/2, 1]$, $[-1/2, 1]$, $[1, 0]$, $[0, 0,]^2$, $[-1, 0]$, $[1/2, -1]$, $[-1/2, -1]$.



The anti-quarks belong to the conjugate triplet $(0, 1)$.