11. Roots of Lie algebras

With the scalar product defined in terms of the inverse of the Killing form on the Cartan subalgebra, it can be proved that $\alpha \cdot \alpha > 0$, for any root α. It has been shown that, for a semi-simple Lie algebra, if α is a root, then $-\alpha$ is also a root. It can also be shown that 2α is not a root.

> [The obvious result $[E_\alpha, E_\alpha] = 0$ is consistent with this statement, but is not sufficient to prove it, since it is conceivable that $N_{\alpha\alpha} = 0$ but $[E_{\alpha+\beta}, E_{\alpha-\beta}] \neq 0$.]

The only multiples of a root α which are also roots are 0 and $\pm\alpha$. It is also possible to prove that all non-zero roots are non-degenerate, i.e. there is only one root vector for each root. The degeneracy of the zero root, which corresponds to the vectors of the Cartan subalgebra, is equal to the rank of the algebra.

Suppose α, β are roots, such that $\alpha + \beta$ is not a root, i.e. $[E_\alpha, E_\beta] = 0$. Then $[E_{-\alpha}, E_\beta] \equiv E'_{\beta-\alpha}$ is proportional to $E_{\beta-\alpha}$ and so $[E_{-\alpha}, E'_{\beta-j\alpha}] \equiv E'_{\beta-(j+1)\alpha}$ is proportional to $E_{\beta-(j+1)\alpha}$. (It is assumed that some normalisation has been chosen for the root vectors E_α. The normalisation of the vectors E'_α is then fixed by their definition.) Since there is only a finite number of roots, there must be a maximum value of j, denoted M, such that $[E_{-\alpha}, E'_{\beta-M\alpha}] = 0 \Longrightarrow E'_{\beta-(M+1)\alpha} = 0$. Now consider $[E_\alpha, E'_{\beta-j\alpha}] = \mu_j E'_{\beta-(j-1)\alpha}$, which defines the factor of proportionality μ_j, since the normalisations of E_α and E'_α are fixed. It follows that

$$\begin{aligned}
\mu_{j+1} E'_{\beta-j\alpha} &= [E_\alpha, E'_{\beta-(j+1)\alpha}] \\
&= [E_\alpha, [E_{-\alpha}, E'_{\beta-j\alpha}]] \\
&= -[E_{-\alpha}, [E'_{\beta-j\alpha}, E_\alpha]] - [E'_{\beta-j\alpha}, [E_\alpha, E_{-\alpha}]] \\
&= \mu_j [E_{-\alpha}, E'_{\beta-(j-1)\alpha}] + g_{\alpha,-\alpha}[H_\alpha, E'_{\beta-j\alpha}] \\
&= \mu_j E'_{\beta-j\alpha} + g_{\alpha,-\alpha}\alpha \cdot (\beta - j\alpha) E'_{\beta-j\alpha},
\end{aligned}$$

which implies $\mu_{j+1} = \mu_j + (\alpha \cdot \beta - j\alpha \cdot \alpha) g_{\alpha,-\alpha}$. This is a recursion relation for the coefficient μ_j, with the initial condition $\mu_0 = 0$.

> [This follows from $[E_\alpha, E'_{\beta-\alpha}] = \mu_1 E_\beta = [E_\alpha, [E_{-\alpha}, E_\beta]]$ $= -[E_{-\alpha}, [E_\beta, E_\alpha]] - [E_\beta, [E_\alpha, E_{-\alpha}]] = g_{\alpha,-\alpha}[H_\alpha, E_\beta]$ $= g_{\alpha,-\alpha}\alpha \cdot \beta E_\beta \Longrightarrow \mu_1 = \alpha \cdot \beta g_{\alpha,-\alpha} = \mu_0 + \alpha \cdot \beta g_{\alpha,-\alpha}$.]

The recursion relation has the solution $\mu_j = g_{\alpha,-\alpha}\left(j\alpha \cdot \beta - \alpha \cdot \alpha j(j-1)/2\right)$, satisfying $\mu_0 = 0$. But $\mu_{M+1} E'_{\beta-M\alpha} = [E_\alpha, E'_{\beta-(M+1)\alpha}] = 0 \Longrightarrow \mu_{M+1} = 0 \Longrightarrow \alpha \cdot \beta = M\alpha \cdot \alpha/2 \Longrightarrow M = 2\alpha \cdot \beta/\alpha \cdot \alpha$.

To recapitulate, starting from a root β such that $\beta + \alpha$ is not a root, a sequence of roots $\beta - j\alpha$ was generated, terminating with the root $\beta - M\alpha$, where $M = 2\alpha \cdot \beta/\alpha \cdot \alpha$. This sequence is called an *α-string* of roots.

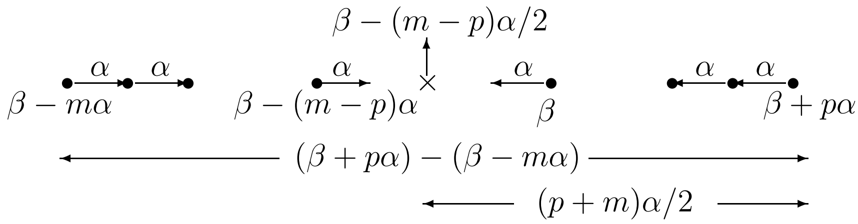
In general, any root γ belongs to an α-string of roots, ranging from $\gamma - m\alpha$ to $\gamma + p\alpha$. Here, m and p are non-negative integers, which may be zero. The "top" of this string is at $\beta = \gamma + p\alpha$, and it ends at $\beta - M\alpha = \gamma - m\alpha$, so $M = m + p$. Hence $m + p = M = 2\alpha \cdot (\gamma + p\alpha)/\alpha \cdot \alpha = 2\alpha \cdot \gamma/\alpha \cdot \alpha + 2p$. Hence

$$m - p = 2\alpha \cdot \gamma/\alpha \cdot \alpha.$$

This is a major result, with many implications and applications. It will be referred to from now on as *the "magic formula"*.

According to the "magic formula", for any two roots α, β, the quantity $2\alpha \cdot \beta/\alpha \cdot \alpha$ is an integer. If β is at the top of an α-string (i.e. $\beta + \alpha$ is not a root, so $p = 0$), then the quantity is non-negative. If $\alpha \cdot \alpha = 0$, then $\alpha \cdot \beta = 0$ for all β, which implies $\sum_i \alpha_i g^{ij} = 0$. But this implies $\det g = 0$, which is impossible for semi-simple Lie algebras.

If β belongs to an α-string from $\beta - m\alpha$ to $\beta + p\alpha$, then the midpoint of the string is at $\beta - (m - p)\alpha/2 = \beta - (\beta \cdot \alpha/\alpha \cdot \alpha)\alpha$ and the "mirror image" of β on the string is $\beta - 2(\beta \cdot \alpha/\alpha \cdot \alpha)\alpha$, and is also a root.



This procedure for producing a root from a given root is called a *Weyl reflection*, and the set of all such operations constitutes the *Weyl group*.

There can be at most four roots in a string.

> [Suppose there are five roots, chosen without loss of generality as $\beta - 2\alpha, \beta - \alpha, \beta, \beta + \alpha, \beta + 2\alpha$. Since $\beta + 2\alpha - \beta = 2\alpha$, and 2α is not a root, and since $\beta + 2\alpha + \beta = 2(\beta + \alpha)$, and $2(\beta + \alpha)$ is not a root, the root $\beta + 2\alpha$ belongs to a β-string of length 1, so $m = 0 = p$ and $(\beta + 2\alpha) \cdot \beta = 0$. Similarly, $(\beta - 2\alpha) \cdot \beta = 0$, hence $\beta \cdot \beta = 0$, which is impossible.]

The only possible values of $m-p$ in a 4-string are ± 3 and ± 1; in a 3-string they are ± 2 or 0; in a 2-string they are ± 1; and in a 1-string the only possible value is 0. So, from the "magic formula", $2\alpha \cdot \beta/\alpha \cdot \alpha$ can only take on the values $0, \pm 1, \pm 2, \pm 3$.

It is possible to adopt a simple geometrical picture in which $\alpha \cdot \alpha$ is the square of the length of the root α and $\alpha \cdot \beta$ is proportional to the cosine of the angle between the roots α and β. In fact, $\cos^2 \phi_{\alpha\beta} = (\alpha \cdot \beta)^2/(\alpha \cdot \alpha)(\beta \cdot \beta) = rs/4$, where r, s are integers, the appropriate values of $m-p$ for the relevant α- and β-strings.

> [Note that $\cos^2 \phi_{\alpha\beta} \leq 1$, because of the Schwarz inequality $(\alpha \cdot \beta)^2 \leq (\alpha \cdot \alpha)(\beta \cdot \beta)$. This inequality also correlates the values of $m-p$ for α- and β-strings.]

It may be concluded that $\cos^2 \phi_{\alpha\beta}$ takes only the values $0, 1/4, 1/2, 3/4, 1$, so the only possible angles between two roots are $90°, 60°$ or $120°, 45°$ or $135°$, $30°$ or $150°$, with $\cos^2 \phi_{\alpha\beta} = 1 \Longrightarrow \beta = \pm\alpha$. Since $r/s = \beta \cdot \beta/\alpha \cdot \alpha$, it is easy to deduce that the ratio of lengths of two roots is 1 if the angle between them is $60°$ or $120°$, $\sqrt{2}$ if the angle is $45°$ or $135°$, $\sqrt{3}$ if the angle is $30°$ or $150°$ and indeterminate if the angle is $90°$. Each Lie algebra can then be represented by a vector diagram of roots in an ℓ-dimensional space, where ℓ is the rank of the algebra.

If a basis set of roots $\{\alpha^{(i)}\}$ is chosen in the root space, then any root can be written as $\beta = \sum_i q_i \alpha^{(i)}$, where the expansion coefficients q_i are rational numbers.

> [$2\beta \cdot \alpha^{(j)}/\alpha^{(j)} \cdot \alpha^{(j)} = \sum_i q_i 2\alpha^{(i)} \cdot \alpha^{(j)}/\alpha^{(j)} \cdot \alpha^{(j)}$ is a set of linear equations for the $\{q_i\}$, with integer coefficients, so the solutions are rational.]

This is consistent with the earlier comment that real linear combinations of roots constitute a sufficient root space. The scalar products $\alpha \cdot \alpha$ and $\alpha \cdot \beta$ are also rational numbers.

> [$\alpha \cdot \alpha = \sum_{ij} \alpha_i g^{ij} \alpha_j = \sum_{ijkl} \alpha_i g^{ik} g_{kl} g^{lj} \alpha_j = \sum_{ijkl} \sum_\beta \alpha_i g^{ik} \beta_k \beta_l g^{lj} \alpha_j = \sum_\beta (\alpha \cdot \beta)(\beta \cdot \alpha) = \sum_\beta (\alpha \cdot \beta)^2 = \sum_\beta (m-p)^2_\beta (\alpha \cdot \alpha)^2/4 \Longrightarrow \alpha \cdot \alpha = 4/\sum_\beta (m-p)^2_\beta$, which is rational and manifestly positive, justifying the use of the term "scalar product". Then $\alpha \cdot \beta = (m-p)_\beta \alpha \cdot \alpha/2$, also rational.]

An ordering can be introduced into the real root space by choosing an arbitrary basis and defining a root as positive if the first non-zero coefficient in its expansion in terms of the basis is positive. Then $\alpha > \beta$ if $\alpha - \beta > 0$. Since $\alpha > 0 \iff -\alpha < 0$, half of the non-zero roots will be positive.

A positive root is said to be *simple* if it cannot be written as the sum of two positive roots. If two roots α, β are simple, then their difference $\alpha - \beta$ is not a root.

> [If $\alpha - \beta$ is a root, then $\alpha - \beta > 0$ or $\beta - \alpha > 0$. But $\alpha = (\alpha - \beta) + \beta$ and $\beta = (\beta - \alpha) + \alpha$, so either α or β is not simple.]

For the α-string containing β, $m - p = 2\alpha \cdot \beta / \alpha \cdot \alpha$. But $\beta - \alpha$ is not a root, so $m = 0$. Therefore $\alpha \cdot \beta \leq 0$ for simple roots.

The set of simple roots is linearly independent.

> [If the simple roots $\{\alpha^{(i)}\}$ were linearly dependent, then $\sum_i c_i \alpha^{(i)} = 0$ for some coefficients $\{c_i\}$. Since all $\alpha^{(i)} > 0$, some of the $\{c_i\}$ are negative. Rewrite the dependence as $\sum_i a_i \alpha^{(i)} = \sum_j b_j \alpha^{(j)}$, where all $a_i, b_j > 0$. Then $(\sum_i a_i \alpha^{(i)}) \cdot (\sum_i a_i \alpha^{(i)}) = (\sum_i a_i \alpha^{(i)}) \cdot (\sum_j b_j \alpha^{(j)}) = \sum_{i \neq j} a_i b_j \alpha^{(i)} \cdot \alpha^{(j)}$. The left hand side is positive, while the right hand side is a sum of negative terms.]

It can be shown that the number of simple roots is equal to ℓ, the rank of the algebra, so the simple roots form a basis for the real root space $\mathcal{H}_0^*$.

Every positive root $\alpha > 0$ is a positive sum of simple roots (i.e. a sum of positive integer multiples of simple roots).

> [If α is simple, the result is immediate. If α is not simple, choose the smallest such α which is not a positive sum of simple roots. Since it is not simple, it is the sum of two smaller positive roots, at least one of which is not a positive sum of simple roots. But α was chosen to be the smallest such positive root, a contradiction.The result follows.]

Finally, it can be proved that the simple roots of a semi-simple Lie algebra are of at most two different lengths.

Appendix

It has been pointed out that the scalar product in root space, defined with the inverse of the Killing form on the Cartan subalgebra as metric, is independent of the choice of basis for the Cartan subalgebra. As a result, all quantities defined in terms of scalar products of roots are also independent of this choice of basis. The specific numerical values of the components of the simple roots do depend on the basis, but any set of ℓ linearly independent ℓ-tuplets (for a rank-ℓ algebra) can serve as simple roots, by an appropriate choice of basis.

As a concrete example of this feature, consider the algebra $\mathsf{su}(3)$, a rank-2 algebra of dimension 8. Let the two simple roots be $\alpha^{(1)} = (a, b)$ and $\alpha^{(2)} = (c, d)$. The remaining positive root is $\alpha^{(1)} + \alpha^{(2)} = (a + c, b + d)$. The Killing form on the Cartan subalgebra is given by $g_{ij} = \sum_\alpha \alpha_i \alpha_j$, so

$$\begin{aligned} g_{11} &= 2[a^2 + c^2 + (a + c)^2] = 4(a^2 + c^2 + ac) \\ g_{22} &= 2[b^2 + d^2 + (b + d)^2] = 4(b^2 + d^2 + bd) \\ g_{12} = g_{21} &= 2[ab + cd + (a + c)(b + d)] = 2(2ab + 2cd + ad + bc) \end{aligned}$$

and $\det g = 16(a^2 + c^2 + ac)(b^2 + d^2 + bd) - 4(2ab + 2cd + ad + bc)^2 = 12a^2d^2 + 12b^2d^2 - 24abcd = 12(ad - bc)^2$. The condition for $\det g \neq 0$ is $ad \neq bc$, precisely the condition for (a, b) and (c, d) to be linearly independent.

The inverse of the Killing form is

$$\begin{aligned} g^{11} &= 2(b^2 + d^2 + bd)/6(ad - bc)^2 \\ g^{22} &= 2(a^2 + c^2 + ac)/6(ad - bc)^2 \\ g^{12} = g^{21} &= -(2ab + 2cd + ad + bc)/6(ad - bc)^2. \end{aligned}$$

The scalar product is defined by $\alpha \cdot \beta = \sum_{ij} \alpha_i g^{ij} \beta_j$, so that

$$\begin{aligned} \alpha^{(1)} \cdot \alpha^{(1)} = a^2 g^{11} + 2ab g^{12} + b^2 g^{22} &= 1/3 \\ \alpha^{(2)} \cdot \alpha^{(2)} = c^2 g^{11} + 2cd g^{12} + d^2 g^{22} &= 1/3 \\ \alpha^{(1)} \cdot \alpha^{(2)} = ac g^{11} + (ad + bc) g^{12} + bd g^{22} &= -1/6 \end{aligned}$$

and these are explicitly independent of the components a, b, c, d.