10. Cartan–Weyl basis

From this point on, the discussion will be restricted to semi-simple Lie algebras, which are the ones of principal interest in physics.

In dealing with the algebra of angular momentum $\mathbf{J}$, it is useful to single out an operator (usually J_z) to be diagonalised and then to re-express the other operators in terms of eigenoperators of the chosen diagonal operator (the step operators $J_\pm = J_x \pm iJ_y$, which satisfy $[J_z, J_\pm] = \pm J_\pm$). It turns out that this procedure can be generalized to the case of any Lie algebra $\mathcal{A}$ and that it leads to the definition of a standard basis in terms of which the representations of the algebra can be characterized completely.

The first step is to choose an appropriate maximal Abelian subalgebra of mutually commuting elements, called the *Cartan subalgebra* and denoted $\mathcal{H} \subset \mathcal{A}$. This subalgebra is *maximal* because there is no additional element of the algebra which commutes with all the elements of the subalgebra. In practice, it is usually fairly simple to pick out a maximal set of mutually commuting generators, which span the Cartan subalgebra. The dimension ℓ of the Cartan subalgebra $\mathcal{H} \subset \mathcal{A}$ is called the *rank* of the algebra $\mathcal{A}$.

> [The notion of Cartan subalgebra is actually rather more subtle than suggested here. Some of the issues, in slightly more technical language, are spelled out in the appendix. For now, it is sufficient that every semi-simple Lie algebra over the real or complex numbers has a maximal Abelian Cartan subalgebra, unique up to isomorphism, whose dimension is the rank of the algebra.]

In principle, a Cartan subalgebra is obtained by finding an element of the adjoint representation with the least number of zero eigenvalues and appending to it all elements of the algebra which commute with this special element and with one another. (Note that every element of the adjoint representation has at least one zero eigenvalue, since it commutes with itself. It also commutes with any other eigenvector belonging to a zero eigenvalue.)

Let $\{H_i\}$, with $i = 1, \ldots, \ell$, be a basis for the Cartan subalgebra $\mathcal{H}$. Since all the H_i commute with one another, they may be simultaneously diagonalized. The simultaneous eigenvectors of all the H_i are called *root vectors*. The set of ℓ eigenvalues associated with a given root vector constitutes a vector in the space of complex numbers, called a *root*.

[**Note:** Do not confuse *roots*, which are vectors in an abstract space of complex numbers, with *root vectors*, which are elements of the Lie algebra. Note also that the process of diagonalization generally introduces complex numbers, both as eigenvalues and as coefficients of basis vectors in the expression of eigenvectors, even if the original algebra was defined over the reals. The Cartan-Weyl approach automatically introduces the *complex extensions* of the Lie algebras dealt with, namely the algebra obtained by expanding the real field to the complex field.]

The roots are linear functionals on the Cartan subalgebra $\mathcal{H}$ (they assign an ℓ-tuple of complex numbers to every element of $\mathcal{H}$) and they can be shown to span the dual space $\mathcal{H}^*$ of linear functionals on $\mathcal{H}$, called the *root space*. It turns out that the real subspace of $\mathcal{H}^*$, denoted $\mathcal{H}_0^*$, is sufficient for all practical purposes. It, too, is of dimension ℓ.

So, given a Cartan subalgebra $\mathcal{H} \subset \mathcal{A}$, the remaining generators of the Lie algebra $\mathcal{A}$ can be regrouped in terms of root vectors E_α belonging to roots $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, such that

$$[H_i, E_\alpha] = \alpha_i E_\alpha. \tag{1}$$

Since $[H_i, H_j] = 0$ for all i, j, all elements of $\mathcal{H}$ are root vectors with root zero. The root vectors are defined by a homogeneous eigenvalue equation, so they are not normalized. To complete their definition unambiguously, some sort of normalization condition is required. This will be dealt with explicitly whenever the need arises. (It should also be noted that the basis of the Cartan subalgebra can be freely rescaled, with a corresponding rescaling of the roots.) The algebra $\mathcal{A}$ is decomposed into the Cartan subalgebra and the complementary space of root vectors with non-zero roots.

Now $[H_i, [E_\alpha, E_\beta]] = [E_\alpha, [H_i, E_\beta]] + [[H_i, E_\alpha], E_\beta] = [E_\alpha, \beta_i E_\beta] + [\alpha_i E_\alpha, E_\beta] = (\alpha_i + \beta_i)[E_\alpha, E_\beta]$ (where the Jacobi identity has been used). This implies that $[E_\alpha, E_\beta]$ is a root vector with root $\alpha + \beta$, or else $[E_\alpha, E_\beta] = 0$. (Note that, if $\alpha + \beta = 0$, i.e. $\beta = -\alpha$, then $[E_\alpha, E_\beta]$ commutes with every H_i, i.e. $[E_\alpha, E_\beta] \in \mathcal{H}$.) It may be concluded that

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}, \tag{2}$$

where (i) the coefficient of proportionality $N_{\alpha\beta}$ depends on the normalization of the root vectors; (ii) $N_{\alpha\beta} = 0$ if $\alpha + \beta$ is not a root; and (iii) E_0 represents an element of the Cartan subalgebra.

The convention will be adopted that generators of the algebra are identified by indices $i, j, k, \ldots$ when they belong to the Cartan subalgebra, by indices $\alpha, \beta, \gamma, \ldots$ when they belong to the complementary subalgebra, and by indices $\rho, \sigma, \tau, \ldots$ when they could belong to either subalgebra. (**N.B.** This means that what was previously denoted g_{ij} would now be written $g_{\rho\sigma}$.)

The Killing form $g_{\rho\alpha} = \sum_{\sigma\tau} c^{\sigma}_{\rho\tau} c^{\tau}_{\alpha\sigma} = \sum_{ij} c^{i}_{\rho j} c^{j}_{\alpha i} + \sum_{i\beta} c^{i}_{\rho\beta} c^{\beta}_{\alpha i} + \sum_{\beta i} c^{\beta}_{\rho i} c^{i}_{\alpha\beta} + \sum_{\beta\gamma} c^{\beta}_{\rho\gamma} c^{\gamma}_{\alpha\beta}$. The first of the four double sums contains the factor $c^{j}_{\alpha i} = 0$, from (1). The second sum contains the factors $c^{\beta}_{\alpha i} \propto \delta_{\alpha\beta}$ (from (1)) and $c^{i}_{\rho\alpha} \propto \delta_{\alpha+\rho,0}$. In the same way, the third sum contains factors proportional to $\delta_{\alpha+\rho,0}$. Finally, the fourth sum contains factors $c^{\gamma}_{\alpha\beta} \propto \delta_{\alpha+\beta,\gamma}$ (from (1)) and $c^{\beta}_{\alpha+\beta,\rho} \propto \delta_{\alpha+\beta+\rho,\beta}$ (from (1)). So $g_{\rho\alpha} \propto \delta_{\alpha+\rho,0}$. If $-\alpha$ is not a root, then $g_{\rho\alpha} = 0$ for all ρ and $\det g = 0$, so the algebra is not semi-simple. It follows that, for a semi-simple Lie algebra, if α is a root, then $-\alpha$ is a root.

The above argument also indicates that the only non-vanishing elements of the Killing form are g_{ij} and $g_{\alpha,-\alpha}$. The Killing form is block diagonal (assuming the roots are ordered as $\alpha^{(1)}, -\alpha^{(1)}, \alpha^{(2)}, -\alpha^{(2)}, \ldots$), and its determinant is $\pm \det(g_{ij}) \prod_{\{\alpha,-\alpha\}} g^2_{\alpha,-\alpha}$, where the product is over all pairs of non-zero roots. So for a semi-simple Lie algebra, $g_{\alpha,-\alpha} \neq 0$ and $\det(g_{ij}) \neq 0$. The inverse of the Killing form on the Cartan subalgebra is well-defined in this case, and is denoted g^{ij}. It can serve as a metric tensor and allows the definition of a *scalar product in root space*,

$$\alpha \cdot \beta = \sum_{i,j=1}^{\ell} \alpha_i g^{ij} \beta_j. \tag{3}$$

In matrix notation, this can be written $\alpha \cdot \beta = A^{\mathrm{tr}} G^{-1} B$, where A and B are column vectors representing the roots α and β and G is the matrix representing the Killing form on the Cartan subalgebra. This form makes it clear that the scalar product is invariant under a change of basis of the Cartan subalgebra.

[Let $H'_i = \sum_j S_{ij} H_j$ be a new basis for $\mathcal{H}$. The matrix S is non-singular. The root α becomes α', with $\alpha'_i = \sum_j S_{ij}\alpha_j$. The root vectors remain unchanged. The Killing form on the Cartan subalgebra becomes

$$\begin{aligned} g'_{ij} &= \sum_{kl} (ad(H'_i))_{kl} (ad(H'_j))_{lk} \\ &= \sum_{kl} (ad(\sum_m S_{im} H_m))_{kl} (ad(\sum_n S_{jn} H_n))_{lk} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{klmn} S_{im} S_{jn} (ad(H_m))_{kl} (ad(H_n))_{lk} \\
&= \sum_{mn} S_{im} S_{jn} g_{mn} \\
&= \sum_{mn} S_{im} g_{mn} S^{\mathrm{tr}}_{nj},
\end{aligned}
$$

i.e. $G' = SGS^{\mathrm{tr}}$. Finally, $\alpha' \cdot \beta' = (A')^{\mathrm{tr}} (G')^{-1} B' = A^{\mathrm{tr}} S^{\mathrm{tr}} (S^{\mathrm{tr}})^{-1} G^{-1} S^{-1} S B = A^{\mathrm{tr}} G^{-1} B = \alpha \cdot \beta$.]

Also note that

$$
\begin{aligned}
g_{ij} &= \sum_{\rho\sigma} c^{\rho}_{i\sigma} c^{\sigma}_{j\rho} \\
&= \sum_{kl} c^{k}_{il} c^{l}_{jk} + \sum_{k\alpha} c^{k}_{i\alpha} c^{\alpha}_{jk} + \sum_{\alpha k} c^{\alpha}_{ik} c^{k}_{j\alpha} + \sum_{\alpha\beta} c^{\alpha}_{i\beta} c^{\beta}_{j\alpha} \\
&= \sum_{\alpha\beta} \beta_i \delta_{\alpha\beta} \alpha_j \delta_{\alpha\beta} \quad (\text{since } c^{\rho}_{jk} = 0 = c^{\rho}_{ik}) \\
&= \sum_{\alpha} \alpha_i \alpha_j . \qquad (4)
\end{aligned}
$$

Let r be an arbitrary vector in root space, with components $\{r_i\}$. It can be associated uniquely with an element H_r of the Cartan subalgebra, where H_r is defined with the help of the Killing form by

$$g_{ri} = \mathrm{tr}(ad(H_r) ad(H_i)) = r_i . \qquad (5)$$

In terms of the basis vectors, $H_r = \sum_i \lambda^{(r)}_i H_i$, so that $r_i = g_{ri} = \sum_j \lambda^{(r)}_j g_{ji} \Longrightarrow \lambda^{(r)}_j = \sum_i r_i g^{ij}$. Therefore

$$H_r = \sum_{i,j=1}^{\ell} r_i g^{ij} H_j \qquad (6)$$

for any element r of the root space. This can be written schematically as $H_r = r \cdot H$. Since, by the properties of the Killing form, $g_{[E_\alpha, E_{-\alpha}] H_i} = g_{E_\alpha [E_{-\alpha}, H_i]} = \alpha_i g_{\alpha,-\alpha}$, it may be concluded that $H_\alpha = [E_\alpha, E_{-\alpha}]/g_{\alpha,-\alpha}$.

The semi-simple Lie algebra can now be rewritten in terms of the alternate basis determined by the $\{H_i, i = 1, \ldots, \ell\}$ and the $\{E_\alpha\}$ (of which there are $d - \ell$, where d is the dimension of the algebra and ℓ is its rank). This is called the *Cartan-Weyl basis* and its defining Lie products are

$$[H_i, H_j] = 0 \qquad (7)$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{8}$$

$$[E_\alpha, E_{-\alpha}] = g_{\alpha,-\alpha} H_\alpha \tag{9}$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}, \quad (\alpha + \beta \neq 0). \tag{10}$$

It should be recalled that $N_{\alpha\beta} = 0$ when $\alpha + \beta$ is not a root and that $H_\alpha = \sum_{ij} \alpha_i g^{ij} H_j$. Also, for a given choice of basis, the H_i can be freely rescaled (affecting the numerical values of the roots) and the E_α can be arbitrarily normalized. A convenient and frequently used normalization is $g_{\alpha,-\alpha} = 1$.

Note that

$$\begin{gathered}[H_\alpha, E_\alpha] = \textstyle\sum_{ij} \alpha_i g^{ij} \alpha_j E_\alpha = \alpha \cdot \alpha E_\alpha, \\ [H_\alpha, E_{-\alpha}] = -\alpha \cdot \alpha E_{-\alpha}, \\ [E_\alpha, E_{-\alpha}] = g_{\alpha,-\alpha} H_\alpha.\end{gathered}$$

Up to normalization, these are precisely the commutators of the angular momentum algebra, or the Lie products of $\mathsf{su}(2)$. So every pair of roots $\pm\alpha$ in the complementary subspace of the Cartan subalgebra is associated with an $\mathsf{su}(2)$ subalgebra of the Lie algebra. The different $\mathsf{su}(2)$'s are connected via eq. (10). (Note also that $g_{H_\alpha,H_\beta} = \sum_{ijkl} \alpha_i g^{ij} \beta_k g^{kl} g_{jl} = \alpha \cdot \beta$.)

Appendix — Cartan subalgebras

Here is a brief compendium of relevant definitions and results concerning Cartan subalgebras.

- A Lie algebra is *nilpotent* if, for some k, all k-fold Lie products of elements $V, W, \ldots, Z$ of the algebra $\underbrace{[[[\ldots[V,W],\ldots,X],Y],Z]}_{k}$ vanish.
- A Lie subalgebra $\mathcal{H} \subset \mathcal{A}$ is *self-normalising* if every element $Y \in \mathcal{A}$ whose Lie product with all elements of $\mathcal{H}$ is in $\mathcal{H}$ belongs to $\mathcal{H}$ (i.e. $[X, Y] \in \mathcal{H}$ for all $X \in \mathcal{H} \Longrightarrow Y \in \mathcal{H}$).
- An element $X \in \mathcal{A}$ is *regular* if the dimension of its *centraliser* (the subalgebra of all elements of $\mathcal{A}$ whose Lie products with X vanish) is minimal among all centralisers of elements of $\mathcal{A}$.
- A field K with multiplicative identity 1 has *characteristic zero* if all the numbers $1+1$, $1+1+1$, $1+1+1+1$, … are different. The fields of rational, real and complex numbers all have characteristic zero.

- A field K is *algebraically closed* if every polynomial with coefficients in K has a root in K.

Given a finite-dimensional Lie algebra $\mathcal{A}$ over a field K, any nilpotent self-normalising subalgebra of $\mathcal{A}$ is a Cartan subalgebra. A Cartan subalgebra is a maximal self-normalising subalgebra of a Lie algebra.

If the field K is infinite, then $\mathcal{A}$ has Cartan subalgebras. If K is of characteristic zero, then there is a Cartan subalgebra for every regular element of the algebra $\mathcal{A}$, each regular element belongs to one and only one Cartan subalgebra and all Cartan subalgebras have the same dimension (called the *rank* of $\mathcal{A}$).

If the field K is algebraically closed, then all Cartan subalgebras are isomorphic. If the algebra $\mathcal{A}$ is semi-simple, all Cartan subalgebras are Abelian.

For the purposes of this course, which concentrates on semi-simple Lie algebras over the real or complex fields, every algebra has a maximal Abelian Cartan subalgebra (unique up to isomorphism) whose dimension is the rank of the algebra.