0. Intro & Motivation

This course is called "Introductory Algebra for Physicists" and each of the words in the title is significant. As a course in Algebra it is essentially a mathematics course. Outside of this chapter, there is no reference to physics applications. As an Introductory course it aims to introduce the students to various techniques which have proved useful in many topics in physics, without probing the subject in great depth. Some results are proved, others are simply quoted. The intention is to produce algebraic plumbers, familiar with useful tools and knowing how to select the correct ones, not algebraic engineers, with deep understanding of the foundations of the subject. Finally, as a course for Physicists, no pretensions are made of mathematical rigor and formal proofs may occasionally be skipped in favour of plausible reasoning.

To get some indication of how algebra can be relevant to physics, consider a physical system whose quantal states belong to a Hilbert space $\mathcal{S}$ and whose dynamics is governed by a Hamiltonian operator H. In particular, let ψ be an eigenstate of H, satisfying $H\psi = E\psi$.

Suppose the system preserves certain symmetries, such as rotational or translational invariance. This means there are operators on the Hilbert space which rotate and/or translate the states of the system while leaving the Hamiltonian unchanged. These operators, referred to as transformations T, commute with the Hamiltonian, $[T, H] = 0$. The following sequence of manipulations

$$\begin{aligned} H(T\psi) &= ([H,T] + TH)\,\psi \\ &= (0 + TE)\,\psi \\ &= E(T\psi) \end{aligned}$$

then shows that $T\psi$ is an eigenstate of H with the same eigenvalue E as ψ. The degeneracies of Hamiltonian eigenstates are associated with symmetries of the system.

Further states degenerate in energy with ψ could be produced by acting again with the transformation T on the eigenstate $T\psi$, and so on, or by acting with several different transformations. The vector space generated by linear combinations of independent degenerate eigenstates is invariant under the action of the transformations and will support eigenstates of these transformations. The associated eigenvalues serve as quantum numbers labeling the states of $\mathcal{S}$.

Similar arguments (which will be encountered in later chapters) connect the symmetries of the system to matrix elements of the transformation operators and to selection rules. It is such connections that make the study of symmetries, via the theory of groups and algebras, relevant to physics.