

Angular Momentum Algebra

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(Dated: January 2002)

Abstract

This brief summary of the quantum theory of angular momentum is intended as a heuristic, reasonably self-contained presentation of useful results. It has no pretensions to rigour and certainly does not presume to supplant the existing detailed texts on the subject, which should be consulted for a more thorough treatment of the topics touched on below. A brief bibliography is given at the end. In addition, most texts on quantum mechanics have at least some discussion of the quantum mechanics of angular momentum, some of them quite extensive.

I. BASIC DEFINITIONS

In classical mechanics, the angular momentum of a point object is defined as the vector product of its position and momentum vectors, $\vec{L} = \vec{r} \times \vec{p}$. In quantum mechanics, where $\vec{r}$ and $\vec{p}$ are operators, one for each component of each vector, this same definition produces a set of three operators, L_x , L_y and L_z . From the standard commutation relation $[r_\alpha, p_\beta] = i\hbar\delta_{\alpha\beta}$, where the subscripts α and β denote the indices x , y or z and $\delta_{\alpha\beta}$ is the Krönecker delta, it follows that

$$[L_x, L_y] = i\hbar L_z \quad (1.1)$$

and similarly for any cyclic permutation of the indices x , y and z . These results generalise easily to collections of independent particles. Choosing $\hbar$ as the unit of angular momentum, and regarding these commutation relations as the defining characteristic of angular momentum, the general quantal definition of angular momentum will be taken to be as follows:

Angular momentum is a physical observable represented by three hermitian operators j_x , j_y and j_z which satisfy the commutation relations $[j_x, j_y] = ij_z$, and cyclic permutations. These operators are the components of a vector $\vec{j}$.

The change in notation from $\vec{L}$ to $\vec{j}$ is intended to indicate the possibility of generalisation of the concept of angular momentum beyond that associated with classical orbital motion.

The operators of angular momentum generate an algebra (the commutator of any two operators in the set is a linear combination of operators from the same set). Since the significance of operators in quantum mechanics lies in their matrix elements, there is obvious interest in establishing the matrix representations of the angular momentum algebra in terms of standard basis functions.

Though no two components of the angular momentum operator commute with one another, all three components compute with the quadratic form $\vec{j}^2 = j_x^2 + j_y^2 + j_z^2$, and it may be established that this is the most general angular momentum operator with this property. According to the general principles of quantum mechanics, $\vec{j}^2$ may be diagonalised simultaneously with any one component of $\vec{j}$, and their eigenvalues may be used to label quantum states. The standard choice is to diagonalise $\vec{j}^2$ and j_z , though any other choice of the component to be diagonalised would be completely equivalent.

It turns out to be useful to look for what may be thought of as eigenoperators of $\vec{j}^2$ and j_z , namely operators Θ which satisfy $[\vec{j}^2, \Theta] = \lambda\Theta$ and $[j_z, \Theta] = m\Theta$, where λ and m are numbers. Any linear combination of the operators j_α satisfies the first relation, with $\lambda = 0$, but the second is less trivial. It is easily established that the only linear combinations of j_α satisfying the second eigenoperator relation are j_z itself, trivially, with $m = 0$, and the two operators $j_\pm = j_x \pm ij_y$, with $m = \pm 1$. (Of course, any multiples of these operators will satisfy the same relationship, since it is linear.) These operators satisfy the commutation relations

$$[\vec{j}^2, j_\pm] = 0 \quad (1.2)$$

$$[j_z, j_\pm] = \pm j_\pm \quad (1.3)$$

$$[j_+, j_-] = 2j_z \quad (1.4)$$

Consider a set of states $|\lambda m\rangle$ which are simultaneous eigenstates of $\vec{j}^2$ and j_z with eigenvalues λ and m , respectively, i.e.

$$\vec{j}^2|\lambda m\rangle = \lambda|\lambda m\rangle \quad (1.5)$$

$$j_z|\lambda m\rangle = m|\lambda m\rangle \quad (1.6)$$

Applying to these states the commutation relations for $j_\pm$ one obtains

$$\vec{j}^2 j_\pm |\lambda m\rangle = [\vec{j}^2, j_\pm] |\lambda m\rangle + j_\pm \vec{j}^2 |\lambda m\rangle = \lambda j_\pm |\lambda m\rangle$$

and

$$j_z j_\pm |\lambda m\rangle = [j_z, j_\pm] |\lambda m\rangle + j_\pm j_z |\lambda m\rangle = (m \pm 1) j_\pm |\lambda m\rangle$$

from which it follows that

$$j_\pm |\lambda m\rangle = \alpha_{\lambda m}^{(\pm)} |\lambda, m \pm 1\rangle \quad (1.7)$$

where $\alpha_{\lambda m}^{(\pm)}$ is a numerical coefficient and it is assumed that the states $|\lambda m\rangle$ are all normalised. The operators $j_\pm$ are therefore referred to as step operators, changing the eigenvalue of j_z by one unit, up or down respectively, while leaving the eigenvalue of $\vec{j}^2$ unchanged. It should be noted that $j_\pm$ are not hermitian operators, but that each is the hermitian conjugate of the other.

The relations

$$j_+ j_- = j_x^2 + j_y^2 + i[j_y, j_x] = \vec{j}^2 - j_z^2 + j_z \quad (1.8)$$

and

$$j_- j_+ = \vec{j}^2 - j_z^2 - j_z \quad (1.9)$$

which follow from the definition of $j_\pm$, may be used to determine the coefficients $\alpha_{\lambda m}^{(\pm)}$, since

$$\begin{aligned} |\alpha_{\lambda m}^{(\pm)}|^2 &= \|j_\pm |\lambda m\rangle\|^2 \\ &= \langle \lambda m | j_\mp j_\pm | \lambda m \rangle \\ &= \langle \lambda m | \vec{j}^2 - j_z^2 \mp j_z | \lambda m \rangle \\ &= \lambda - m^2 \mp m. \end{aligned}$$

As always, quantal states are defined, even when normalised, only up to an overall phase. There is thus complete freedom to choose the phases of the basis states $|\lambda m\rangle$ in order to simplify the above results as far as possible. The standard choice, which can be shown always to be possible, is known as the Condon-Shortley phase convention and involves choosing the coefficients $\alpha_{\lambda m}^{(\pm)}$ to be real and positive. So

$$\alpha_{\lambda m}^{(\pm)} = \sqrt{\lambda - m^2 \mp m}. \quad (1.10)$$

Since the norm of a quantal state must be non-negative, it follows that

$$\begin{aligned} 0 &\leq \langle \lambda m | j_+^\dagger j_+ + j_-^\dagger j_- | \lambda m \rangle \\ &= \langle \lambda m | j_- j_+ + j_+ j_- | \lambda m \rangle \\ &= \langle \lambda m | 2\vec{j}^2 - 2j_z^2 | \lambda m \rangle \\ &= 2(\lambda - m^2) \end{aligned}$$

so that $m^2 \leq \lambda$ and, for any finite value of λ , there is a limitation on the possible values of m .

Consider a specific state $|\lambda m\rangle$ and apply to it the angular-momentum raising (step-up) operator j_+ . This produces a state $|\lambda, m+1\rangle$. Repeated applications of j_+ will produce a succession of states $|\lambda, m+n\rangle$, where n is an arbitrary positive integer. For any initial values of λ and m , there will eventually be some value of n for which $(m+n)^2 > \lambda$, which is forbidden. Thus there must exist some maximum value of n , denoted n_{max} , such that

$j_+|\lambda, m + n_{max}\rangle = 0$. Then

$$j_-j_+|\lambda, m + n_{max}\rangle = [\lambda - (m + n_{max})^2 - (m + n_{max})]|\lambda, m + n_{max}\rangle = 0$$

i.e. $\lambda = (m + n_{max})(m + n_{max} + 1)$.

Similarly, repeated applications of the step-down operator j_- to the state $|\lambda m\rangle$ will ultimately produce a state $|\lambda, m - n'\rangle$ such that $(m - n')^2 > \lambda$, which is again forbidden. There must thus also exist some maximum value of the positive integer n' , denoted n_{min} , such that $j_-|\lambda, m - n_{min}\rangle = 0$. Then

$$j_+j_-|\lambda, m - n_{min}\rangle = [\lambda - (m - n_{min})^2 + (m - n_{min})]|\lambda, m - n_{min}\rangle = 0$$

i.e. $\lambda = (m - n_{min})(m - n_{min} - 1)$. Therefore

$$(m + n_{max})(m + n_{max} + 1) = (m - n_{min})(m - n_{min} - 1),$$

from which $m + n_{max} = -(m - n_{min})$ and so $2m = n_{min} - n_{max}$. Recalling that n_{max} and n_{min} are non-negative integers, it follows that m must be an integer or half an odd integer (positive or negative), with a maximum allowed value and a minimum allowed value for any fixed value of λ . Denoting the maximum allowed value of m by j , the above results imply that the minimum allowed value is $-j$ and that $\lambda = j(j + 1)$.

To summarise, it is inherent in the angular momentum algebra that the simultaneous eigenstates of $\vec{j}^2$ and j_z are determined by the quantum numbers j and m , where j is a non-negative number, either an integer or half an odd integer, and m takes the $2j + 1$ values $-j$ to j in integer steps. The normalised states $|jm\rangle$ satisfy

$$\vec{j}^2|jm\rangle = j(j + 1)|jm\rangle \quad (1.11)$$

$$j_z|jm\rangle = m|jm\rangle \quad (1.12)$$

$$j_{\pm}|jm\rangle = \alpha_{jm}^{(\pm)}|j, m \pm 1\rangle \quad (1.13)$$

$$\alpha_{jm}^{(\pm)} = \sqrt{j(j + 1) - m(m \pm 1)} \quad (1.14)$$

where the last equation embodies a phase convention for the states. For each j there is a $(2j + 1)$ -dimensional matrix representation for the three angular momentum operators j_x , j_y and j_z , where the rows and columns are labelled by the quantum number m . Since $j_{\pm}$ (and hence j_x and j_y) connect every state to a neighbouring state, there is no smaller subset

of linear combinations of the states $|jm\rangle$, for given j , which is closed under the operation of all three operators j_α (i.e. such that any member of the subset is converted into a linear combination only of members of the subset by any of the operators j_α). Such representations are said to be irreducible.

Since the whole of the above discussion has been based only on the commutation relations of the operators, it holds true for any set of operators satisfying the same commutator algebra and, in particular, for the orbital angular momentum operators $\vec{\ell} = (\vec{r} \times \vec{p})/\hbar = -i\vec{r} \times \vec{\nabla}$. It is well known that the simultaneous eigenfunctions of $\vec{\ell}^2$ and ℓ_z are the spherical harmonics $Y_{\ell m}(\theta, \phi)$, where ℓ is restricted to be an integer in order for the function to be single-valued over the range of its arguments. The Condon-Shortley phase convention implies $Y_{\ell m}^*(\theta, \phi) = (-)^m Y_{\ell, -m}(\theta, \phi)$.

A very useful general result can now be established concerning the matrix elements, in the angular momentum basis, of any operator which commutes with all the components of the angular momentum operator $\vec{j}$. Denoting the operator by Θ , the condition $[\Theta, j_\alpha] = 0$ for $\alpha = x, y, z$ implies also that $[\Theta, \vec{j}^2] = 0$. Matrix elements of this equation may be taken between states $|\xi jm\rangle$, where ξ symbolises all other quantum numbers required to specify the states completely. These quantum numbers are irrelevant to the present discussion and will simply remain unchanged at each step of the argument. Thus

$$\begin{aligned} 0 &= \langle \xi jm | [\Theta, \vec{j}^2] | \xi' j' m' \rangle \\ &= \langle \xi jm | \Theta \vec{j}^2 - \vec{j}^2 \Theta | \xi' j' m' \rangle \\ &= j'(j' + 1) \langle \xi jm | \Theta | \xi' j' m' \rangle - j(j + 1) \langle \xi jm | \Theta | \xi' j' m' \rangle \\ &= [j'(j' + 1) - j(j + 1)] \langle \xi jm | \Theta | \xi' j' m' \rangle \end{aligned}$$

where the hermiticity of $\vec{j}^2$ has been used in operating to the left on the state $|\xi jm\rangle$ and in taking real eigenvalues. It follows that

$$\langle \xi jm | \Theta | \xi' j' m' \rangle = 0 \text{ unless } j' = j. \quad (1.15)$$

In precisely the same way, taking matrix elements of $[\Theta, j_z] = 0$ leads to

$$\langle \xi jm | \Theta | \xi' j' m' \rangle = 0 \text{ unless } m' = m. \quad (1.16)$$

Finally, since $[\Theta, j_+] = 0$, it follows that

$$\begin{aligned}
\langle \xi j, m+1 | \Theta | \xi' j, m+1 \rangle &= \langle \xi j, m+1 | \Theta j_+ | \xi' j m \rangle / \alpha_{jm}^{(+)} \\
&= \langle \xi j, m+1 | j_+ \Theta | \xi' j m \rangle / \alpha_{jm}^{(+)} \\
&= \alpha_{j, m+1}^{(-)} \langle \xi j m | \Theta | \xi' j m \rangle / \alpha_{jm}^{(+)} \\
&= \langle \xi j m | \Theta | \xi' j m \rangle,
\end{aligned}$$

from the explicit form of $\alpha_{jm}^{(\pm)}$. Therefore, any operator which commutes with all the components of $\vec{j}$ is diagonal in the quantum numbers j and m and its matrix elements in the basis $|\xi j m\rangle$ are independent of m .

In particular, the unit operator commutes with all components of $\vec{j}$. Its matrix elements simply express the overlap between different states. The general result here then ensures that the overlap between any two eigenstates of a given angular momentum operator $\vec{j}$, no matter what other quantum numbers are required to specify the states completely, will be zero if the states belong to different eigenvalues j or m and will be independent of m when the states belong to the same eigenvalues j and m .

II. COMBINATION OF ANGULAR MOMENTA

Consider a number of independent angular momenta, i.e. a number of triplets of operators $\vec{j}^{(a)}$, where a is an index labelling a particular triplet, such that the components of different operators $\vec{j}^{(a)}$ commute,

$$[j_\alpha^{(a)}, j_\beta^{(b)}] = i\delta_{ab} j_\gamma^{(a)} \quad (2.1)$$

where (α, β, γ) is a cyclic permutation of (x, y, z) . These may be the orbital angular momenta of different particles, or the orbital and spin angular momentum of a single particle, or some more general combination. Defining the resultant or total angular momentum operator $\vec{j}$ by

$$j_\alpha = \sum_a j_\alpha^{(a)}, \quad (2.2)$$

it is easily checked that the three components of $\vec{j}$ do in fact satisfy the angular momentum algebra, hence justifying the nomenclature. There will thus be a set of eigenstates $|jm\rangle$ with all the properties established above.

In this case, it is easy to find additional operators which commute with $\vec{j}^2$ and j_z and with one another and whose eigenvalues can be used to supply additional quantum numbers characterising the total angular momentum eigenstates. In fact, any operator $(\vec{j}^{(a)})^2$ commutes with all the components of any $\vec{j}^{(b)}$ and hence with $\vec{j}^2$, j_z and any $(\vec{j}^{(b)})^2$. It is therefore possible to diagonalise simultaneously all the $(\vec{j}^{(a)})^2$ operators, together with $\vec{j}^2$ and j_z , and to produce the states $|j_1 j_2 j_3 \dots j_n j m\rangle$, with

$$\vec{j}^2 |j_1 j_2 j_3 \dots j_n j m\rangle = j(j+1) |j_1 j_2 j_3 \dots j_n j m\rangle \quad (2.3)$$

$$j_z |j_1 j_2 j_3 \dots j_n j m\rangle = m |j_1 j_2 j_3 \dots j_n j m\rangle \quad (2.4)$$

$$(\vec{j}^{(a)})^2 |j_1 j_2 j_3 \dots j_n j m\rangle = j_a(j_a+1) |j_1 j_2 j_3 \dots j_n j m\rangle. \quad (2.5)$$

No single component $j_\alpha^{(a)}$ of the individual angular momenta commutes with $\vec{j}^2$ and j_z , because of the cross terms in $\vec{j}^2$. However, partial sums of angular momenta, like $(\vec{j}^{(a)} + \vec{j}^{(b)})^2$ or $(\vec{j}^{(a)} + \vec{j}^{(b)} + \vec{j}^{(c)})^2$, etc., do commute with all the $(\vec{j}^{(a)})^2$ and with $\vec{j}^2$ and j_z , so many more angular momentum quantum numbers can be simultaneously specified. Some caution is required, though, since two partial sums with overlapping sets of indices $a, b, \dots$ will not commute with one another unless one is wholly contained as a sub-sum in the other. Clearly, several different sets of mutually commuting operators can be found, all including $\vec{j}^2$ and j_z and each defining a different basis of eigenvectors.

If the operator $\vec{j}^2$ is omitted, many more bases can be defined, the simplest being that in which $(\vec{j}^{(a)})^2$ and $j_z^{(a)}$, for all a , are diagonal, namely $|j_1 m_1 j_2 m_2 j_3 m_3 \dots j_n m_n\rangle$. Others would involve various partial sums of angular momenta, such as $|j_1 j_2 j_3 j_{123} m_{123} j_4 j_5 j_{45} m_{45} \dots j_n m_n\rangle$, for example. There is an embarrassing richness of alternatives, not all of which are really different from one another (they may differ simply by a relabeling of the $\vec{j}^{(a)}$, for instance), and not all of which are of practical importance. In practice, detailed calculations seldom deal explicitly with more than four angular momenta at one time, and then the number of really different bases is relatively small and the relations between them quite easily established. It is both simplest and most instructive to tackle, in order, the case of two, then three and then four angular momenta, after which the generalisation to larger numbers is straightforward.

In the case of two independent angular momenta $\vec{j}^{(1)}$ and $\vec{j}^{(2)}$, there are only two complete sets of mutually commuting angular momentum operators, the first being

$(\vec{j}^{(1)})^2, j_z^{(1)}, (\vec{j}^{(2)})^2, j_z^{(2)}$ and the second being $(\vec{j}^{(1)})^2, (\vec{j}^{(2)})^2, \vec{j}^2, j_z$, where $\vec{j} = \vec{j}^{(1)} + \vec{j}^{(2)}$. The corresponding eigenvectors are $|j_1 m_1 j_2 m_2\rangle$ and $|j_1 j_2; j m\rangle$, the former being referred to as uncoupled, the latter as coupled. For given values of the quantum numbers j_1 and j_2 , these two sets of eigenvectors constitute different orthonormal bases for the representation of the angular momentum operators and must be connected by a unitary transformation. (In accordance with the general result obtained at the end of the preceding section, eigenvectors corresponding to different values of the quantum number j_1 or j_2 are orthogonal to one another.)

The transformation coefficients between bases are called Clebsch-Gordan coefficients or vector-coupling coefficients and are most easily written down in Dirac notation:

$$|j_1 m_1 j_2 m_2\rangle = \sum_{jm} \langle j_1 j_2; j m | j_1 m_1 j_2 m_2 \rangle |j_1 j_2; j m\rangle \quad (2.6)$$

$$|j_1 j_2; j m\rangle = \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | j_1 j_2; j m \rangle |j_1 m_1 j_2 m_2\rangle \quad (2.7)$$

where $\langle j_1 m_1 j_2 m_2 | j_1 j_2; j m \rangle = \langle j_1 j_2; j m | j_1 m_1 j_2 m_2 \rangle^*$ and the unitarity conditions are

$$\sum_{m_1 m_2} \langle j_1 j_2; j' m' | j_1 j_2 m_1 m_2 \rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2; j m \rangle = \delta_{jj'} \delta_{mm'} \quad (2.8)$$

$$\sum_{jm} \langle j_1 m'_1 j_2 m'_2 | j_1 j_2; j m \rangle \langle j_1 j_2; j m | j_1 m_1 j_2 m_2 \rangle = \delta_{m_1 m'_1} \delta_{m_2 m'_2}. \quad (2.9)$$

For given j_1 and j_2 , the quantum numbers m_1 and m_2 have their standard ranges (from $-j_i$ to j_i in unit steps) and for given j , the quantum number m has its standard range, but it has not yet been determined what are the possible values of j . For this, it is necessary to look into some of the detailed properties of the vector coupling process.

Intuitively, since $j_z = j_z^{(1)} + j_z^{(2)}$, it is reasonable to expect $m = m_1 + m_2$. Therefore, the largest possible value attainable by m is $m_{max} = j_1 + j_2$. It follows that the largest value attainable by j is also $j_1 + j_2$, since if there were a larger value of j it would be associated with larger values of m . There is only one basis function $|j_1 m_1 j_2 m_2\rangle$ with $m_1 + m_2 = j_1 + j_2$, so that, necessarily, $|j_1 j_2; j_1 + j_2, j_1 + j_2\rangle = |j_1 j_1 j_2 j_2\rangle$ and the corresponding Clebsch-Gordan coefficient is unity. (Note the off-hand introduction of a further phase convention.) It can be checked by straightforward, if lengthy, calculation that the uncoupled state $|j_1 j_1 j_2 j_2\rangle$ is an eigenstate of $\vec{j}^2$ and of j_z , with quantum numbers $j_1 + j_2$ and $j_1 + j_2$. The state $|j_1 j_2; j_1 + j_2, j_1 + j_2 - 1\rangle$ is now produced by acting on this extreme state with the step-down operator j_- . Since the value $m = j_1 + j_2 - 1$ can be produced either by $m_1 = j_1, m_2 = j_2 - 1$

or by $m_1 = j_1 - 1, m_2 = j_2$, there are two independent linear combinations of uncoupled states with this m value. One of them will be produced by $j_-|j_1j_2; j_1 + j_2, j_1 + j_2\rangle$, and the orthogonal linear combination must then be the state $|j_1j_2; j_1 + j_2 - 1, j_1 + j_2 - 1\rangle$ (with once again a free choice of phase).

This procedure may now be repeated — $m = j_1 + j_2 - 2$ can be produced in three different ways: $m_1 = j_1, m_2 = j_2 - 2$; $m_1 = j_1 - 1, m_2 = j_2 - 1$; $m_1 = j_1 - 2, m_2 = j_2$; so there are three independent linear combinations of uncoupled states with this m value. Two of them are produced by operating with the step-down operator j_- on the two coupled states found in the preceding steps, and are $|j_1j_2; j_1 + j_2, j_1 + j_2 - 2\rangle$ and $|j_1j_2; j_1 + j_2 - 1, j_1 + j_2 - 2\rangle$. The third linear combination, orthogonal to both of these, is then necessarily $|j_1j_2; j_1 + j_2 - 2, j_1 + j_2 - 2\rangle$. Continuing in this way, the coupled states $|j_1j_2; jm\rangle$ may be produced in stepwise fashion, each step automatically determining the corresponding Clebsch-Gordan coefficients (and requiring one further phase choice, which can obviously be used to make all the Clebsch-Gordan coefficients real, since the Condon-Shortley convention makes all the $\alpha_{jm}^{(\pm)}$ coefficients real).

The above process produces one extra state at each step, with a j value decreased by unity from that of the previous step, and will terminate when all the uncoupled basis states have been used up, which happens when either m_1 or m_2 reaches the lower end of its range. At this point, the number of different ways of making the appropriate value of m is equal to the number of different j values produced in the preceding steps. At the end of the process, all $(2j_1 + 1)(2j_2 + 1)$ uncoupled basis functions $|j_1m_1j_2m_2\rangle$ will have been used to produce an equal number of coupled basis functions $|j_1j_2; jm\rangle$:

$$\begin{aligned} (2j_1 + 1)(2j_2 + 1) &= \sum_{j=j_{min}}^{j_1+j_2} (2j + 1) \\ &= (j_1 + j_2 + 1)^2 - j_{min}^2 \end{aligned}$$

from which $j_{min} = |j_1 - j_2|$. Thus the two angular momenta j_1 and j_2 may be coupled to a total angular momentum $|j_1 - j_2|, |j_1 - j_2| + 1, |j_1 - j_2| + 2, \dots, j_1 + j_2 - 1, j_1 + j_2$, each j value occurring once.

The same results may be obtained more formally in the following fashion. Applying j_z to the defining equation

$$|j_1j_2; jm\rangle = \sum_{m_1m_2} \langle j_1m_1j_2m_2 | j_1j_2; jm \rangle |j_1m_1j_2m_2\rangle$$

produces

$$m|j_1j_2; jm\rangle = \sum_{m_1m_2} \langle j_1m_1j_2m_2|j_1j_2; jm\rangle (m_1 + m_2)|j_1m_1j_2m_2\rangle.$$

On substituting the defining equation again in the left hand side, this may be rearranged into the form

$$\sum_{m_1m_2} (m - m_1 - m_2) \langle j_1m_1j_2m_2|j_1j_2; jm\rangle |j_1m_1j_2m_2\rangle = 0.$$

The states being summed over are mutually orthogonal and hence linearly independent, so the sum can vanish only if the coefficient of every state in the sum vanishes, i.e. $\langle j_1m_1j_2m_2|j_1j_2; jm\rangle = 0$ unless $m = m_1 + m_2$. Thus the additive condition on m assumed intuitively above arises as a selection rule on the Clebsch-Gordan coefficients. (Note that the state $|j_1m_1j_2m_2\rangle$ is an eigenstate of j_z with eigenvalue $m_1 + m_2$, while $|j_1j_2; jm\rangle$ is an eigenstate of j_z with eigenvalue m . The Clebsch-Gordan coefficient is an overlap between these two states, so vanishes unless their j_z eigenvalues are equal. This is an alternative proof of the m selection rule.)

The same technique may be used to produce a pair of recursion relations for the Clebsch-Gordan coefficients, by applying the operators $j_{\pm}$ to the defining equation, substituting the defining equation again on the left hand side of the result, rearranging and using the linear independence of the uncoupled basis functions. The properties of the coefficients $\alpha_{jm}^{(\pm)}$ and of the basis vectors $|j_im_i\rangle$ at the extremes of the range of m are important in ensuring that the ranges of summation on the two sides of the equation are compatible. The result is

$$\begin{aligned} \alpha_{jm}^{(\pm)} \langle j_1m_1j_2m_2|j_1j_2; j, m \pm 1\rangle &= \alpha_{j_1, m_1 \mp 1}^{(\pm)} \langle j_1, m_1 \mp 1, j_2m_2|j_1j_2; jm\rangle \\ &+ \alpha_{j_2, m_2 \mp 1}^{(\pm)} \langle j_1m_1j_2, m_2 \mp 1|j_1j_2; jm\rangle. \end{aligned} \quad (2.10)$$

These recursion relations can be used to express all the Clebsch-Gordan coefficients for given j_1 and j_2 in terms of either $\langle j_1j_1j_2, j - j_1|j_1j_2; jj\rangle$ or $\langle j_1, j - j_2, j_2j_2|j_1j_2; jj\rangle$. (Consider, for example, the case $m = j$, with the upper sign, for which the left hand side vanishes and the right hand side, with the aid of the m selection rule, produces a recursion relation in m_1 alone.) The magnitude of the single remaining undetermined coefficient can be found by applying the unitarity condition in the form $\sum_{m_1m_2} |\langle j_1m_1j_2m_2|j_1j_2; jm\rangle|^2 = 1$, and its phase can be chosen arbitrarily. For convenience, the phase is chosen so that the relevant coefficient is real and positive, $\langle j_1j_1j_2, j - j_1|j_1j_2; jj\rangle > 0$, and the recursion relation then

ensures that all the Clebsch-Gordan coefficients are real. Therefore, $\langle j_1 m_1 j_2 m_2 | j_1 j_2; j m \rangle = \langle j_1 j_2; j m | j_1 m_1 j_2 m_2 \rangle$ and only one form of the coefficient will be used henceforth. Since the quantum numbers j_1 and j_2 remain the same on both sides of the coefficient, the abbreviated form $\langle j_1 m_1 j_2 m_2 | j m \rangle$ is sufficient and will be adopted from now on.

Given the limitations on the projection quantum numbers m , it is clearly necessary that $-j_2 \leq j - j_1 \leq j_2$, or else all the Clebsch-Gordan coefficients vanish. Since the whole exercise could equally well have been carried out in terms of $\langle j_1, j - j_2, j_2 j_2 | j_1 j_2; j j \rangle$ (or, for that matter, in terms of the two alternative forms with $m_1 = -j_1$ and $m_2 = -j_2$), it is also necessary that $-j_1 \leq j - j_2 \leq j_1$. Combining the two sets of restrictions, it is seen that the values of j must satisfy the triangle inequality

$$|j_1 - j_2| \leq j \leq j_1 + j_2. \quad (2.11)$$

From the general requirement that j be an integer or half an odd integer, and that m have the same character as j , and from the selection rule $m = m_1 + m_2$, it follows that j will be an integer if j_1 and j_2 are both integers or both half odd integers, and will be half an odd integer if only one of j_1 and j_2 is an integer. It is also worth noting that the whole treatment of the coupling of j_1 and j_2 was completely symmetric between the two component angular momenta until the choice of the phase of the determining coefficient in the recursion relation. The choice $\langle j_1 j_1 j_2, j - j_1 | j_1 j_2; j j \rangle > 0$, rather than $\langle j_1, j - j_2, j_2 j_2 | j_1 j_2; j j \rangle > 0$, introduces an asymmetry between j_1 and j_2 , so that the coefficients $\langle j_1 m_1 j_2 m_2 | j m \rangle$ and $\langle j_2 m_2 j_1 m_1 | j m \rangle$ can differ at most by a sign. Similarly, since the recursion relation could have been used to relate all Clebsch-Gordan coefficients to those with $m = -j$, instead of $m = j$, there will be a possible sign difference between coefficients with m_1, m_2, m or with $-m_1, -m_2, -m$.

Detailed discussion of the recursion relation, or inspection of the complicated explicit formula for the Clebsch-Gordan coefficients derived from group theory by various authors and quoted in the references, allow the extraction of the following symmetry properties of the coefficients:

$$\langle j_1 m_1 j_2 m_2 | j m \rangle = (-)^{j_1 + j_2 - j} \langle j_2 m_2 j_1 m_1 | j m \rangle \quad (2.12)$$

$$= (-)^{j_1 + j_2 - j} \langle j_1, -m_1, j_2, -m_2 | j, -m \rangle \quad (2.13)$$

$$= (-)^{j_2 + m_2} \sqrt{(2j + 1)/(2j_1 + 1)} \langle j_2, -m_2 j m | j_1 m_1 \rangle \quad (2.14)$$

$$= (-)^{j_1 - m_1} \sqrt{(2j + 1)/(2j_2 + 1)} \langle j m j_1, -m_1 | j_2 m_2 \rangle. \quad (2.15)$$

These properties suggest a basic symmetry among the three angular momenta involved — the two constituent angular momenta j_1 and j_2 and the total angular momentum j . The same underlying symmetry is indicated by the triangle inequality satisfied by j_1 , j_2 and j , which is equally valid for any division of the three angular momenta into two constituents and a resultant. This symmetry is effectively exploited in an alternative form of the vector-coupling coefficients, the Wigner 3-j symbol, defined by

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-)^{j_1-j_2-m_3} \langle j_1 m_1 j_2 m_2 | j_3, -m_3 \rangle / \sqrt{2j_3 + 1}. \quad (2.16)$$

The 3-j symbol vanishes unless j_1, j_2, j_3 satisfy the triangle inequality and $m_1 + m_2 + m_3 = 0$. It is left unchanged in value by any cyclic permutation of its three columns, but is multiplied by the phase factor $(-)^{j_1+j_2+j_3}$ if any two columns are interchanged or if the signs of all three projection quantum numbers m_i are reversed. Thus $\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = 0$ unless $j_1 + j_2 + j_3$ is even. The symbol also has the useful property that

$$\sum_{m_1 m_2 m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} |j_1 m_1 j_2 m_2 j_3 m_3\rangle$$

represents a state of three angular momenta coupled to total angular momentum zero which is symmetric under cyclic permutation of the three angular momenta.

Some consequences of the angular-momentum coupling formalism are of general interest. In the case $j = 0$, the recursion relation for the Clebsch-Gordan coefficients becomes

$$0 = \alpha_{j_1, m_1-1}^{(+)} \langle j_1, m_1 - 1, j_2, m_2 + 1 | 00 \rangle + \alpha_{j_2, m_2}^{(+)} \langle j_1 m_1 j_2 m_2 | 00 \rangle.$$

From the selection rules on the coefficients, necessarily $j_1 = j_2$ and $m_1 = -m_2$. But from the explicit expression for the $\alpha_{jm}^{(\pm)}$ coefficients, $\alpha_{j, m-1}^{(+)} = \alpha_{j, -m}^{(+)}$, so the result reduces to $\langle j m j, -m | 00 \rangle = -\langle j, m-1, j, -(m-1) | 00 \rangle$, from which it follows that $\langle j m j, -m | 00 \rangle = (-)^{j-m} \langle j j j, -j | 00 \rangle$. But $\sum_m \langle j m j, -m | 00 \rangle^2 = 1$, so that $\langle j j j, -j | 00 \rangle = 1/\sqrt{2j+1}$, where the standard phase convention has been used to select the positive square root. Finally,

$$\langle j_1 m_1 j_2 m_2 | 00 \rangle = (-)^{j_1-m_1} \delta_{j_1 j_2} \delta_{m_1, -m_2} / \sqrt{2j_1 + 1}. \quad (2.17)$$

Using the symmetry relation of the Clebsch-Gordan coefficient, this may be rewritten in the form $\langle j_1 m_1 00 | j_2 m_2 \rangle = \delta_{j_1 j_2} \delta_{m_1 m_2}$, which is self-evident. Alternatively,

$$\begin{pmatrix} j_1 & j_2 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} = (-)^{j_1-m_1} \delta_{j_1 j_2} \delta_{m_1, -m_2} / \sqrt{2j_1 + 1}. \quad (2.18)$$

Another special case of the Clebsch-Gordan coefficient which is frequently useful can be obtained straightforwardly from the explicit formula for the coefficient, alluded to previously, namely

$$\langle jjj, -j | J0 \rangle = (2j)! \sqrt{(2J+1)/(2j+J+1)!(2j-J)!}. \quad (2.19)$$

Two identical systems, defined by their angular momentum j and by additional quantum numbers collectively denoted by ξ , can be coupled to a state of well-defined total angular momentum J as follows:

$$|(\xi j)^2 JM\rangle = \sum_{m_1 m_2} \langle j m_1 j m_2 | JM \rangle |\xi j m_1 \xi j m_2\rangle.$$

The dummy indices m_i have the same range and may be exchanged to produce

$$\begin{aligned} |(\xi j)^2 JM\rangle &= \sum_{m_1 m_2} \langle j m_2 j m_1 | JM \rangle |\xi j m_2 \xi j m_1\rangle \\ &= (-)^{j+j-J} \sum_{m_1 m_2} \langle j m_1 j m_2 | JM \rangle |\xi j m_2 \xi j m_1\rangle, \end{aligned}$$

by the symmetry property of the Clebsch-Gordan coefficient. This may be rewritten

$$|(\xi j)^2 JM\rangle = (-)^{2j-J} \mathcal{P}_{12} \sum_{m_1 m_2} \langle j m_1 j m_2 | JM \rangle |\xi j m_1 \xi j m_2\rangle$$

where $\mathcal{P}_{12}$ is a permutation operator which exchanges the two systems in any state on which it acts (but has no effect on numerical coefficients like the Clebsch-Gordan coefficients). This final result states that

$$\mathcal{P}_{12} |(\xi j)^2 JM\rangle = (-)^{2j-J} |(\xi j)^2 JM\rangle, \quad (2.20)$$

so that the coupled state $|(\xi j)^2 JM\rangle$ is automatically symmetric or antisymmetric under interchange of the two systems, according as $2j - J$ is even or odd, respectively. (Note the importance of the identity of the systems. For $\xi_1 \neq \xi_2$, this result would not hold.)

Considering that identical bosons, which have integer angular momentum, are required to form totally symmetric states, while identical fermions, which have half-odd-integer angular momentum, are required to form totally antisymmetric states, it follows that a pair of identical fermions or a pair of identical bosons can couple only to even total angular momentum J . This argument can be extended, in the case where each system is characterised by two or more independent angular momenta, to produce a correspondence between the sign of the phase factor $(-)^{J_1+J_2+\dots+J_n}$ and the symmetry of the state $|(\xi j_1 j_2 \dots j_n)^2 J_1 M_1 J_2 M_2 \dots J_n M_n\rangle$. Thus for systems characterised by angular momentum

and isospin, for example, selection rules arise for $J + T$, where T is the total isospin of the pair of identical systems.

III. RECOUPLING OF ANGULAR MOMENTA

As indicated at the beginning of the previous section, there will generally be more than two different bases applicable to problems involving more than two angular momenta. In the case of three angular momenta, the bases of interest are the totally uncoupled basis $|j_1 m_1 j_2 m_2 j_3 m_3\rangle$ and the totally coupled basis $|j_1 j_2 j_3 j m\rangle$, where the latter is not yet completely specified, being determined by the eigenvalues of only five, rather than six, mutually commuting operators. The totally coupled basis may be fully specified by defining one intermediate coupling, which can be done in three different ways — $|(j_1 j_2) j_{12} j_3 j m\rangle$, $|(j_1 j_3) j_{13} j_2 j m\rangle$ or $|j_1 (j_2 j_3) j_{23} j m\rangle$, where the notation should be self-explanatory. The transition from the fully uncoupled to the fully coupled bases goes through the intermediate partially coupled basis, with Clebsch-Gordan coefficients being used to couple pairs of angular momenta at each step. So

$$|(j_1 j_2) j_{12} j_3 j m\rangle = \sum_{m_{12} m_3} \langle j_{12} m_{12} j_3 m_3 | j_{12} m \rangle |(j_1 j_2) j_{12} m_{12} j_3 m_3\rangle \quad (3.1)$$

$$= \sum_{m_1 m_2 m_3 m_{12}} \langle j_{12} m_{12} j_3 m_3 | j m \rangle \langle j_1 m_1 j_2 m_2 | j_{12} m_{12} \rangle |j_1 m_1 j_2 m_2 j_3 m_3\rangle, \quad (3.2)$$

and similarly for each of the other possible intermediate couplings. However, the existence of different bases does not imply a proliferation of states — any one of the three fully coupled bases is equally valid, so there must exist unitary transformations between them.

The vectors belonging to a particular basis, such as $|(j_1 j_2) j_{12} j_3 j m\rangle$, are distinguished by the values of the six quantum numbers listed and are all mutually orthogonal, according to the general result proved at the end of section 1. However, vectors from different bases are not necessarily orthogonal, provided they have the same values of the common quantum numbers j_1 , j_2 , j_3 , j and m . The overlaps between functions belonging to different bases are the coefficients of the unitary transformation between the bases, as is clear in Dirac notation:

$$|(j_1 j_2) j_{12} j_3 j m\rangle = \sum_{j_{23}} \langle j_1 (j_2 j_3) j_{23} | (j_1 j_2) j_{12} j_3 j \rangle |j_1 (j_2 j_3) j_{23} j m\rangle. \quad (3.3)$$

The transformation coefficient, again by the general result proved at the end of section 1,

is diagonal in j and m and independent of the value of m , which is thus dropped from the symbol for the overlap.

Using the decomposition of the totally coupled states into totally uncoupled states in terms of Clebsch-Gordan coefficients, the transformation coefficient is explicitly written as

$$\begin{aligned} \langle j_1(j_2j_3)j_{23}j | (j_1j_2)j_{12}j_3j \rangle = & \sum_{m_1m_2m_3m_{12}m_{23}} \langle j_1m_1j_2m_2 | j_{12}m_{12} \rangle \langle j_{12}m_{12}j_3m_3 | jm \rangle \\ & \times \langle j_1m_1j_{23}m_{23} | jm \rangle \langle j_2m_2j_3m_3 | j_{23}m_{23} \rangle. \end{aligned} \quad (3.4)$$

Since the Clebsch-Gordan coefficients are real in the standard phase convention, so are these transformation (or recoupling) coefficients. Note that the recoupling coefficients are determined purely by the Clebsch-Gordan coefficients and are independent of any property of the states considered except their angular momentum quantum numbers.

The recoupling coefficient is commonly written in terms of a more symmetrical quantity, the Wigner 6-j symbol, as follows:

$$\langle (j_1j_2)j_{12}j_3j | j_1(j_2j_3)j_{23}j \rangle = (-)^{j_1+j_2+j_3+j} \sqrt{(2j_{12}+1)(2j_{23}+1)} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}. \quad (3.5)$$

An alternative notation is in terms of the Racah coefficient

$$W(j_1j_2j_3; j_{12}j_{23}) = (-)^{j_1+j_2+j_3+j} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}. \quad (3.6)$$

The 6-j symbol is real and its selection rules are the four triangle inequalities, $\Delta(j_1j_2j_{12})$, $\Delta(j_{12}j_3j)$, $\Delta(j_2j_3j_{23})$ and $\Delta(j_1j_{23}j)$, where

$$\Delta(j_1j_2j_3) \iff |j_1 - j_2| \leq j_3 \leq j_1 + j_2. \quad (3.7)$$

Its symmetry properties may be read off from those of the Clebsch-Gordan coefficients (which is much more easily done when the latter are replaced by 3-j symbols). It turns out to be invariant under those permutations of its six arguments which leave the set of four triangle inequalities unchanged. Depicting these inequalities in the self-explanatory form

$$\left\{ \begin{array}{ccc} \bullet & \bullet & \bullet \end{array} \right\} \quad \left\{ \begin{array}{ccc} & & \bullet \\ \bullet & \bullet & \end{array} \right\} \quad \left\{ \begin{array}{ccc} & \bullet & \\ \bullet & & \bullet \end{array} \right\} \quad \left\{ \begin{array}{ccc} \bullet & & \\ & \bullet & \bullet \end{array} \right\}$$

it is quickly seen that this set of four pictures, and hence the 6-j symbol, is unchanged under arbitrary permutation of its three columns or under simultaneous inversion $\begin{matrix} a \\ b \end{matrix} \longrightarrow \begin{matrix} b \\ a \end{matrix}$ of any

pair of columns. Since the 6-j symbol is entirely independent of any projection quantum numbers m , it may be regarded as a rotational invariant.

Other transformation coefficients can be expressed in terms of the one explicitly investigated above by suitable reorderings of the angular momenta being coupled, using the symmetries of the Clebsch-Gordan coefficients. For instance,

$$\begin{aligned}
\langle (j_1 j_3) j_{13} j_2 j | (j_1 j_2) j_{12} j_3 j \rangle &= (-)^{j_{13}+j_2-j} \langle j_2 (j_1 j_3) j_{13} j | (j_1 j_2) j_{12} j \rangle \\
&= (-)^{j_{13}+j_2-j} (-)^{j_1+j_2-j_{12}} \langle j_2 (j_1 j_3) j_{13} j | (j_2 j_1) j_{12} j_3 j \rangle \\
&= (-)^{j_{12}+j_{13}-j_1-j} (-)^{j_2+j_1+j_3+j} \\
&\quad \times \sqrt{(2j_{12}+1)(2j_{13}+1)} \begin{Bmatrix} j_2 & j_1 & j_{12} \\ j_3 & j & j_{13} \end{Bmatrix} \\
&= (-)^{j_2+j_3+j_{12}+j_{13}} \sqrt{(2j_{12}+1)(2j_{13}+1)} \begin{Bmatrix} j_2 & j_1 & j_{12} \\ j_3 & j & j_{13} \end{Bmatrix}.
\end{aligned}$$

(It should be noted that the 6-j symbol is defined by an overlap in which the three constituent angular momenta appear in precisely the same order in both totally coupled states. The reality of the 6-j symbol makes the order of the two totally coupled states in the overlap unimportant.)

In manipulating phase factors, as has been done above, the following observations are useful. When j_1 and j_2 are coupled to produce j_3 , any of the three may be, in principle, an integer or a half-odd-integer. However, as remarked above, in all cases either all three or only one of the j 's must be integral. This is sufficient to ensure that the combinations $j_1 + j_2 \pm j_3$ are integers, so that $(-)^{j_1+j_2-j_3}$, for instance, is real and is equal to its inverse, $(-)^{-j_1-j_2+j_3}$. However, $(-)^{j_1+j_2-j_3}$ is not necessarily equal to $(-)^{j_1+j_2+j_3}$ — these two phases are equal if j_3 is an integer, but opposite if it is a half-odd-integer. Similar arguments hold for j_1 , j_2 and j_3 coupled to produce j , where $j_1 + j_2 + j_3 + j$ is always an integer and $(-)^{j_1+j_2+j_3+j} = (-)^{-j_1-j_2-j_3-j}$ is real. Also, for any angular momentum j and its projection m , $(-)^{4j} = 1 = (-)^{4m}$ and $(-)^{j-m} = (-)^{m-j}$ is real, as is $(-)^{j+m} = (-)^{-j-m}$.

From the unitarity of the transformation between bases and the definition of the 6-j symbol, the following useful relationships may be written down.

$$\sum_{j_{23}} (2j_{23}+1) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j'_{12} \\ j_3 & j & j_{23} \end{Bmatrix} = \delta_{j_{12}j'_{12}} / (2j_{12}+1) \quad (3.8)$$

$$\begin{aligned} \sum_{m_{12}} \langle j_1 m_1 j_2 m_2 | j_{12} m_{12} \rangle \langle j_{12} m_{12} j_3 m_3 | j m \rangle &= \sum_{j_{23} m_{23}} (-)^{j_1+j_2+j_3+j} \sqrt{(2j_{12}+1)(2j_{23}+1)} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \\ &\times \langle j_1 m_1 j_{23} m_{23} | j m \rangle \langle j_2 m_2 j_3 m_3 | j_{23} m_{23} \rangle \end{aligned} \quad (3.9)$$

$$\begin{aligned} &\sum_{m_2 m_3 m_{12}} \langle j_1 m_1 j_2 m_2 | j_{12} m_{12} \rangle \langle j_{12} m_{12} j_3 m_3 | j m \rangle \langle j_2 m_2 j_3 m_3 | j_{23} m_{23} \rangle \\ &= (-)^{j_1+j_2+j_3+j} \sqrt{(2j_{12}+1)(2j_{23}+1)} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \langle j_1 m_1 j_{23} m_{23} | j m \rangle \end{aligned} \quad (3.10)$$

where the second equation is simply a rewriting of the definition of the unitary transformation coefficient and the third is derived from it by use of the orthogonality of the Clebsch-Gordan coefficients.

A sum rule for the 6-j symbol may be derived by considering different ways of achieving the same ultimate recoupling of three angular momenta. Clearly,

$$\langle (j_1 j_2) j_{12} j_3 j | j_2 (j_3 j_1) j_{13} j \rangle = \sum_{j_{23}} \langle (j_1 j_2) j_{12} j_3 j | j_1 (j_2 j_3) j_{23} j \rangle \langle j_1 (j_2 j_3) j_{23} j | j_2 (j_3 j_1) j_{13} j \rangle.$$

The transformation coefficients on both sides can be reduced to standard form by changing the order of coupling of pairs of angular momenta, with the appropriate phase factors, and rewritten in terms of 6-j symbols to yield

$$\sum_{j_{23}} (-)^{j_{12}+j_{13}+j_{23}} (2j_{23}+1) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \begin{Bmatrix} j_1 & j_3 & j_{13} \\ j_2 & j & j_{23} \end{Bmatrix} = \begin{Bmatrix} j_1 & j_3 & j_{13} \\ j & j_2 & j_{12} \end{Bmatrix}. \quad (3.11)$$

A somewhat more complex sum rule is obtained by considering the recoupling of four angular momenta, three at a time, in different ways. In an obvious notation, $(j_1 j_2) j_{12} j_3 j_{123} j_4 j \rightarrow (j_2 j_3) j_{23} j_1 j_{123} j_4 j \rightarrow (j_2 j_3) j_{23} (j_1 j_4) j_{14} j$; or $(j_1 j_2) j_{12} j_3 j_{123} j_4 j \rightarrow (j_1 j_2) j_{12} j_4 j_{124} j_3 j \rightarrow (j_1 j_4) j_{14} j_2 j_{124} j_3 j \rightarrow (j_1 j_4) j_{14} (j_2 j_3) j_{23} j$. The resulting equality can be converted into the Biedenharn-Elliott sum rule for 6-j symbols,

$$\begin{aligned} &\sum_{j_{124}} (-)^{j_1+j_2+j_3+j_4+j_{12}+j_{23}+j_{14}+j_{123}+j_{124}+j} (2j_{124}+1) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_{124} & j_4 & j_{14} \end{Bmatrix} \begin{Bmatrix} j_{12} & j_3 & j_{123} \\ j & j_4 & j_{124} \end{Bmatrix} \\ &\times \begin{Bmatrix} j_2 & j_3 & j_{23} \\ j & j_{14} & j_{124} \end{Bmatrix} = \begin{Bmatrix} j_1 & j_{23} & j_{123} \\ j & j_4 & j_{14} \end{Bmatrix} \begin{Bmatrix} j_1 & j_{23} & j_{123} \\ j_3 & j_{12} & j_2 \end{Bmatrix}. \end{aligned} \quad (3.12)$$

The value of the 6-j symbol in the special case where one of its arguments is zero is easily determined by substitution in the equation relating the 6-j symbol to the Clebsch-Gordan coefficients, or by observing that

$$\langle (j_1 j_2) j_{12} 0 j | j_1 (j_2 0) j_{23} j \rangle = \delta_{j_{12} j} \delta_{j_{23} j}.$$

Either procedure produces the result

$$\left\{ \begin{matrix} j_1 & j_2 & j \\ j'_2 & j'_1 & 0 \end{matrix} \right\} = (-)^{j_1+j_2+j} \delta_{j_1 j'_1} \delta_{j_2 j'_2} / \sqrt{(2j_1+1)(2j_2+1)}. \quad (3.13)$$

The symmetry properties of the 6-j symbol can then be used to evaluate a 6-j symbol any of whose arguments is zero.

Inserting this special value of the 6-j symbol into the orthogonality relation or any of the sum rules for these symbols, by setting one of the free angular momenta (not an index of summation) equal to zero, will generally produce a trivial identity, but in a few cases it produces further useful sum rules. Setting $j'_{12} = 0$ in the orthogonality relation leads to

$$\sum_{j_{12}} (-)^{j_1+j_2+j_{12}} (2j_{12}+1) \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_2 & j_1 & j \end{matrix} \right\} = \sqrt{(2j_1+1)(2j_2+1)} \delta_{j0}. \quad (3.14)$$

Similarly, setting $j_{13} = 0$ in the first sum rule above produces

$$\sum_j (2j+1) \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_1 & j_2 & j \end{matrix} \right\} = (-)^{2(j_1+j_2)},$$

which can be rewritten in the form

$$\sum_j (-)^{2j} (2j+1) \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_1 & j_2 & j \end{matrix} \right\} = 1. \quad (3.15)$$

Setting an argument equal to zero in the Biedenharn-Elliott sum rule produces no new results.

For the case of four coupled angular momenta, the uncoupled basis $|j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4\rangle$ is related to the fully coupled basis $|j_1 j_2 j_3 j_4 j' j'' j m\rangle$ (where j' and j'' are intermediate couplings required to complete a full set of eight mutually commuting operators) by a unitary transformation. The number of possible sets of fully coupled basis functions is now quite

large (illustrative examples are $j' = j_{12}, j'' = j_{123}; j' = j_{13}, j'' = j_{134}; j' = j_{12}, j'' = j_{34}$), but they fall into two classes — bases where two angular momenta are coupled together, the result coupled to a third and that result to the fourth angular momentum, or bases where two distinct pairs of angular momenta are each coupled together and then their resultants are coupled to produce the total angular momentum. Any two such bases are related by a unitary transformation. The first type is of considerably less interest than the second and can be handled by successive uses of the 6-j symbol. The second type is sufficiently useful to have warranted the introduction of a new recoupling coefficient.

The prototype transformation coefficient for four angular momenta coupled pairwise is $\langle (j_1 j_2) j_{12} (j_3 j_4) j_{34} j | (j_1 j_3) j_{13} (j_2 j_4) j_{24} j \rangle$, where again the overlap between states from different bases is diagonal in j and m and independent of m , which is dropped from the transformation bracket. (The overlap is also, of course, diagonal in the common quantum numbers j_1, j_2, j_3, j_4 .) This can be written in terms of Clebsch-Gordan coefficients as

$$\sum_{m_1 m_2 m_3 m_4 m_{12} m_{34} m_{13} m_{24}} \langle j_1 m_1 j_2 m_2 | j_{12} m_{12} \rangle \langle j_3 m_3 j_4 m_4 | j_{34} m_{34} \rangle \langle j_{12} m_{12} j_{34} m_{34} | j m \rangle \\ \times \langle j_1 m_1 j_3 m_3 | j_{13} m_{13} \rangle \langle j_2 m_2 j_4 m_4 | j_{24} m_{24} \rangle \langle j_{13} m_{13} j_{24} m_{24} | j m \rangle,$$

and is hence real. Once again it turns out to be convenient to introduce a slightly different recoupling coefficient, the Wigner 9-j symbol, defined by

$$\langle (j_1 j_2) j_{12} (j_3 j_4) j_{34} j | (j_1 j_3) j_{13} (j_2 j_4) j_{24} j \rangle = \\ \sqrt{(2j_{12} + 1)(2j_{34} + 1)(2j_{13} + 1)(2j_{24} + 1)} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix}. \quad (3.16)$$

Converting the Clebsch-Gordan coefficients to 3-j symbols and exploiting the m -independence of the recoupling coefficient, it is found that

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} = \sum_{\text{all } m\text{'s}} \begin{pmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{pmatrix} \begin{pmatrix} j_3 & j_4 & j_{34} \\ m_3 & m_4 & m_{34} \end{pmatrix} \begin{pmatrix} j_{13} & j_{24} & j \\ m_{13} & m_{24} & m \end{pmatrix} \\ \times \begin{pmatrix} j_1 & j_3 & j_{13} \\ m_1 & m_3 & m_{13} \end{pmatrix} \begin{pmatrix} j_2 & j_4 & j_{24} \\ m_2 & m_4 & m_{24} \end{pmatrix} \begin{pmatrix} j_{12} & j_{34} & j \\ m_{12} & m_{34} & m \end{pmatrix} \quad (3.17)$$

where each row and each column of the 9-j symbol is represented by a 3-j symbol and all

nine projection quantum numbers m are summed over. The result, independent of projection quantum numbers, may be regarded as a rotational invariant.

The selection rules for the 9-j symbol are now trivial — each row and each column of the symbol contributes a triangle inequality which must be satisfied if the symbol is not to vanish. The symmetries of the 9-j symbol are also easily read off: the interchange of a pair of rows or of a pair of columns produces a phase factor $(-)^{j_1+j_2+j_3+j_4+j_{12}+j_{34}+j_{13}+j_{24}+j}$, while transposition of the symbol (writing the rows as columns and the columns as rows by reflecting in the main diagonal) simply reorders the six 3-j symbols inside the summation and so leaves the symbol unchanged in value. The former symmetry introduces a further selection rule — if any two columns or any two rows of the symbol are identical, then it vanishes unless the sum of its nine arguments is even (or, equivalently, unless the sum of the angular momenta in the remaining column or row is even).

As before, the unitarity of the transformation imposes orthogonality conditions on the 9-j symbols, in the form

$$\sum_{j_{13}j_{24}} (2j_{13}+1)(2j_{24}+1) \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{matrix} \right\} \left\{ \begin{matrix} j_1 & j_2 & j'_{12} \\ j_3 & j_4 & j'_{34} \\ j_{13} & j_{24} & j \end{matrix} \right\} = \delta_{j_{12}j'_{12}} \delta_{j_{34}j'_{34}} / (2j_{12}+1)(2j_{34}+1). \quad (3.18)$$

The 9-j symbol can be expressed in terms of 6-j symbols by recoupling four angular momenta in stages, with three angular momenta being recoupled at each stage, as follows:

$$\begin{aligned} \langle (j_1 j_2) j_{12} (j_3 j_4) j_{34} j | (j_1 j_3) j_{13} (j_2 j_4) j_{24} j \rangle &= \sum_{j_{123}} \langle (j_1 j_2) j_{12} (j_3 j_4) j_{34} j | (j_1 j_2) j_{12} j_3 j_{123} j_4 j \rangle \\ &\times \langle (j_1 j_2) j_{12} j_3 j_{123} j_4 j | (j_1 j_3) j_{13} j_2 j_{123} j_4 j \rangle \langle (j_1 j_3) j_{13} j_2 j_{123} j_4 j | (j_1 j_3) j_{13} (j_2 j_4) j_{24} j \rangle \end{aligned}$$

from which it follows that

$$\left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{matrix} \right\} = \sum_{j_{123}} (-)^{2j_{123}} (2j_{123}+1) \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_{123} & j_3 & j_{13} \end{matrix} \right\} \left\{ \begin{matrix} j_3 & j_4 & j_{34} \\ j & j_{12} & j_{123} \end{matrix} \right\} \left\{ \begin{matrix} j_{13} & j_{24} & j \\ j_4 & j_{123} & j_2 \end{matrix} \right\}. \quad (3.19)$$

As in the case of the expression for the 6-j symbol in terms of Clebsch-Gordan coefficients, this result may be manipulated, using the orthogonality of the 6-j symbols, to express a product of two 6-j symbols as a sum of products of a 6-j and a 9-j symbol. Successive recouplings of four angular momenta can be used to obtain a sum rule expressing a single

9-j symbol as a sum of products of pairs of 9-j symbols. The value of the 9-j symbol when one of its arguments vanishes is also easily obtained from the expression in terms of 6-j symbols. Given the symmetries of the 9-j symbol, it is sufficient to evaluate

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & 0 \end{Bmatrix} = (-)^{j_2+j_{12}+j_3+j_{13}} \delta_{j_{12}j_{34}} \delta_{j_{13}j_{24}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_4 & j_3 & j_{13} \end{Bmatrix} / \sqrt{(2j_{12}+1)(2j_{13}+1)}. \quad (3.20)$$

A useful special case arises when a whole row or column of arguments vanishes. This is covered by the specific result

$$\begin{Bmatrix} j_1 & j'_1 & 0 \\ j_2 & j'_2 & 0 \\ j_3 & j'_3 & 0 \end{Bmatrix} = \delta_{j_1 j'_1} \delta_{j_2 j'_2} \delta_{j_3 j'_3} / \sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}, \quad (3.21)$$

where j_1, j_2 and j_3 satisfy the triangle relation.

IV. SPHERICAL TENSOR OPERATORS

Now that the quantum states of a system have been characterized in terms of their angular momentum properties, it becomes of interest to investigate the effect on these properties of various operators which act in the space of states. Consider, in the simplest case, the position and momentum operators of a single spinless particle with orbital angular momentum operator $\vec{L} = (\vec{r} \times \vec{p})/\hbar$. Using the standard commutation relation $[r_\alpha, p_\beta] = i\hbar\delta_{\alpha\beta}$, where the indices α, β represent the cartesian components x, y, z , it is easily established that the commutator $[L_\alpha, r_\beta]$ is a simple linear combination of the components of $\vec{r}$, while $[L_\alpha, p_\beta]$ is the same linear combination of the corresponding components of $\vec{p}$. It is possible to define specific linear combinations r_m and p_m of the components of $\vec{r}$ and $\vec{p}$ respectively, with $m = -1, 0, +1$, such that

$$[L_z, v_m] = m v_m \quad (4.1)$$

$$[L_\pm, v_m] = \alpha_{1m}^{(\pm)} v_{m\pm 1} \quad (4.2)$$

where $\vec{v}$ represents either $\vec{r}$ or $\vec{p}$ and where

$$v_0 = v_z \quad (4.3)$$

$$v_{\pm 1} = \mp \frac{1}{\sqrt{2}}(v_x \pm i v_y). \quad (4.4)$$

These eigenoperators of $\vec{L}$ have properties reminiscent of the defining properties of a set of states of angular momentum 1, projection m , with the commutator of $\vec{L}$ with the operators v_m playing the role of the action of the operator $\vec{L}$ on the states $|1m\rangle$. This property of the operators $\vec{r}$ and $\vec{p}$ can be generalized and is extremely useful in all applications of the angular momentum algebra.

Before going on to the general case, consider the set of operators $r_\alpha p_\beta$, the nine possible products of the cartesian components ($\alpha, \beta = x, y, z$) of $\vec{r}$ and $\vec{p}$. Once again, the commutators of $\vec{L}$ with these operators produce only linear combinations of operators in the same set. This is guaranteed by the fact that $\vec{r}$ and $\vec{p}$ have this property, while commutators satisfy the identity $[A, BC] = [A, B]C + B[A, C]$. The operator products $r_\alpha p_\beta$ fall naturally into three subsets — the combination $\vec{r} \cdot \vec{p}$ (the sum of the diagonal elements of the set, called the trace), the vector product $\vec{r} \times \vec{p}$ (the three antisymmetric combinations $r_\alpha p_\beta - r_\beta p_\alpha$) and the five independent symmetric combinations $r_\alpha p_\beta + r_\beta p_\alpha - \frac{2}{3}\delta_{\alpha\beta}\vec{r} \cdot \vec{p}$, with vanishing trace. Each product $r_\alpha p_\beta$ can be written as a sum of terms from the three subsets. It is then found that each of these subsets is closed under commutation with $\vec{L}$, i.e. the commutator of any component of $\vec{L}$ with a member of one of the subsets is a linear combination of members of the same subset. The scalar product $\vec{r} \cdot \vec{p}$ commutes with all the components of $\vec{L}$; the vector product $\vec{r} \times \vec{p}$ has the same commutation relations with $\vec{L}$ as the individual vectors $\vec{r}$ and $\vec{p}$; while specific linear combinations $(\vec{r}\vec{p})_m$ of the elements of the symmetric subset can be found for which

$$[L_z, (\vec{r}\vec{p})_m] = m(\vec{r}\vec{p})_m \quad (4.5)$$

$$[L_\pm, (\vec{r}\vec{p})_m] = \alpha_{2m}^{(\pm)} (\vec{r}\vec{p})_{m\pm 1}. \quad (4.6)$$

These results generalize to other operators than $\vec{r}$, $\vec{p}$ and their products and to more general angular momentum operators than $\vec{L}$. Given any set of operators which is closed under commutation with the components of the angular momentum $\vec{j}$, it can always be broken up into subsets, each of which is itself closed under commutation with $\vec{j}$ but cannot be broken up into smaller subsets with the same property of closure. Any of the original operators can be written as a linear combination of operators from these subsets. Within each such irreducible subset, specific linear combinations can be chosen so that their commutation relations with $\vec{j}$ have the standard form encountered twice above, analogous to

that of sets of states $|jm\rangle$ acted upon by the operator $\vec{j}$. Further it can be demonstrated that any arbitrary operator can be written as a linear combination of operators belonging to such standard sets. These general statements are systematized in the definition of an (irreducible) spherical tensor operator.

An (irreducible) spherical tensor operator $T_m^{(j)}$ is one of a set of $2j + 1$ operators, corresponding to different values of $m = -j, -j + 1, \dots, j - 1, j$, which satisfy

$$[j_z, T_m^{(j)}] = mT_m^{(j)} \quad (4.7)$$

$$[j_{\pm}, T_m^{(j)}] = \alpha_{jm}^{(\pm)} T_{m\pm 1}^{(j)}. \quad (4.8)$$

The spherical tensor operator $T_m^{(j)}$ is said to be of rank j , projection m . This definition essentially generalizes the notion of angular momentum of states to include that of angular momentum of operators.

It is now possible to investigate the effect on the angular momentum of a state of the action of a spherical tensor operator. Consider the state produced by operating with $T_m^{(j)}$ on $|j'm'\rangle$. It is still an eigenstate of j_z , since

$$\begin{aligned} j_z T_m^{(j)} |j'm'\rangle &= [j_z, T_m^{(j)}] |j'm'\rangle + T_m^{(j)} j_z |j'm'\rangle \\ &= mT_m^{(j)} |j'm'\rangle + T_m^{(j)} m' |j'm'\rangle \\ &= (m + m') T_m^{(j)} |j'm'\rangle, \end{aligned}$$

but its eigenvalue has become $m + m'$. It is tedious but straightforward to check that $T_m^{(j)} |j'm'\rangle$ is not, however, an eigenstate of $\vec{j}^2$. But it is essentially a product of two objects of well-defined angular momenta j and j' , with well-defined projection quantum numbers m and m' . The analogy with the product state $|j_1 m_1 j_2 m_2\rangle$ suggests that an appropriate linear combination of such products using, of course, the Clebsch-Gordan coefficients may have well-defined angular momentum. Consider, therefore, the state

$$|JM\rangle_{jj'} = \sum_{mm'} \langle jmj'm' | JM \rangle T_m^{(j)} |j'm'\rangle.$$

Because of the Clebsch-Gordan selection rule $M = m + m'$, this is, in fact, an eigenstate of

j_z with eigenvalue M , as suggested by the notation. Now,

$$\begin{aligned}
j_{\pm}|JM\rangle_{jj'} &= \sum_{mm'} \langle jmj'm'|JM\rangle \{[j_{\pm}, T_m^{(j)}] + T_m^{(j)}j_{\pm}\}|j'm'\rangle \\
&= \sum_{mm'} \langle jmj'm'|JM\rangle \{\alpha_{jm}^{(\pm)}T_{m\pm 1}^{(j)}|j'm'\rangle + \alpha_{j'm'}^{(\pm)}T_m^{(j)}|j', m' \pm 1\rangle\} \\
&= \sum_{mm'} [\alpha_{j, m\mp 1}^{(\pm)}\langle j, m \mp 1, j'm'|JM\rangle + \alpha_{j', m'\mp 1}^{(\pm)}\langle jmj', m' \mp 1|JM\rangle] T_m^{(j)}|j'm'\rangle \\
&= \sum_{mm'} \alpha_{JM}^{(\pm)} \langle jmj'm'|J, M \pm 1\rangle T_m^{(j)}|j'm'\rangle \\
&= \alpha_{JM}^{(\pm)}|J, M \pm 1\rangle_{jj'}
\end{aligned}$$

where use has been made of the Clebsch-Gordan coefficient recursion relations and of the restrictions on the ranges of the projection quantum numbers. But this result is precisely what is required to establish $|JM\rangle_{jj'}$ as an eigenstate of $\vec{j}^2$ with eigenvalue $J(J+1)$, thus fully justifying the notation used.

The Clebsch-Gordan coefficients thus couple the rank of a spherical tensor operator to the angular momentum of the state on which it acts so that the state so produced is a well-defined angular momentum eigenstate. The orthogonality of the Clebsch-Gordan coefficients can now be used to write

$$T_m^{(j)}|j'm'\rangle = \sum_{JM} \langle jmj'm'|JM\rangle |JM\rangle_{jj'},$$

which can be used to evaluate the matrix element

$$\begin{aligned}
\langle j''m''|T_m^{(j)}|j'm'\rangle &= \sum_{JM} \langle jmj'm'|JM\rangle \langle j''m''|JM\rangle_{jj'} \\
&= \sum_{JM} \langle jmj'm'|JM\rangle \Theta_{j''jj'} \delta_{j''J} \delta_{m''M} \\
&= \langle jmj'm'|j''m''\rangle \Theta_{j''jj'},
\end{aligned}$$

where $\Theta_{j''jj'}$, the overlap between the states $|j''m''\rangle$ and $|JM\rangle_{jj'}$, is diagonal in $j''(J)$ and in $m''(M)$ and independent of M . Therefore the dependence of the matrix element on the projection quantum numbers is contained entirely in the Clebsch-Gordan coefficient and is independent of the detailed dynamics of the states and of the operator involved. This very powerful and extremely useful result is known as the Wigner-Eckart theorem. It should be emphasized that all that is required for the theorem to hold is for the two states concerned to be eigenstates of the same angular momentum operator relative to which the operator involved is a spherical tensor operator. Any such matrix element can then be regarded as

the product of a purely geometrical factor, the Clebsch-Gordan coefficient, containing all the dependence on the projection quantum numbers, and a reduced matrix element, independent of the projection quantum numbers, which contains the dynamics of the situation. There are rival definitions of the reduced matrix element, but the one probably most widely used is

$$\langle j''m''|T_m^{(j)}|j'm'\rangle = (-)^{j''-m''} \begin{pmatrix} j'' & j & j' \\ -m'' & m & m' \end{pmatrix} \langle j''||T^{(j)}||j'\rangle \quad (4.9)$$

where the double-barred matrix element without projection quantum numbers is the reduced matrix element. In terms of Clebsch-Gordan coefficients, this definition reads

$$\langle j''m''|T_m^{(j)}|j'm'\rangle = (-)^{j''+j-j'} \langle jmj'm'|j''m''\rangle \langle j''||T^{(j)}||j'\rangle / \sqrt{2j''+1}. \quad (4.10)$$

Some useful reduced matrix elements are easily derived. From the basic definitions, the unit operator is a spherical scalar (a spherical tensor of rank zero), while j_m , defined analogously to v_m above, are the components of a spherical vector (a spherical tensor of rank one). Thus

$$\langle jm|1|j'm'\rangle = \delta_{jj'}\delta_{mm'} = (-)^{j-m} \begin{pmatrix} j & 0 & j' \\ -m & 0 & m' \end{pmatrix} \langle j||1||j'\rangle$$

implies that

$$\langle j||1||j'\rangle = \delta_{jj'}\sqrt{2j+1}, \quad (4.11)$$

while

$$\langle jm|j_z|j'm'\rangle = m\delta_{jj'}\delta_{mm'} = (-)^{j-m} \begin{pmatrix} j & 1 & j' \\ -m & 0 & m' \end{pmatrix} \langle j||\vec{j}||j'\rangle$$

and

$$\begin{pmatrix} j & 1 & j \\ -m & 0 & m \end{pmatrix} = (-)^{j-m} m / \sqrt{j(j+1)(2j+1)} \quad (4.12)$$

imply that

$$\langle j||\vec{j}||j'\rangle = \delta_{jj'}\sqrt{j(j+1)(2j+1)}. \quad (4.13)$$

The simplest and most immediate consequence of the Wigner-Eckart theorem is that the matrix element $\langle j''m''|T_m^{(j)}|j'm'\rangle$ is governed by the same selection rules as the Clebsch-Gordan coefficient $\langle jmj'm'|j''m''\rangle$ (and possibly additional selection rules arising from the reduced matrix element), namely $m'' = m + m'$ and $|j' - j''| \leq j \leq j' + j''$. The latter

condition ensures that a state of angular momentum j can have no static multipole moment of order $\lambda > 2j$, since the static multipole moment of order λ is defined in terms of the diagonal matrix element $\langle j, m = j | T_0^{(\lambda)} | j, m = j \rangle$, which vanishes, by the triangle inequality, unless $\lambda \leq 2j$. (A single matrix element is sufficient to define the multipole moment of a state, since all other matrix elements can be related to any one of them by the Wigner-Eckart theorem.) Hence a state of angular momentum zero cannot have a non-zero dipole moment (or moment of any higher order), a state of angular momentum $\frac{1}{2}$ cannot have a non-zero quadrupole moment (or moment of any higher order), etc. The same condition allows a quick classification of the possible operators on a system. For example, a single spin- $\frac{1}{2}$ particle can be acted upon only by spin operators of rank zero or one, any operator of higher rank having only vanishing matrix elements. From this and the properties of the spin operators $\vec{s}$ it is easy to conclude that the only available spin operators for a single spin- $\frac{1}{2}$ particle are the unit operator and the operator $\vec{s}$. For a system of two spin- $\frac{1}{2}$ particles, the total spin of the system can be only 0 or 1, so only spin operators of ranks 0, 1 and 2 will be effective, the last-named only in the spin-1 subspace.

Perhaps the best-known application of the Wigner-Eckart theorem is in deriving the Landé formula for the matrix elements of an arbitrary vector operator. In general, given two spherical tensor operators of the same rank and projection operating in the same space, the ratio of their non-vanishing matrix elements between a pair of angular momentum eigenstates will be independent of all projection quantum numbers, i.e.

$$\langle jm | T_M^{(L)} | j' m' \rangle = A_{jj'}^{(L)} \langle jm | U_M^{(L)} | j' m' \rangle$$

when neither matrix element is zero. The factor $A_{jj'}^{(L)}$ is a ratio of reduced matrix elements. Thus, the matrix elements of an arbitrary vector operator $\vec{V}$ can be related to those of a specific vector operator, the angular momentum itself, provided the value of the quantum number j is the same on both sides of the matrix element (since $\vec{j}$ is always diagonal in j). So

$$\langle jm | \vec{V} | j m' \rangle = A_j \langle jm | \vec{j} | j m' \rangle$$

To evaluate A_j , consider the diagonal matrix element

$$\begin{aligned}
\langle jm | \vec{V} \cdot \vec{j} | jm \rangle &= \sum_{j'm'} \langle jm | \vec{V} | j'm' \rangle \cdot \langle j'm' | \vec{j} | jm \rangle \\
&= \sum_{j'm'} \langle jm | \vec{V} | j'm' \rangle \cdot \langle j'm' | \vec{j} | jm \rangle \delta_{jj'} \\
&= \sum_{j'm'} A_j \langle jm | \vec{j} | j'm' \rangle \cdot \langle j'm' | \vec{j} | jm \rangle \\
&= A_j \langle jm | \vec{j}^2 | jm \rangle \\
&= A_j j(j+1),
\end{aligned}$$

where $|j'm'\rangle$ is a complete set of intermediate states. Finally,

$$\langle jm | \vec{V} | jm' \rangle = \langle jm | \vec{V} \cdot \vec{j} | jm \rangle \langle jm | \vec{j} | j'm' \rangle / j(j+1), \quad (4.14)$$

the Landé formula.

In the same way that the product of two angular momentum eigenstates is not automatically an angular momentum eigenstate, with Clebsch-Gordan coefficients being required to generate linear combinations of such products which are angular momentum eigenstates, so the product of two spherical tensor operators of given ranks and projections is not automatically a spherical tensor operator of well-defined rank. However, as might be expected, a linear combination of such products, again using the same Clebsch-Gordan coefficients, is in fact a well-defined spherical tensor operator. Such a tensor product is given by

$$[T^{(j_1)} \otimes T^{(j_2)}]_m^{(j)} = \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | jm \rangle T_{m_1}^{(j_1)} T_{m_2}^{(j_2)} \quad (4.15)$$

and is a spherical tensor operator of rank j and projection m , as may be demonstrated in the following way:

$$\begin{aligned}
[j_z, [T^{(j_1)} \otimes T^{(j_2)}]_m^{(j)}] &= \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | jm \rangle \{ [j_z, T_{m_1}^{(j_1)}] T_{m_2}^{(j_2)} + T_{m_1}^{(j_1)} [j_z, T_{m_2}^{(j_2)}] \} \\
&= \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | jm \rangle (m_1 + m_2) T_{m_1}^{(j_1)} T_{m_2}^{(j_2)} \\
&= m [T^{(j_1)} \otimes T^{(j_2)}]_m^{(j)},
\end{aligned}$$

using the selection rules of the Clebsch-Gordan coefficients. A similar manipulation, making explicit use of the Clebsch-Gordan coefficient recursion relations, demonstrates that

$$[j_\pm, [T^{(j_1)} \otimes T^{(j_2)}]_m^{(j)}] = \alpha_{jm}^{(\pm)} [T^{(j_1)} \otimes T^{(j_2)}]_{m\pm 1}^{(j)},$$

which proves the assertion made above.

It is conventional to define the scalar product of two spherical tensors of the same rank by

$$T^{(j)} \cdot U^{(j)} = \sum_m (-)^m T_{-m}^{(j)} U_m^{(j)}, \quad (4.16)$$

which reduces to the usual definition of the scalar product for vector (rank-1) operators. By inspection of the Clebsch-Gordan coefficient $\langle jm_1 jm_2 | 00 \rangle$, it is seen that

$$T^{(j)} \cdot U^{(j)} = (-)^j \sqrt{2j+1} [T^{(j)} \otimes U^{(j)}]_0^{(0)}. \quad (4.17)$$

(This definition of the scalar product is not particularly convenient for operators of half-odd-integer rank, where it implies imaginary phase factors. In that case, it is more convenient to define the scalar product as

$$T^{(j)} \cdot U^{(j)} = \sum_m (-)^{j+m} T_{-m}^{(j)} U_m^{(j)} = \sqrt{2j+1} [T^{(j)} \otimes U^{(j)}]_0^{(0)}.)$$

Similarly, it may be checked that the usual vector product of two vector operators is related to the rank-1 tensor product of two rank-1 spherical tensor operators by

$$\vec{T} \times \vec{V} = -i\sqrt{2} [T^{(1)} \otimes V^{(1)}]^{(1)}. \quad (4.18)$$

For products of more than two spherical tensor operators, precisely the same recoupling coefficients can be used as were introduced to deal with the coupling of more than two angular momenta, since they are determined by the Clebsch-Gordan coefficients in exactly the same way.

As a useful illustration of the manipulation of tensor products, consider the well-known tensor force operator $S_{12} = (\vec{\sigma}_1 \cdot \hat{r})(\vec{\sigma}_2 \cdot \hat{r}) - \vec{\sigma}_1 \cdot \vec{\sigma}_2/3$, where $\vec{\sigma}_i$ are spin operators and $\hat{r}$ a unit position vector. This is related in form to the interaction between a pair of magnetic dipoles. The first term may be rewritten as

$$3[[\vec{\sigma}_1 \otimes \hat{r}]^{(0)} \otimes [\vec{\sigma}_2 \otimes \hat{r}]^{(0)}]^{(0)} = 3 \sum_{LS} \sqrt{(2L+1)(2S+1)} \begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ S & L & 0 \end{Bmatrix} [[\vec{\sigma}_1 \otimes \vec{\sigma}_2]^{(S)} \otimes [\hat{r} \otimes \hat{r}]^{(L)}]^{(0)}$$

where the 9-j symbol has been used to recouple the four spherical tensors of rank 1 and where use has been made of the fact that the spin operators $\vec{\sigma}_i$ commute with the position operator $\hat{r}$. This 9-j symbol is equal to $\delta_{LS}/3\sqrt{2L+1}$, so the double summation reduces to a

single sum over S , which can take on the values 0, 1 and 2, from the 9-j selection rules. The corresponding terms in the sum contain, respectively, $[\hat{r} \otimes \hat{r}]^{(0)} = -\frac{1}{\sqrt{3}}$, $[\hat{r} \otimes \hat{r}]^{(1)} = 0$ (since the vector product $\vec{v} \times \vec{v}$ vanishes when the components of v commute with one another) and $[\hat{r} \otimes \hat{r}]^{(2)}$. So the first term in the tensor force operator becomes

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 / 3 + \sqrt{5} [[\vec{\sigma}_1 \otimes \vec{\sigma}_2]^{(2)} \otimes [\hat{r} \otimes \hat{r}]^{(2)}]^{(0)}$$

and the tensor force operator itself can be rewritten

$$S_{12} = [\vec{\sigma}_1 \otimes \vec{\sigma}_2]^{(2)} \cdot [\hat{r} \otimes \hat{r}]^{(2)}. \quad (4.19)$$

It is thus an overall scalar operator, formed by the scalar product of a rank-2 tensor operator in spin space and a rank-2 tensor operator in position space.

Since tensor products are themselves spherical tensor operators, they obey the Wigner-Eckart theorem and the quantity of interest for practical applications is their reduced matrix element between angular momentum states. Note that, from the definition of the reduced matrix element,

$$\langle j'' \| T^{(j)} \| j' \rangle = (-)^{2j} \sqrt{2j'' + 1} \sum_{mm'} \langle j' m' j m | j'' m'' \rangle \langle j'' m'' | T_m^{(j)} | j' m' \rangle, \quad (4.20)$$

by the orthogonality of the Clebsch-Gordan coefficients.

Now suppose that $T^{(j_1)}$ and $T^{(j_2)}$ operate in different spaces and consider the relevant reduced matrix element

$$\begin{aligned} \langle j_1'' j_2'' j'' \| [T^{(j_1)} \otimes T^{(j_2)}]^{(j)} \| j_1' j_2' j' \rangle &= (-)^{2j} \sqrt{2j'' + 1} \sum_{mm'} \langle j' m' j m | j'' m'' \rangle \\ &\quad \times \langle j_1'' j_2'' j'' m'' | [T^{(j_1)} \otimes T^{(j_2)}]_m^{(j)} | j_1' j_2' j' m' \rangle \\ &= (-)^{2j} \sqrt{2j'' + 1} \sum_{m_1 m_2 m_1' m_2' m_1'' m_2'' m m'} \langle j' m' j m | j'' m'' \rangle \\ &\quad \times \langle j_1 m_1 j_2 m_2 | j m \rangle \langle j_1'' m_1'' j_2'' m_2'' | j'' m'' \rangle \langle j_1' m_1' j_2' m_2' | j' m' \rangle \\ &\quad \times (-)^{2j_1} \langle j_1' m_1' j_1 m_1 | j_1'' m_1'' \rangle \langle j_1'' \| T^{(j_1)} \| j_1' \rangle \\ &\quad \times (-)^{2j_2} \langle j_2' m_2' j_2 m_2 | j_2'' m_2'' \rangle \langle j_2'' \| T^{(j_2)} \| j_2' \rangle / \sqrt{(2j_1'' + 1)(2j_2'' + 1)} \\ &= \sqrt{((2j + 1)(2j' + 1)(2j'' + 1))} \left\{ \begin{matrix} j_1'' & j_2'' & j'' \\ j_1' & j_2' & j' \\ j_1 & j_2 & j \end{matrix} \right\} \\ &\quad \times \langle j_1'' \| T^{(j_1)} \| j_1' \rangle \langle j_2'' \| T^{(j_2)} \| j_2' \rangle, \end{aligned} \quad (4.21)$$

using the expression for the recoupling coefficient of four angular momenta in terms of a sum of products of six Clebsch-Gordan coefficients. This is easily specialised to the interesting cases

$$\langle j_1'' j_2'' j'' \| T^{(j)} \cdot U^{(j)} \| j_1' j_2' j' \rangle = \delta_{j_1'' j_2''} \sqrt{2j' + 1} (-)^{j_1' + j_2' + j''} \left\{ \begin{matrix} j_1'' & j_2'' & j'' \\ j_2' & j_1' & j \end{matrix} \right\} \langle j_1'' \| T^{(j)} \| j_1' \rangle \langle j_2'' \| U^{(j)} \| j_2' \rangle \quad (4.22)$$

and

$$\langle j_1'' j_2'' j'' \| T^{(j)} \| j_1' j_2' j' \rangle = \delta_{j_2'' j_2'} \sqrt{(2j'' + 1)(2j' + 1)} (-)^{j_1'' + j_2'' + j' + j} \left\{ \begin{matrix} j_1'' & j_2'' & j'' \\ j' & j & j_1' \end{matrix} \right\} \langle j_1'' \| T^{(j)} \| j_1' \rangle \quad (4.23)$$

where, in the last equation, $T^{(j)}$ acts only in the j_1 space.

If, however, $T^{(j_1)}$ and $T^{(j_2)}$ operate in the same space, then

$$\begin{aligned} \langle j'' \| [T^{(j_1)} \otimes T^{(j_2)}]^{(j)} \| j' \rangle &= (-)^{2j} \sqrt{2j'' + 1} \sum_{mm'm_1m_2LM} \langle j' m' j m | j'' m'' \rangle \langle j_1 m_1 j_2 m_2 | j m \rangle \\ &\quad \times (-)^{2j_1} \langle L M j_1 m_1 | j'' m'' \rangle \langle j'' \| T^{(j_1)} \| L \rangle (-)^{2j_2} \langle j' m' j_2 m_2 | L M \rangle \\ &\quad \times \langle L \| T^{(j_2)} \| j' \rangle / \sqrt{(2j'' + 1)(2L + 1)} \\ &= (-)^{j'' + j' + j} \sqrt{2j + 1} \sum_L \left\{ \begin{matrix} j_1 & j_2 & j \\ j' & j'' & L \end{matrix} \right\} \langle j'' \| T^{(j_1)} \| L \rangle \langle L \| T^{(j_2)} \| j' \rangle \end{aligned}$$

where a sum over a complete set of states has been inserted to separate the operators $T^{(j_1)}$ and $T^{(j_2)}$.

V. ROTATION OPERATORS

Classically, angular momentum is clearly associated with rotations, the component of angular momentum along a given axis being the classical variable conjugate to the angle of rotation about the same axis. In quantum mechanics, rotation of a system is an operation which induces a change in the description of the states of the system within its Hilbert space. Given a possible state ψ of the system, rotation of the system in this state through some angle about some axis produces another possible state ψ' of the system. Such properties as the norms of states and their mutual orthogonality (in general, the absolute squares of their overlaps) cannot be changed by such an operation, so it must be equivalent to a unitary transformation in the Hilbert space.

Hence, to every rotation of a physical system there corresponds a unitary operator in the Hilbert space of states of the system. Such an operator can be represented as $e^{-i\theta M}$, where θ is real and M is a hermitian operator. Then θ determines the angle of rotation (with $\theta = 0$, i.e. no rotation, producing the unit operator) and M is an operator associated with the axis of rotation. Since a three-dimensional rotation is determined by three angles, two to define the direction of the rotation axis and one to define the angle of rotation about the axis, there must be three rotational parameters and three associated operators M to describe a rotation fully. A possible representation of a general rotation is then $e^{-i\vec{\theta}\cdot\vec{M}}$, where M_x , M_y and M_z are operators associated with rotations about the corresponding coordinate axes and $\vec{\theta}$ is a vector in the direction of the rotation axis and with magnitude equal to the angle of rotation. Implicit in the notation adopted here is the recognition that an arbitrary rotation can be built up by an infinite number of infinitesimal rotations about the same axis. Thus the infinitesimal rotations $e^{-i\delta\theta_\alpha M_\alpha} \simeq 1 - i\delta\theta_\alpha M_\alpha$ (where $\delta\theta_\alpha$ is an infinitesimal angle) are sufficient to determine the properties of the operators M_α , which are referred to as the infinitesimal generators of the rotations.

Euler's theorem states that any motion of a rigid body with one point fixed is equivalent to a rotation about some axis through the fixed point. So the result of two successive rotations about arbitrary axes through a given fixed origin is itself a rotation about some axis through that same origin. This is sufficient to prove that rotations form a group (the existence of an identity transformation and of inverses being trivially obvious) and it follows, as will become clear below, that the infinitesimal generators form an algebra, determined by their commutation relations. Since finite rotations do not generally commute (imagine the result of rotating some object, say a book, through ninety degrees about each of two mutually orthogonal axes in turn, and compare with the result of performing the same two rotations in the opposite order), the commutators of the infinitesimal generators will be non-zero. These commutation relations may be derived by consideration of a specific simple example.

Let $P(x, y, z)$ be an arbitrary point in three-dimensional space and consider the effect on its coordinates of successive rotations of the position vector from the origin to the point P . (The following conventions will be adopted throughout in describing rotations: (i) a right-handed coordinate system is used; (ii) a positive rotation advances a right-handed screw along the axis of rotation, i.e. it produces a counter-clockwise motion in the plane

to which the axis of rotation is the positive normal; and (iii) the physical system is rotated relative to a fixed coordinate system, i.e. the rotations are “active”. All three of these conventions can be individually reversed, so caution is required in interpreting signs of angles in other references.) Rotating about the x axis by an angle θ_x , the point (x, y, z) moves to $(x, y \cos \theta_x - z \sin \theta_x, z \cos \theta_x + y \sin \theta_x)$. Rotation by a further angle θ_y about the y axis moves the point to the final position with coordinates $(x \cos \theta_y + z \cos \theta_x \sin \theta_y + y \sin \theta_x \sin \theta_y, y \cos \theta_x - z \sin \theta_x, z \cos \theta_x \cos \theta_y + y \sin \theta_x \cos \theta_y - x \sin \theta_y)$. Now consider the same pair of rotations, but in the opposite order. First, under θ_y , the point (x, y, z) moves to $(x \cos \theta_y + z \sin \theta_y, y, z \cos \theta_y - x \sin \theta_y)$; then, under θ_x , it moves to the final position $(x \cos \theta_y + z \sin \theta_y, y \cos \theta_x - z \cos \theta_y \sin \theta_x + x \sin \theta_y \sin \theta_x, z \cos \theta_y \cos \theta_x - x \sin \theta_y \cos \theta_x + y \sin \theta_x)$. The difference between the final positions is

$$(z \sin \theta_y (\cos \theta_x - 1) + y \sin \theta_x \sin \theta_y, z \sin \theta_x (\cos \theta_y - 1) - x \sin \theta_y \sin \theta_x, \\ y \sin \theta_x (\cos \theta_y - 1) + x \sin \theta_y (\cos \theta_x - 1)).$$

Now let θ_x and θ_y become infinitesimally small and retain only the lowest-order non-vanishing terms in the infinitesimals. The process described above is then denoted

$$[(1 - i\theta_y M_y)(1 - i\theta_x M_x) - (1 - i\theta_x M_x)(1 - i\theta_y M_y)](x, y, z) = (y\theta_x\theta_y, -x\theta_x\theta_y, 0).$$

But this may be rewritten, to the same order in infinitesimals,

$$\begin{aligned} \theta_x\theta_y[M_x, M_y](x, y, z) &= (x, y, z) - (x - \theta_x\theta_y y, y + \theta_x\theta_y x, z) \\ &= (x, y, z) - e^{-i\theta_x\theta_y M_z}(x, y, z) \\ &= i\theta_x\theta_y M_z(x, y, z). \end{aligned}$$

(Note that including the second-order infinitesimals in the expansion of the rotation operators on the left hand side would not have altered this result.)

Since P is an arbitrary point and θ_x and θ_y are arbitrary infinitesimal angles, it follows that $[M_x, M_y] = iM_z$. Since cyclic permutation of the indices x, y, z simply corresponds to relabeling the axes while leaving them a right-handed system, this commutation relation will continue to hold under cyclic permutation of the indices x, y, z . Thus the infinitesimal generators of the rotation group satisfy the algebra of angular momentum and this may be

taken as the true fundamental definition of the angular momentum operators in quantum mechanics — they are the infinitesimal generators of rotations.

Euler supplied a convenient definition of the three angles required to specify a general rotation, and his parametrisation is almost universally used in quantum mechanics. An arbitrary rotation about any axis through the origin of the coordinate system can be built up from a rotation through the angle γ about the fixed z axis, followed by a rotation through the angle β about the fixed y axis, followed by a further rotation by the angle α about the fixed z axis again. The rotation is fully specified by the magnitudes of the three Euler angles α, β, γ and all possible rotations are encompassed in the range $0 \leq \alpha \leq 2\pi, 0 \leq \beta \leq \pi, 0 \leq \gamma \leq 2\pi$. Thus a general rotation can conveniently be represented by the operator

$$D(\alpha, \beta, \gamma) = e^{-i\alpha j_z} e^{-i\beta j_y} e^{-i\gamma j_z} \quad (5.1)$$

and a matrix representation is obtained by taking matrix elements of this unitary operator in a suitable basis.

Since the angular momentum eigenstates $|jm\rangle$ form a basis for a $(2j+1)$ -dimensional irreducible representation of the angular momentum algebra, where all the matrix elements of $\vec{j}$ are known, they also form a useful basis for the $(2j+1)$ -dimensional irreducible representation of the rotation group given by

$$\mathcal{D}_{mm'}^{(j)}(\alpha, \beta, \gamma) = \langle jm | D(\alpha, \beta, \gamma) | jm' \rangle, \quad (5.2)$$

known as the Wigner rotation matrices. Being built from angular momentum operators, D is necessarily diagonal in j . Then a given angular momentum eigenstate $|jm\rangle$ transforms under rotation as

$$D(\alpha, \beta, \gamma) |jm\rangle = \sum_{m'} \mathcal{D}_{m'm}^{(j)}(\alpha, \beta, \gamma) |jm'\rangle. \quad (5.3)$$

Note carefully the order of the subscripts on $\mathcal{D}^{(j)}$.

The completeness of the current description of the rotation group is expressed by the useful formula, which will not be derived here,

$$\int_0^{2\pi} d\alpha \int_0^\pi \sin\beta d\beta \int_0^{2\pi} d\gamma \mathcal{D}_{m_1 m'_1}^{(j_1)*}(\alpha, \beta, \gamma) \mathcal{D}_{m_2 m'_2}^{(j_2)}(\alpha, \beta, \gamma) = 8\pi^2 \delta_{j_1 j_2} \delta_{m_1 m_2} \delta_{m'_1 m'_2} / (2j_1 + 1). \quad (5.4)$$

The inverse of $D(\alpha, \beta, \gamma)$ is clearly $D(-\gamma, -\beta, -\alpha)$, and by the unitarity of the transformations this is equal to $D^\dagger(\alpha, \beta, \gamma)$. Thus

$$\mathcal{D}_{mm'}^{(j)*}(\alpha, \beta, \gamma) = \mathcal{D}_{m'm}^{(j)}(-\gamma, -\beta, -\alpha) \quad (5.5)$$

and there exist orthogonality relations

$$\sum_m \mathcal{D}_{mm'}^{(j)*}(\alpha, \beta, \gamma) \mathcal{D}_{mm''}^{(j)}(\alpha, \beta, \gamma) = \delta_{m'm''} = \sum_m \mathcal{D}_{m'm}^{(j)*}(\alpha, \beta, \gamma) \mathcal{D}_{m''m}^{(j)}(\alpha, \beta, \gamma). \quad (5.6)$$

Since the Wigner matrices are defined in a basis of states $|jm\rangle$, they are trivially rewritten in the form

$$\mathcal{D}_{mm'}^{(j)}(\alpha, \beta, \gamma) = e^{-im\alpha} d_{mm'}^{(j)}(\beta) e^{-im'\gamma} \quad (5.7)$$

where $d_{mm'}^{(j)}(\beta) = \langle jm | e^{-i\beta j_y} | jm' \rangle$ is known as a reduced rotation matrix. The relations given above for $\mathcal{D}$ then imply

$$d_{mm'}^{(j)*}(\beta) = d_{m'm}^{(j)}(-\beta) \quad (5.8)$$

and

$$\sum_m d_{mm'}^{(j)*}(\beta) d_{mm''}^{(j)}(\beta) = \delta_{m'm''}. \quad (5.9)$$

Consider the application of a rotation operator to a coupled state of two angular momenta. Since

$$D(\alpha, \beta, \gamma) |j_1 j_2 j m\rangle = \sum_{m'} \mathcal{D}_{m'm}^{(j)}(\alpha, \beta, \gamma) |j_1 j_2 j m'\rangle$$

and

$$D(\alpha, \beta, \gamma) |j_1 m_1 j_2 m_2\rangle = \sum_{m'_1 m'_2} \mathcal{D}_{m'_1 m_1}^{(j_1)}(\alpha, \beta, \gamma) \mathcal{D}_{m'_2 m_2}^{(j_2)}(\alpha, \beta, \gamma) |j_1 m'_1 j_2 m'_2\rangle$$

it follows that

$$\begin{aligned} & \sum_{m'} \mathcal{D}_{m'm}^{(j)}(\alpha, \beta, \gamma) \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | jm' \rangle |j_1 m_1 j_2 m_2\rangle = \\ & \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | jm \rangle \sum_{m'_1 m'_2} \mathcal{D}_{m'_1 m_1}^{(j_1)}(\alpha, \beta, \gamma) \mathcal{D}_{m'_2 m_2}^{(j_2)}(\alpha, \beta, \gamma) |j_1 m'_1 j_2 m'_2\rangle. \end{aligned}$$

Using the orthogonality of the Clebsch-Gordan coefficients and the linear independence of the basis functions $|j_1 m_1 j_2 m_2\rangle$, this produces the Clebsch-Gordan series,

$$\mathcal{D}_{m'_1 m_1}^{(j_1)}(\alpha, \beta, \gamma) \mathcal{D}_{m'_2 m_2}^{(j_2)}(\alpha, \beta, \gamma) = \sum_{j m m'} \langle j_1 m_1 j_2 m_2 | jm \rangle \langle j_1 m'_1 j_2 m'_2 | jm' \rangle \mathcal{D}_{m'm}^{(j)}(\alpha, \beta, \gamma) \quad (5.10)$$

or the alternative form

$$\mathcal{D}_{m'm}^{(j)}(\alpha, \beta, \gamma) = \sum_{m_1 m'_1 m_2 m'_2} \langle j_1 m_1 j_2 m_2 | jm \rangle \langle j_1 m'_1 j_2 m'_2 | jm' \rangle \mathcal{D}_{m'_1 m_1}^{(j_1)}(\alpha, \beta, \gamma) \mathcal{D}_{m'_2 m_2}^{(j_2)}(\alpha, \beta, \gamma). \quad (5.11)$$

This last result can be regarded as a recursion relation for the rotation matrices.

Since the selection rules $m_1 + m_2 = m$ and $m'_1 + m'_2 = m'$ of the Clebsch-Gordan coefficients ensure that the exponential α, γ dependence of this last equation is automatically correct, the equation may be rewritten as a recursion relation for the reduced rotation matrices,

$$d_{m'm}^{(j)}(\beta) = \sum_{m_1 m'_1 m_2 m'_2} \langle j_1 m_1 j_2 m_2 | j m \rangle \langle j_1 m'_1 j_2 m'_2 | j m' \rangle d_{m'_1 m_1}^{(j_1)}(\beta) d_{m'_2 m_2}^{(j_2)}(\beta). \quad (5.12)$$

For this recursion relation to be useful, some initial $d^{(j)}$ must be known. The trivial case $d_{00}^{(0)}(\beta) = 1$ is of no help, but the case $j = \frac{1}{2}$ is almost equally trivial. In this case, the angular momentum operators are represented by the Pauli matrices, $\vec{j} = \frac{1}{2}\vec{\sigma}$, which satisfy $\sigma_\rho^2 = 1$ for $\rho = x, y, z$, so that $e^{-i\beta j_y} = e^{-i\frac{1}{2}\beta\sigma_y} = \cos(\beta/2) - i\sin(\beta/2)\sigma_y$, upon expanding the exponential and resumming separately the terms independent of σ_y and those linear in σ_y . But the standard representation of σ_y is $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ (where the rows and columns are labelled in decreasing order of $m = \pm\frac{1}{2}$), so

$$d^{(\frac{1}{2})}(\beta) = \begin{pmatrix} \cos(\beta/2) & \sin(\beta/2) \\ -\sin(\beta/2) & \cos(\beta/2) \end{pmatrix}, \quad (5.13)$$

i.e. $d_{m'm}^{(\frac{1}{2})}(\beta) = (m - m')\sin(\beta/2) + \delta_{mm'}\cos(\beta/2)$, and is real. Since all $d^{(j)}$, for any j , can be built up from this explicit form of $d^{(\frac{1}{2})}$ by the use of the recursion relation, whose coefficients are the (real) Clebsch-Gordan coefficients, it follows that all reduced rotation matrices $d^{(j)}$ are real and orthogonal.

Certain properties of the reduced rotation matrix $d^{(\frac{1}{2})}$, which follow directly from the explicit formula given above, can be generalised, with the help of the recursion relation, to all reduced rotation matrices $d^{(j)}$. For example, $d_{m'm}^{(\frac{1}{2})}(\beta) = d_{-m, -m'}^{(\frac{1}{2})}(\beta)$ leads directly to $d_{m'm}^{(j)}(\beta) = d_{-m, -m'}^{(j)}(\beta)$, with the help of the recursion relation and the symmetry properties of the Clebsch-Gordan coefficients. Similarly, $d_{m'm}^{(\frac{1}{2})}(\pi) = m - m' = (-)^{\frac{1}{2}-m}\delta_{m', -m}$ leads to the general result $d_{m'm}^{(j)}(\pi) = (-)^{j-m}\delta_{m', -m}$, while $d_{-m', -m}^{(\frac{1}{2})}(\beta) = (-)^{m-m'}d_{m'm}^{(\frac{1}{2})}(\beta)$ leads directly to $d_{-m', -m}^{(j)}(\beta) = (-)^{m-m'}d_{m'm}^{(j)}(\beta)$. All of these relations can then be translated into analogous relations for the full rotation matrix $\mathcal{D}^{(j)}(\alpha, \beta, \gamma)$ by incorporating the simple exponential dependence on α and γ .

The Clebsch-Gordan series, together with the completeness integral for the rotation ma-

trices, allows the straightforward derivation of the formula

$$\begin{aligned} \int_0^{2\pi} d\alpha \int_0^\pi \sin \beta d\beta \int_0^{2\pi} d\gamma \mathcal{D}_{m'_1 m_1}^{(j_1)*}(\alpha, \beta, \gamma) \mathcal{D}_{m'_2 m_2}^{(j_2)}(\alpha, \beta, \gamma) \mathcal{D}_{m'_3 m_3}^{(j_3)}(\alpha, \beta, \gamma) \\ = 8\pi^2 \langle j_2 m_2 j_3 m_3 | j_1 m_1 \rangle \langle j_2 m'_2 j_3 m'_3 | j_1 m'_1 \rangle / (2j_1 + 1) \end{aligned}$$

which, using $\mathcal{D}_{m'_m}^{(j)}(\alpha, \beta, \gamma) = (-)^{m-m'} \mathcal{D}_{-m', -m}^{(j)*}(\alpha, \beta, \gamma)$, can be rewritten in the form

$$\begin{aligned} \int_0^{2\pi} d\alpha \int_0^\pi \sin \beta d\beta \int_0^{2\pi} d\gamma \mathcal{D}_{m'_1 m_1}^{(j_1)}(\alpha, \beta, \gamma) \mathcal{D}_{m'_2 m_2}^{(j_2)}(\alpha, \beta, \gamma) \mathcal{D}_{m'_3 m_3}^{(j_3)}(\alpha, \beta, \gamma) \\ = 8\pi^2 \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix}. \quad (5.14) \end{aligned}$$

This technique can clearly be generalized to deal with the integral of a product of any number of rotation matrices. These formulas can then be specialized to give the integral of products of spherical harmonics by first deriving a useful relation between spherical harmonics and rotation matrices.

The spherical harmonic $Y_{LM}(\theta, \phi) = \langle \theta, \phi | LM \rangle$, where $|LM\rangle$ is the usual angular momentum eigenstate. Under rotation,

$$D(\alpha, \beta, \gamma) |LM\rangle = \sum_{M'} \mathcal{D}_{M'M}^{(L)}(\alpha, \beta, \gamma) |LM'\rangle,$$

so that

$$\langle \theta, \phi | D(\alpha, \beta, \gamma) | LM \rangle = \sum_{M'} \mathcal{D}_{M'M}^{(L)}(\alpha, \beta, \gamma) \langle \theta, \phi | LM' \rangle.$$

But

$$\langle \theta, \phi | D(\alpha, \beta, \gamma) | LM \rangle = \langle D^\dagger(\alpha, \beta, \gamma)(\theta, \phi) | LM \rangle = \langle \theta', \phi' | LM \rangle,$$

with $D(\alpha, \beta, \gamma)$ being the rotation which carries the unit vector in the direction $(\theta', \phi') = D^\dagger(\alpha, \beta, \gamma)(\theta, \phi)$ into the direction (θ, ϕ) . Thus

$$Y_{LM}(\theta', \phi') = \sum_{M'} \mathcal{D}_{M'M}^{(L)}(\alpha, \beta, \gamma) Y_{LM'}(\theta, \phi).$$

If the direction (θ, ϕ) is chosen to be along the z axis ($\theta = 0, \phi$ arbitrary), then the Euler angles needed are $\gamma = -\phi', \beta = -\theta', \alpha$ arbitrary. But the spherical harmonics satisfy $Y_{LM}(0, \phi) = \delta_{M0} \sqrt{(2L+1)/4\pi}$, so that

$$Y_{LM}(\theta', \phi') = \sqrt{(2L+1)/4\pi} \mathcal{D}_{0M}^{(L)}(-\alpha, -\theta', -\phi') = \sqrt{(2L+1)/4\pi} \mathcal{D}_{M0}^{(L)*}(\phi', \theta', \alpha),$$

and the apparent dependence on α is fictitious. Thus, finally,

$$\mathcal{D}_{M0}^{(L)}(\alpha, \beta, \gamma) = \sqrt{4\pi/(2L+1)} Y_{LM}^*(\beta, \alpha) \quad (5.15)$$

and hence

$$\begin{aligned} & \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta Y_{L_1 M_1}(\theta, \phi) Y_{L_2 M_2}(\theta, \phi) Y_{L_3 M_3}(\theta, \phi) \\ &= \sqrt{(2L_1+1)(2L_2+1)(2L_3+1)/4\pi} \begin{pmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3 \end{pmatrix} \begin{pmatrix} L_1 & L_2 & L_3 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (5.16)$$

The Wigner-Eckart theorem is easily derived in the rotation group framework by noting that an operator T which transforms states $|\Psi\rangle$ according to $|\Psi\rangle \Rightarrow T|\Psi\rangle$ will transform operators Θ according to $\Theta \Rightarrow T\Theta T^{-1}$ (so that simultaneous transformation of states and operators will leave matrix elements unaltered). A spherical tensor operator $T^{(j)}$ can then be defined as a set of $(2j+1)$ operators which transform among themselves under rotations according to the j representation of the rotation group, namely

$$D(\alpha, \beta, \gamma) T_m^{(j)} D^\dagger(\alpha, \beta, \gamma) = \sum_{m'} \mathcal{D}_{m'm}^{(j)}(\alpha, \beta, \gamma) T_{m'}^{(j)}. \quad (5.17)$$

Once again, the fundamental requirement is that the set of operators be closed under rotations, with the precise form given here corresponding to a special standard form of the components. (That this definition of a spherical tensor operator is in fact the same as that given previously in terms of commutation relations with the angular momentum operators becomes evident upon considering an infinitesimal rotation. When the equation is expanded in powers of the infinitesimal angle of rotation, the zero order term is the same on both sides of the equation. The term of first order on the left hand side of the equation involves the commutator of the operator $T_m^{(j)}$ with the infinitesimal generator of the rotation, while the term of first order on the right hand side contains just the appropriate matrix element of the angular momentum to reproduce the earlier definition of a spherical tensor operator.)

Now consider

$$\begin{aligned} D(\alpha, \beta, \gamma) T_m^{(j)} |j'm'\rangle &= D(\alpha, \beta, \gamma) T_m^{(j)} D^\dagger(\alpha, \beta, \gamma) D(\alpha, \beta, \gamma) |j'm'\rangle \\ &= \sum_{nn'} \mathcal{D}_{nm}^{(j)}(\alpha, \beta, \gamma) \mathcal{D}_{n'm'}^{(j')}(\alpha, \beta, \gamma) T_n^{(j)} |j'n'\rangle \\ &= \sum_{nn' JMM'} \langle jn j'n' | JM' \rangle \langle jm j'm' | JM \rangle \mathcal{D}_{M'M}^{(J)}(\alpha, \beta, \gamma) T_n^{(j)} |j'n'\rangle, \end{aligned}$$

using the Clebsch-Gordan series for the product of two rotation matrices. From the orthogonality of the Clebsch-Gordan coefficients it follows that

$$D(\alpha, \beta, \gamma) \sum_{mm'} \langle jmj'm'|JM\rangle T_m^{(j)}|j'm'\rangle = \sum_{M'} \mathcal{D}_{M'M}^{(J)}(\alpha, \beta, \gamma) \sum_{nn'} \langle jnj'n'|JM'\rangle T_n^{(j)}|j'n'\rangle. \quad (5.18)$$

Therefore, the linear combination $\sum_{mm'} \langle jmj'm'|JM\rangle T_m^{(j)}|j'm'\rangle$ transforms under rotation as a state of angular momentum J , projection M , and its overlap with any other eigenstate $|J'M'\rangle$ will be diagonal in J and M and independent of M , from which the Wigner-Eckart theorem follows as before.

Some further useful consequences of the results proved using rotation operators are the following. The integral of the product of three spherical harmonics is simply related to the matrix element $\langle L_1 M_1 | Y_{L_2 M_2} | L_3 M_3 \rangle$, and from it, using the Wigner-Eckart theorem, can be read the reduced matrix element

$$\langle L_1 || Y_{L_2} || L_3 \rangle = (-)^{L_1} \sqrt{(2L_1 + 1)(2L_2 + 1)(2L_3 + 1)/4\pi} \begin{pmatrix} L_1 & L_2 & L_3 \\ 0 & 0 & 0 \end{pmatrix}. \quad (5.19)$$

There follows the additional selection rule on the matrix element that the sum $L_1 + L_2 + L_3$ must be even. Using again the relation between the spherical harmonics and the rotation matrices, the Clebsch-Gordan series implies

$$\begin{aligned} Y_{L_1 M_1}(\theta, \phi) Y_{L_2 M_2}(\theta, \phi) &= \sqrt{(2L_1 + 1)(2L_2 + 1)} \mathcal{D}_{M_1 0}^{(L_1)*}(\phi, \theta, \gamma) \mathcal{D}_{M_2 0}^{(L_2)*}(\phi, \theta, \gamma) / 4\pi \\ &= \frac{\sqrt{(2L_1 + 1)(2L_2 + 1)}}{4\pi} \sum_{LM} \langle L_1 M_1 L_2 M_2 | LM \rangle \langle L_1 0 L_2 0 | L 0 \rangle \mathcal{D}_{M 0}^{(L)*}(\phi, \theta, \gamma) \\ &= \sum_{LM} \sqrt{(2L_1 + 1)(2L_2 + 1)/4\pi(2L + 1)} \\ &\quad \times \langle L_1 M_1 L_2 M_2 | LM \rangle \langle L_1 0 L_2 0 | L 0 \rangle Y_{LM}(\theta, \phi) \end{aligned}$$

from which it follows that

$$\begin{aligned} \sum_{M_1 M_2} \langle L_1 M_1 L_2 M_2 | LM \rangle Y_{L_1 M_1}(\theta, \phi) Y_{L_2 M_2}(\theta, \phi) &= \\ (-)^{L_1 - L_2} \sqrt{(2L_1 + 1)(2L_2 + 1)/4\pi} \begin{pmatrix} L_1 & L_2 & L \\ 0 & 0 & 0 \end{pmatrix} Y_{LM}(\theta, \phi). \end{aligned} \quad (5.20)$$

So, for example,

$$[\hat{r} \otimes \hat{r}]_m^{(2)} = \frac{4\pi}{3} [Y_1(\theta, \phi) \otimes Y_1(\theta, \phi)]_m^{(2)} = \sqrt{\frac{8\pi}{15}} Y_{2m}(\theta, \phi). \quad (5.21)$$

There remains one further refinement of these techniques which is of significance for various applications. Many problems involve the action of operator fields, namely systems of operators defined at every point in space, $T(\vec{r})$. In dealing with the properties of such fields under rotation, it is important to realize that the rotation has two effects — if the field is defined at each point in space in terms of a number of components, then these components will generally transform into linear combinations of one another under rotation (essentially the case considered up to now) while also moving to a different point in space, so that the dependence of the field on position is also affected. This corresponds to the well-known distinction between orbital and intrinsic angular momentum. The intrinsic spin serves as the infinitesimal generator for rotations in spin space (affecting the component indices of the field at a given physical point), the orbital angular momentum as the infinitesimal generator for rotations in position space (affecting the functional dependence of the field components on position). The total angular momentum then determines the full behaviour of the operator field under rotation.

As an example, consider a vector field, defined at every point in space by a set of three components. This may be regarded as a set of three related single-component fields. If each component field is expressed in terms of eigenfunctions of the orbital angular momentum, then standard vector coupling methods can be used to produce a vector field (perhaps better referred to as a physical vector field to emphasize that the term refers to the number of components required to specify the field at a given point) of any desired total rank. The prototype example is the vector spherical harmonics $\vec{\mathcal{Y}}_{LJM}$, obtained by vector coupling the spherical harmonics Y_{LM} to a set of three basis vectors $\vec{v}_m$ which have the property that the scalar product $\vec{V} \cdot \vec{v}_m = V_m$ gives the spherical vector components of an arbitrary vector $\vec{V}$. The three vectors $\vec{v}_m$ ($m = -1, 0, +1$) form the basis for a spin-1 representation of the rotation group and $\vec{\mathcal{Y}}_{LJM} = \sum_{M'm} \langle LM'1m | JM \rangle Y_{LM'} \vec{v}_m$ is a physical vector field of spherical tensor rank J , projection M .

As a further illustration, consider the vector $\vec{r}$, the position vector of a point in space. It is a vector, its three components transforming under rotation according to the $j = 1$ representation of the rotation group. Thus the field of position vectors $\vec{r}$, defined at each point in space by the radius vector from the origin to that point, is a physical vector field. However, under rotation the field as a whole is unchanged, still being defined at each point in space by the radius vector from the origin to that point. The physical vector field $\vec{r}$ is

thus a spherical scalar field.

VI. BIBLIOGRAPHY

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