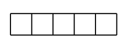
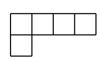
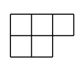
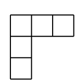
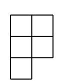
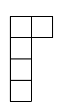
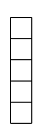
Young diagrams

The conjugacy classes of $\mathcal{S}_n$, the symmetric group on n elements, are determined by their cycle structure. All elements of a class have the same cycle structure; all elements with the same cycle structure belong to the same class. The cycle structure is defined as the set of non-negative integers $\{\nu_r\}$, where $r = 1, 2, \ldots, n$ and ν_r is the number of r-cycles in the unique resolution of a permutation into non-overlapping cycles. Note that $\sum_r r\nu_r = n$.

An alternative characterisation of the classes is in terms of the non-negative integers $\{\mu_r\}$, where $\mu_s = \sum_{r=s}^{n} \nu_r$ and it is evident that $\sum_r \mu_r = n$ and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 0$, i.e. the set $\{\mu_r\}$ is a *partition* of n. The classes can be labeled by the partitions of n; there are as many classes as partitions of n; and the cycle structure can be deduced from $\nu_r = \mu_r - \mu_{r+1}$ (with $\mu_{n+1} = 0$).

A useful and widely adopted pictorial notation for partitions is provided by *Young diagrams*. The Young diagram corresponding to the partition $\{\mu_r\}$ of n consists of left-justified rows of μ_r boxes stacked in decreasing order of length. This definition is best understood by way of a concrete example.

The partitions of $n = 5$, with their associated Young diagrams, are

partition	short form	Young diagram	cycle structure	g_k
5+0+0+0+0	5		(5,0,0,0,0)	1
4+1+0+0+0	41		(3,1,0,0,0)	10
3+2+0+0+0	32		(1,2,0,0,0)	15
3+1+1+0+0	31^2		(2,0,1,0,0)	20
2+2+1+0+0	2^21		(0,1,1,0,0)	20
2+1+1+1+0	21^3		(1,0,0,1,0)	30
1+1+1+1+1	1^5		(0,0,0,0,1)	24

and the cycle structure is given by the number of boxes at each level protruding to the right of the next lower level. These Young diagrams provide a handy label for the conjugacy classes, but no more. Recall that the number of elements in a class of cycle structure $\{\nu_r\}$ is $n!/\prod_r r^{\nu_r}\nu_r!$, listed in the table as g_k.

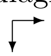
Irreps of $\mathcal{S}_n$

Young diagrams play a more significant role in defining the irreps of the symmetric group $\mathcal{S}_n$. Once again, there is a Young diagram for every partition of n, but now there is an irrep for every Young diagram. [The Young diagrams defining the irreps have no connection with the Young diagrams labeling the classes, except that there are equal numbers of each, as there must be. The same set of Young diagrams serves both to label the classes and to define the irreps.]

Given a Young diagram, one produces *Young tableaux* by filling the n boxes of the diagram with the numbers 1 through n, in such a way that the numbers always increase from left to right along any row and from top to bottom down any column. An example of a Young tableau for the 32 irrep of $\mathcal{S}_5$ is

1	2	4
3	5	

. The dimension of the irrep is the number of different Young tableaux that can be produced in this way. It is given by the so-called *hook-length formula*, as follows.

In a Young diagram (with empty boxes), place in the centre of each box in turn a "hook" , extending to the right past the end of the row and down past the end of the column in which the box is placed. The hook length of the box is the number of boxes lying along the hook, in both directions. (e.g. In the Young tableau in the previous paragraph, the middle box in the top row has hook length 3 and the right-hand box in the second row has hook length 1.) In each box, enter its hook length. The number of distinct Young tableaux, and hence the dimension of the irrep, is $n!$ divided by the product of the n hook lengths.

> [The Young diagram of the 32 irrep of $\mathcal{S}_5$, with the hook lengths of the boxes entered, is
>
> | | | |
> |---|---|---|
> | 4 | 3 | 1 |
> | 2 | 1 | |
>
> . The dimension of the irrep is 5!/4.3.1.2.1=5.]

The irreps of $\mathcal{S}_n$ define the symmetry of vectors under permutation. Roughly speaking, the vectors are symmetric under permutation of the labels in each row of a Young tableau, antisymmetric under permutation of the labels in each column of the tableau. The irrep n (one row of n boxes) is totally symmetric in the n labels, while the irrep 1^n (one column of n

boxes) is totally antisymmetric in the n labels. All other irreps have mixed symmetry.

To each tableau there is a *symmetrizer* which projects out of any vector on which it acts the part belonging to that tableau of the given irrep. If the vector has no part belonging to the relevant tableau, the projector gives zero. The projector is defined by identifying, among the $n!$ permutations P, those which leave the labels in their rows (denoted R) and those which leave the labels in their columns (denoted C). Permutations which move labels between rows or between columns are omitted from this classification. The elements of the group algebra $\rho = \sum R$ and $\kappa = \sum \zeta(C)C$ are formed, where $\zeta(P)$ is the alternating character of the permutation P, and their product $\kappa\rho$ is the Young symmetrizer of the tableau. [Note that it is also possible to define the symmetrizer as $\rho\kappa$ — the result is different but equally acceptable. But whichever convention is chosen, it must be applied consistently.]

Recall the simple case of $\mathcal{S}_3$, with the six permutations $\mathbf{1}$, (12), (13), (23), (123), (132). The irreps are labeled 3, 21, 1^3 and all the Young tableaux are $(i) : \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$, $(ii) : \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, $(iii) : \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ and $(iv) : \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$. Of the six permutations, all leave the labels of tableaux (i) and (iv) in their rows or columns, but only some of them qualify for inclusion in the symmetrizers of tableaux (ii) and (iii). For tableau (ii), the set of R's is $\{\mathbf{1}, (12)\}$ and the set of C's is $\{\mathbf{1}, (13)\}$, so the Young symmetrizer of (ii) is $[\mathbf{1}-(13)][\mathbf{1}+(12)] = \mathbf{1}+(12)-(13)-(123)$. For tableau (iii), the set of R's is $\{\mathbf{1}, (13)\}$ and the set of C's is $\{\mathbf{1}, (12)\}$, so the Young symmetrizer of (iii) is $[\mathbf{1}-(12)][\mathbf{1}+(13)] = \mathbf{1}-(12)+(13)-(132)$. The Young symmetrizers of tableaux (i) and (iv) are $\mathbf{1} \pm (12) \pm (13) \pm (23) + (123) + (132)$ respectively (upper signs for (i), lower signs for (iv)).

> [Note in passing that the hook length formula gives dimensions 3!/3.2.1=1 for irreps 3 and 1^3 and 3!/1.3.1=2 for irrep 21, as found previously. The formula always produces $d = 1$ for the irreps n and 1^n of $\mathcal{S}_n$ and $d = n-1$ for the irrep $n-1, 1$.]

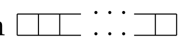
Irreps of $\mathsf{su}(n)$

It is of interest that Young diagrams can also be used to label the irreps of the algebras $\mathsf{su}(n)$. The relevant rules are:

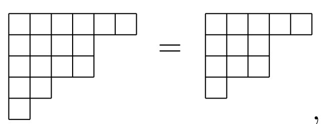
- The Young diagram of an irrep of $\mathsf{su}(n)$ has at most n rows. There is no limit on the total number of boxes.

- The Dynkin indices of an irrep labeled by a given Young diagram are, row by row, the number of blocks by which a row exceeds the length of the following row. This number may be zero.

- A column of n boxes may be omitted from the Young diagram of an irrep of $\mathsf{su}(n)$. (The previous rule implies that the Young diagram consisting of a single column of n boxes labels the irrep with Dynkin indices (0,0,. . . ,0), i.e. the singlet irrep.)

- The irrep of $\mathsf{su}(n)$ labeled by a given Young diagram has the symmetry of the irrep of $\mathcal{S}_r$ with the same Young diagram, where r is the number of boxes in the diagram. [Note the implication that the irrep (r,0,0,. . . ,0), corresponding to the Young diagram  with a single row of r boxes, is totally symmetric. This is relevant to the identification of $\mathsf{su}(3)$ irreps for the isotropic harmonic oscillator.]

The following is an example of a Young diagram labeling the irrep (2,0,2,1) of $\mathsf{su}(5)$.



where the right hand diagram is obtained by removing the column of 5 boxes from the left hand diagram.

Products of $\mathsf{su}(n)$ irreps

Young diagrams may also be used to reduce the product of $\mathsf{su}(n)$ irreps. Each of the two irreps being multiplied together is represented by its Young diagram. The squares of the smaller of the diagrams are filled with labels, the first row being labeled a, the second row b, the third row c, and so on. The labeled squares are then attached, one by one, to the larger diagram, forming new, partly labeled, Young diagrams. (As always, the lengths of the rows of any Young diagram cannot exceed the length of any higher row.) The following restrictions apply at every stage:

- No two squares with the same label may occur in the same column.

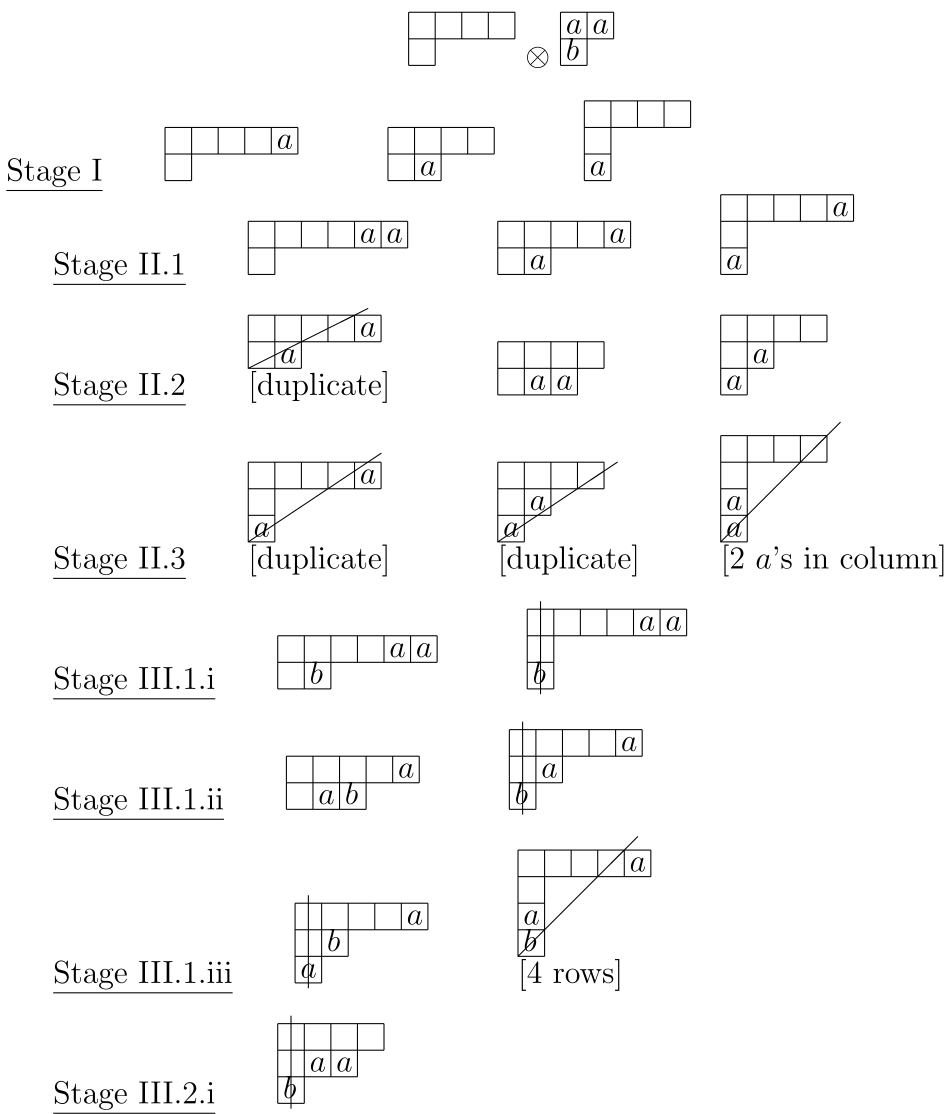
- The total number of labels a, counting by columns from right to left, cannot be less than the number of labels b, which itself cannot be less than the number of labels c, and so on. (The "row counting rule".)

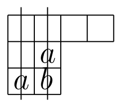
- The total number of labels a, counting by rows from top to bottom, cannot be less than the number of labels b, which itself cannot be less than the number of labels c, and so on. (The "column counting rule".)

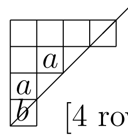
- The same Young diagram may be produced more than once. If the labeling is the same in both diagrams, only one is retained. If the labeling is different, both are retained.

- For application to a specific $\mathsf{su}(n)$, diagrams with more than n rows are discarded and columns of n squares are removed from the diagrams.

A small example follows, for the product $(3,1) \otimes (1,1)$ of $\mathsf{su}(3)$ irreps.



Stage III.2.ii 



[4 rows]

Some comments: (1) Young diagrams with more than 3 rows have been discarded, specifically for $\mathsf{su}(3)$. (2) Columns of 3 squares have been removed from the final set of Young diagrams, again specifically for $\mathsf{su}(3)$. (3) At stage III, a number of Young diagrams have been omitted because they violate the “counting rules”. (4) The same Young diagram occurs in stages III.1.ii and III.1.iii, but the labeling is different, so both diagrams are retained. Duplicates with the same labeling have been discarded along the way.

The final reduction is

$$(3,1) \otimes (1,1) = (4,2) \oplus (5,0) \oplus (2,3) \oplus (3,1) \oplus (3,1) \oplus (1,2) \oplus (2,0).$$

The relevant dimensions are $24 \times 8 = 60+21+42+24+24+15+6$, confirming the reduction.

[Note that the highest weight rule produces (4,2); Dynkin’s second highest weight theorem produces both (2,3) and (5,0), since the extended Dynkin diagram has two chains; and the adjoint rule produces one (3,1). There then remain three irreps, which are found by the Young diagram rules.]