Isotropic harmonic oscillator

The hamiltonian of the isotropic harmonic oscillator is

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\vec{\nabla}^2 + \frac{1}{2}m\omega^2\vec{r}^2 &(1)\\ &= \sum_{\rho=x,y,z}\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{1}{2}m^2\omega^2\rho^2\right], &(2)\end{aligned}$$

a sum of three one-dimensional oscillators with equal masses m and angular frequencies ω. The hamiltonian of the one-dimensional oscillator can be rewritten in terms of dimensionless quantities as

$$H_x = \left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{1}{2}\xi^2\right]\hbar\omega, \tag{3}$$

where $\xi = x/b$ and $b = \sqrt{\hbar/m\omega}$. The expression multiplying $\hbar\omega$ in eq.(3) will be denoted H_ξ. The corresponding hamiltonians for the y and z coordinates are denoted H_η and H_ζ respectively.

The canonical commutation relation $[x, p_x] = [x, -i\hbar\mathrm{d}/\mathrm{d}x] = i\hbar$ can be rewritten as $[\xi, \mathrm{d}/\mathrm{d}\xi] = -1$, which leads to

$$[\xi + \frac{\mathrm{d}}{\mathrm{d}\xi}, \xi - \frac{\mathrm{d}}{\mathrm{d}\xi}] = 2$$

and

$$(\xi - \frac{\mathrm{d}}{\mathrm{d}\xi})(\xi + \frac{\mathrm{d}}{\mathrm{d}\xi}) = \xi^2 - \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - 1 = 2H_\xi - 1.$$

So, in terms of

$$\begin{aligned} a_\xi &= \frac{1}{\sqrt{2}}(\xi + \frac{\mathrm{d}}{\mathrm{d}\xi}) &(4)\\ \text{and}\quad a_\xi^\dagger &= \frac{1}{\sqrt{2}}(\xi - \frac{\mathrm{d}}{\mathrm{d}\xi}), &(5)\end{aligned}$$

(which are hermitian conjugates of one another, since the operator $p_\xi = -i\mathrm{d}/\mathrm{d}\xi$ is hermitian),

$$\begin{aligned} H_\xi &= a_\xi^\dagger a_\xi + \frac{1}{2}, &(6)\\ \text{where}\quad [a_\xi, a_\xi^\dagger] &= 1 &(7)\\ \text{and}\quad [a_\xi, a_\xi] &= 0 = [a_\xi^\dagger, a_\xi^\dagger]. &(8)\end{aligned}$$

Denoting

$$\hat{n}_\xi = a_\xi^\dagger a_\xi, \tag{9}$$

which is called the number operator for ξ, it follows from eqs.(7, 8) that

$$[\hat{n}_\xi, a_\xi^\dagger] = a_\xi^\dagger \quad \text{and} \quad [\hat{n}_\xi, a_\xi] = -a_\xi, \tag{10}$$

using the commutator identity $[AB, C] = A[B, C] + [A, C]B$. The number operator $\hat{n}_\xi$ is hermitian and positive definite, so must have non-negative real eigenvalues. Let $|\alpha\rangle$ be an eigenstate of $\hat{n}_\xi$ with eigenvalue α. Then

$$\hat{n}_\xi a_\xi|\alpha\rangle = \left([\hat{n}_\xi, a_\xi] + a_\xi\hat{n}_\xi\right)|\alpha\rangle = (\alpha - 1)a_\xi|\alpha\rangle,$$

so $a_\xi|\alpha\rangle$ is an eigenstate of $\hat{n}_\xi$ with eigenvalue $\alpha - 1$. It follows that $(a_\xi)^m|\alpha\rangle$ is an eigenstate of $\hat{n}_\xi$ with eigenvalue $\alpha - m$, for any integer m. Whatever the value of α, there will be some m for which $\alpha - m < 0$, which is a contradiction, unless $a_\xi|\alpha - m + 1\rangle = 0$, which implies $\hat{n}_\xi|\alpha - m + 1\rangle = (\alpha - m + 1)|\alpha - m + 1\rangle = 0 \Longrightarrow \alpha = m - 1$. So there must exist a state $|0\rangle$ such that

$$a_\xi|0\rangle = 0. \tag{11}$$

It will satisfy $\hat{n}_\xi|0\rangle = 0$ and is the ground state of the number operator. The eigenvalues of the number operator are non-negative integers, which justifies its name.

Now $\left[\hat{n}_\xi, \left(a_\xi^\dagger\right)^m\right] = [\hat{n}_\xi, a_\xi^\dagger]\left(a_\xi^\dagger\right)^{m-1} + a_\xi^\dagger\left[\hat{n}_\xi, \left(a_\xi^\dagger\right)^{m-1}\right]$, which leads to the recursion relation $\left[\hat{n}_\xi, \left(a_\xi^\dagger\right)^m\right] = \left(a_\xi^\dagger\right)^m + a_\xi^\dagger\left[\hat{n}_\xi, \left(a_\xi^\dagger\right)^{m-1}\right]$ and hence, eventually, to $\left[\hat{n}_\xi, \left(a_\xi^\dagger\right)^m\right] = m\left(a_\xi^\dagger\right)^m$. Therefore, the eigenstates of $\hat{n}_\xi$ are $\left(a_\xi^\dagger\right)^m|0\rangle$, for any integer m, with eigenvalues m. Finally, these are the eigenstates of the one-dimensional harmonic oscillator H_x, with eigenvalues $(m + \frac{1}{2})\hbar\omega$.

The ground state, or vacuum, $|0\rangle$ lies at energy $\hbar\omega/2$ and the excited states are spaced at equal energy intervals of $\hbar\omega$. The operator $a_\xi^\dagger$ increases the energy by one unit of $\hbar\omega$ and can be considered as creating a single excitation, called a quantum or phonon. The operator a_ξ lowers the energy by one unit of $\hbar\omega$ and can be considered as destroying a quantum. Because the creation and destruction operators each commute with themselves, multi-quantum states are unchanged under exchange of quanta, which therefore behave as bosons.

Since the isotropic three-dimensional harmonic oscillator hamiltonian is

$$H = H_x + H_y + H_z, \tag{12}$$

(and the different one-dimensional hamiltonians H_ρ commute with one another) its eigenstates are simultaneous eigenvectors of H_ρ, with $\rho = x, y, z$, and its spectrum is

$$E(n_x, n_y, n_z) = (n_x + n_y + n_z + \frac{3}{2})\hbar\omega, \tag{13}$$

for any non-negative integers n_x, n_y, n_z. Denoting $N = n_x + n_y + n_z$, this can be rewritten $E_N = (N + \frac{3}{2})\hbar\omega$ and each level E_N is degenerate, the degeneracy being the number of ways of writing N as a sum of three non-negative integers, namely $(N + 1)(N + 2)/2$.

> [For given N, choose $n_x = 0, 1, \ldots, N$ and $n_y = 0, 1, \ldots, N - n_x$, with n_z then being determined as $N - n_x - n_y$. For each n_x, there are $(N - n_x + 1)$ choices for n_y, so the total number of choices is $\sum_{n_x=0}^{N}(N + 1 - n_x) = (N + 1)(N + 2)/2$.]

It is generally (though not universally) true that degeneracy in the spectrum of a hamiltonian can be attributed to the existence of a symmetry. The isotropic oscillator is rotationally invariant, so could be solved, like any central force problem, in spherical coordinates. The angular dependence produces spherical harmonics $Y_{\ell m}$ and the radial dependence produces the eigenvalues $E_{n\ell} = (2n+\ell+\frac{3}{2})\hbar\omega$, dependent on the angular momentum ℓ but independent of the projection m. The spherical symmetry is responsible for the $(2\ell + 1)$-fold degeneracy arising from the independence of m, but there remains a further degeneracy of different n, ℓ values with the same value of $2n + \ell$, where n and ℓ are non-negative integers. Denoting $N = 2n + \ell$, it is straightforward to check that the total degeneracy (including the $(2\ell+1)$-fold degeneracy of each ℓ level) is again $(N + 1)(N + 2)/2$, as it must be.

> [For given N, as n takes the values $0, 1, \ldots$, the values of ℓ are $N, N - 2, N - 4, \ldots, 1 \text{ or } 0$, and are all even if N is even, all odd if N is odd. The total degeneracy of the N^{th} level is $\sum_\ell(2\ell + 1)$, where the upper limit on the sum is the largest integer no larger than $N/2$, the lower limit is 0 or 1, for N even or odd, respectively, and ℓ increases by steps of 2.]

This suggests the existence of a larger symmetry, including rotational symmetry but going further.

From eq.(2),

$$H = H_x + H_y + H_z = (\hat{n}_\xi + \hat{n}_\eta + \hat{n}_\zeta + \frac{3}{2})\hbar\omega = (\hat{N} + \frac{3}{2})\hbar\omega, \tag{14}$$

and the three number operators are easily seen to commute with one another, with the total number operator $\hat{N}$ and with H. The creation and destruction operators obey the boson commutation relations

$$\begin{aligned} [a_\rho, a_\sigma^\dagger] &= \delta_{\rho\sigma} \\ [a_\rho, a_\sigma] &= 0 \\ \left[a_\rho^\dagger, a_\sigma^\dagger\right] &= 0, \end{aligned} \tag{15}$$

where $\rho, \sigma = \xi, \eta, \zeta$. The three bosons, ξ, η, ζ, are completely equivalent to one another, differing only in their labels, so interchanging their identities should have no effect. Such an operation is performed by the binary operators $a_\rho^\dagger a_\sigma$, which destroy a quantum of type σ and create a quantum of type ρ, equivalent to replacing a σ boson by a ρ boson. There are nine such operators, the three number operators ($\rho = \sigma$) and six off-diagonal operators ($\rho \neq \sigma$).

The commutators

$$\left[a_\mu^\dagger a_\nu, a_\rho^\dagger a_\sigma\right] = \delta_{\nu\rho} a_\mu^\dagger a_\sigma - \delta_{\mu\sigma} a_\rho^\dagger a_\nu \tag{16}$$

(where use has been made of the commutator identity $[AB, CD] = A[B, C]D + AC[B, D] + [A, C]DB + C[A, D]B$) produce linear combinations of the nine binary operators, which thus form a set closed under commutation. The vector space spanned by these operators therefore constitutes a Lie algebra. The largest mutually commuting subset of operators consists of $\{\hat{n}_\xi, \hat{n}_\eta, \hat{n}_\zeta\}$ and will be chosen as the Cartan subalgebra.

The Lie products shown in eq.(16) imply

$$\begin{array}{lll}
[\hat{n}_\xi, a_\xi^\dagger a_\eta] = a_\xi^\dagger a_\eta; & [\hat{n}_\eta, a_\xi^\dagger a_\eta] = -a_\xi^\dagger a_\eta; & [\hat{n}_\zeta, a_\xi^\dagger a_\eta] = 0; \\
[\hat{n}_\xi, a_\eta^\dagger a_\xi] = -a_\eta^\dagger a_\xi; & [\hat{n}_\eta, a_\eta^\dagger a_\xi] = a_\eta^\dagger a_\xi; & [\hat{n}_\zeta, a_\eta^\dagger a_\xi] = 0; \\
[\hat{n}_\xi, a_\eta^\dagger a_\zeta] = 0; & [\hat{n}_\eta, a_\eta^\dagger a_\zeta] = a_\eta^\dagger a_\zeta; & [\hat{n}_\zeta, a_\eta^\dagger a_\zeta] = -a_\eta^\dagger a_\zeta; \\
[\hat{n}_\xi, a_\zeta^\dagger a_\eta] = 0; & [\hat{n}_\eta, a_\zeta^\dagger a_\eta] = -a_\zeta^\dagger a_\eta; & [\hat{n}_\zeta, a_\zeta^\dagger a_\eta] = a_\zeta^\dagger a_\eta; \\
[\hat{n}_\xi, a_\zeta^\dagger a_\xi] = -a_\zeta^\dagger a_\xi; & [\hat{n}_\eta, a_\zeta^\dagger a_\xi] = 0; & [\hat{n}_\zeta, a_\zeta^\dagger a_\xi] = a_\zeta^\dagger a_\xi; \\
[\hat{n}_\xi, a_\xi^\dagger a_\zeta] = a_\xi^\dagger a_\zeta; & [\hat{n}_\eta, a_\xi^\dagger a_\zeta] = 0; & [\hat{n}_\zeta, a_\xi^\dagger a_\zeta] = -a_\xi^\dagger a_\zeta,
\end{array}$$

i.e. the six off-diagonal products are the root vectors of the algebra, with roots $(1, -1, 0)$, $(-1, 1, 0)$, $(0, 1, -1)$, $(0, -1, 1)$, $(-1, 0, 1)$ and $(1, 0, -1)$, in the order listed above. The Killing form on the Cartan subalgebra is given by

$$g_{ij} = \sum_\alpha \alpha_i \alpha_j = \begin{pmatrix} 8 & -4 & -4 \\ -4 & 8 & -4 \\ -4 & -4 & 8 \end{pmatrix}, \tag{17}$$

which has a vanishing determinant. A singular Killing form means this Lie algebra is not semi-simple.

Since each of the roots α, including the three zero roots, satisfies $\sum_i \alpha_i = 0$, it follows that $\sum_\rho \hat{n}_\rho = \hat{N}$ commutes with all nine generators of the algebra (as can also be seen directly from the list of Lie products), which therefore has a non-trivial center and hence contains an Abelian ideal. This is the reason the algebra is not semi-simple. It is necessary to separate the total number operator $\hat{N}$ from the rest of the generators, leaving a set of eight generators, made up of the six off-diagonal products and two independent linear combinations of the three number operators $\hat{n}_\rho$. A convenient simple choice is

$$\begin{aligned} h_1 &= \hat{n}_\xi - \hat{n}_\eta \\ h_2 &= \hat{n}_\eta - \hat{n}_\zeta, \end{aligned} \tag{18}$$

in terms of which $\hat{n}_\xi = (\hat{N} + 2h_1 + h_2)/3$, $\hat{n}_\eta = (\hat{N} - h_1 + h_2)/3$ and $\hat{n}_\zeta = (\hat{N} - h_1 - 2h_2)/3$. The resulting set of eight operators is closed under commutation and generates a Lie algebra.

> [Closure is evident, by inspection of eq.(16), for all Lie products except the commutators $[a^\dagger_\rho a_\sigma, a^\dagger_\sigma a_\rho] = \hat{n}_\rho - \hat{n}_\sigma$, with $\rho \neq \sigma$. But the difference between any two $\hat{n}$'s contains only h_1 and h_2, so closure is confirmed.]

The original algebra of dimension 9 has been decomposed into the direct sum of an algebra of dimension 1 and an algebra of dimension 8, the former being generated by $\hat{N}$. Since $\hat{N}$, and hence H, commutes with all the generators of the algebra of dimension 8, the latter is a symmetry of the isotropic harmonic oscillator.

The Cartan subalgebra of the algebra of dimension 8 can now be chosen to be $\{h_1, h_2\}$, with the same 6 root vectors as before, but now with the roots $(2,-1)$, $(-2,1)$, $(-1,2)$, $(1,-2)$, $(-1,-1)$ and $(1,1)$. In terms of the root-space basis $\{(1,0),(0,1)\}$, the positive roots are $(2,-1)$, $(1,-2)$ and $(1,1)$. Since $(2,-1) = (1,-2) + (1,1)$, the simple roots are $\alpha^{(1)} = (1,-2)$ and $\alpha^{(2)} = (1,1)$.

The Killing form on the Cartan subalgebra is now $g = \begin{pmatrix} 12 & -6 \\ -6 & 12 \end{pmatrix}$ and is non-singular, so the metric is

$$g^{-1} = \frac{1}{18}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{19}$$

The scalar products of simple roots are $\alpha^{(1)} \cdot \alpha^{(1)} = \frac{1}{3}$, $\alpha^{(2)} \cdot \alpha^{(2)} = \frac{1}{3}$ and $\alpha^{(1)} \cdot \alpha^{(2)} = -\frac{1}{6}$ and the Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{20}$$

with corresponding Dynkin diagram ○—○ . The algebra is $\mathcal{A}_2$, or $\mathsf{su}(3)$. The original dimension-9 algebra can be identified as $\mathsf{u}(3) = \mathsf{u}(1) \oplus \mathsf{su}(3)$.

As pointed out, all the generators of this $\mathsf{su}(3)$ algebra commute with $\hat{N}$ and hence with H, so that, by Schur's lemma, every irrep has a well-defined value of N, the total number of quanta, and a well-defined energy. The lowest-energy irrep has $N = 0$, no quanta of excitation, and an energy $E_0 = \frac{3}{2}\hbar\omega$. It has dimension 1 and is spanned by the vacuum state $|0\rangle$. Since $h_1|0\rangle = 0 = h_2|0\rangle$, the vacuum has weight $(0,0)$ and the irrep is the singlet $(0,0)$ of $\mathsf{su}(3)$.

The irrep of next higher energy must have $N = 1$, energy $E_1 = \frac{5}{2}\hbar\omega$, and its basis states are generated by acting with $a^\dagger_\rho$ on the vacuum. There are 3 independent states, corresponding to $\rho = \xi, \eta, \zeta$. Note that $[h_1, a^\dagger_\xi] = a^\dagger_\xi$, $[h_2, a^\dagger_\xi] = 0$; $[h_1, a^\dagger_\eta] = -a^\dagger_\eta$, $[h_2, a^\dagger_\eta] = a^\dagger_\eta$; $[h_1, a^\dagger_\zeta] = 0$, $[h_2, a^\dagger_\zeta] = -a^\dagger_\zeta$, while $[a^\dagger_\rho a_\sigma, a^\dagger_\tau] = \delta_{\sigma\tau} a^\dagger_\rho$, so that the three creation operators $a^\dagger_\rho$ span a 3-dimensional invariant subspace and have weights $(1,0)$, $(-1,1)$ and $(0,-1)$ respectively. It can be checked that these are also the Dynkin indices of the weights. The creation operators belong to the 3-dimensional irrep $(1,0)$ of $\mathsf{su}(3)$. Since the vacuum is a singlet $(0,0)$, the three one-quantum states $a^\dagger_\rho|0\rangle$ also belong to the $(1,0)$ irrep and are degenerate, at energy E_1.

The two-quantum states are obtained by acting twice with creation operators on the vacuum and might be expected to include 9 states, but because of the boson symmetry $a^\dagger_\rho a^\dagger_\sigma = a^\dagger_\sigma a^\dagger_\rho$ there are only six independent states. Since the commutator of an operator of structure $a^\dagger a$ with an operator of structure $a^\dagger a^\dagger$ is an operator of structure $a^\dagger a^\dagger$, the latter span a 6-dimensional invariant subspace. The $\mathsf{su}(3)$ product decomposition $(1,0) \otimes (1,0) = (2,0) \oplus (0,1)$ contains the 6-dimensional irrep $(2,0)$ and the 3-dimensional irrep $(0,1)$, so the 2-quantum excited states can be identified as belonging to the $(2,0)$ irrep. They are degenerate, at energy $E_2 = \frac{7}{2}\hbar\omega$.

The next step, constructing the 3-quantum states, involves the $\mathsf{su}(3)$ product decomposition $(1,0) \otimes (2,0) = (3,0) \oplus (1,1)$, leading to the 10-dimensional irrep $(3,0)$ and the 8-dimensional irrep $(1,1)$. It is straightforward to confirm that there are ten independent 3-quantum products (again exploiting the boson symmetry between quanta) and that they span an invariant subspace. The 3-quantum states belong to the 10-dimensional irrep $(3,0)$ and are degenerate, at energy $E_3 = \frac{9}{2}\hbar\omega$.

Continuing this process, step by step, establishes that the N-quantum states belong to the irrep $(N,0)$, of dimension $(N+1)(N+2)/2$, and are degenerate, at energy $E_N = (N + \frac{3}{2})\hbar\omega$. The degeneracy of the isotropic harmonic oscillator is entirely due to an $\mathsf{su}(3)$ symmetry of the hamiltonian. The restriction to the $(N,0)$ irreps is a consequence of the exchange symmetry of the multi-quantum system — only states totally symmetric under interchange of quanta are admitted. (This is a concrete example of a general feature of symmetries. Accommodating simultaneously several compatible symmetries will generally constrain the acceptable irreps of the symmetries involved.)

The only remaining issue is the angular momentum ℓ content of the $(N, 0)$ irrep of $\mathsf{su}(3)$. This irrep contains $(N + 1)(N + 2)/2$ weights, all non-degenerate. The canonical subalgebra chain $\mathsf{su}(3) \supset \mathsf{su}(2)$ has the branching rule $(N, 0) \to (N) \oplus (N - 1) \oplus (N - 2) \oplus \ldots \oplus (2) \oplus (1)$, in terms of Dynkin indices, corresponding to $j = N/2, (N - 1)/2, (N - 2)/2, \ldots, 1, 1/2$. This is clearly not relevant to the always-integer angular momentum ℓ of the isotropic oscillator. However, the set of three antisymmetric combinations of off-diagonal operators, $\{a^\dagger_\xi a_\eta - a^\dagger_\eta a_\xi, a^\dagger_\eta a_\zeta - a^\dagger_\zeta a_\eta, a^\dagger_\zeta a_\xi - a^\dagger_\xi a_\zeta\}$ is closed under commutation and generates the algebra $\mathsf{so}(3)$. (This is a special case of the general result that n independent bosons generate the algebra $\mathsf{so}(n)$ via the $n(n - 1)/2$ antisymmetric operators $a^\dagger_i a_j - a^\dagger_j a_i$, where the indices i, j label the bosons and run from 1 to n.) The subalgebra chain $\mathsf{su}(3) \supset \mathsf{so}(3)$ then provides the appropriate ℓ content for the $(N, 0)$ irreps, namely $\ell = N, N - 2, N - 4, \ldots, 1 \,\text{or}\, 0$, as follows.

The orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$ can be rewritten in terms of creation and destruction operators, as defined in eqs.(4) and (5), in the form

$$L_z = xp_y - yp_x = -i\hbar(\xi\frac{\partial}{\partial\eta} - \eta\frac{\partial}{\partial\xi}) = -i\hbar(a^\dagger_\xi a_\eta - a^\dagger_\eta a_\xi), \qquad (21)$$

and cyclically in x, y, z. Up to a factor i and a scale factor $\hbar$, these are just the generators identified above as those of $\mathsf{so}(3)$. It is evident from the structure of the three generators L_ρ that $L_\rho|0\rangle = 0$, so the vacuum state has angular momentum 0.

Straightforward evaluation of commutators leads to the set of relations

$$\begin{array}{lll} [L_x, a^\dagger_\xi] = 0; & [L_x, a^\dagger_\eta] = i\hbar a^\dagger_\zeta; & [L_x, a^\dagger_\zeta] = -i\hbar a^\dagger_\eta; \\ [L_y, a^\dagger_\xi] = -i\hbar a^\dagger_\zeta; & [L_y, a^\dagger_\eta] = 0; & [L_y, a^\dagger_\zeta] = i\hbar a^\dagger_\xi; \\ [L_z, a^\dagger_\xi] = i\hbar a^\dagger_\eta; & [L_z, a^\dagger_\eta] = -i\hbar a^\dagger_\xi; & [L_z, a^\dagger_\zeta] = 0, \end{array}$$

from which it follows that the three quantities

$$a^\dagger_0 = a^\dagger_\zeta; \quad a^\dagger_{+1} = -\frac{1}{\sqrt{2}}\left(a^\dagger_\xi + ia^\dagger_\eta\right); \quad a^\dagger_{-1} = \frac{1}{\sqrt{2}}\left(a^\dagger_\xi - ia^\dagger_\eta\right) \qquad (22)$$

satisfy the equations

$$[L_z, a^\dagger_{\pm 1}] \;=\; \pm\hbar a^\dagger_{\pm 1} \qquad (23)$$

$$[L_z, a_0^\dagger] = 0 \quad (24)$$
$$[L_+, a_{+1}^\dagger] = 0 \quad (25)$$
$$[L_+, a_0^\dagger] = \sqrt{2}\hbar a_{+1}^\dagger \quad (26)$$
$$[L_+, a_{-1}^\dagger] = \sqrt{2}\hbar a_0^\dagger \quad (27)$$
$$[L_-, a_{+1}^\dagger] = \sqrt{2}\hbar a_0^\dagger \quad (28)$$
$$[L_-, a_0^\dagger] = \sqrt{2}\hbar a_{-1}^\dagger \quad (29)$$
$$[L_-, a_{-1}^\dagger] = 0 \quad (30)$$

with the usual step operators $L_\pm = L_x \pm iL_y$. These are recognised as the defining properties of the spherical components of a vector operator $\vec{a}^\dagger$, so that the states $a_\mu^\dagger|0\rangle$, with $\mu = -1, 0, +1$, have angular momentum 1 and projection μ (both in units of $\hbar$).

The angular momentum content of multi-quantum states can now easily be deduced. Each quantum (action of a creation operator $a^\dagger$ on the vacuum) carries a unit of angular momentum and a projection $\mu\hbar$. An N-quantum state has a maximum possible total angular momentum projection of $N\hbar$ (all N quanta having projection $+\hbar$). There is only one such state, which must then have angular momentum $N\hbar$. There will thus be $2N + 1$ states of angular momentum $N\hbar$, with all possible projections from $-N\hbar$ to $+N\hbar$, in unit steps. The next highest projection attainable with N quanta is $(N-1)\hbar$, containing $N-1$ quanta with projection $+\hbar$ and one quantum with projection 0. (Note that the identity of the quanta means that there is no significance to their order — this is characteristic of bosonic excitations.) There is only one way to make such a projection, so only one state of this projection, which must belong to the $L = N$ state already found. There is therefore no state of angular momentum $L = N - 1$.

The next lower projection, $(N - 2)\hbar$, can be made up in two different ways — $N - 2$ times $+\hbar$ and twice 0, or $N - 1$ times $+\hbar$ and once $-\hbar$. Two independent linear combinations of these configurations can be formed, one belonging to the existing state of $L = N$ and constructed by acting twice with the step-down operator L_- on the state with projection $N\hbar$. The other independent linear combination must belong to a state of angular momentum $L = N - 2$.

This process can be continued. At each stage, the relevant value of the projection M is reduced by one and the number of ways of producing that M value is determined. If this is equal to the number of L values already established, then there is no new state with $L = M$; if the number is larger than the number of L values already established, then there are new states of $L = M$. But the number of ways of producing a given value of M is easily established. The given value is obtained by selecting M quanta with projection $+1$ and $N-M$ with projection 0. All other configurations with the same value of M are then obtained by replacing a pair of $\mu=0$ quanta by one with $\mu=+1$ and one with $\mu=-1$. So the number of configurations with a given value of M is just $\lfloor\frac{N-M}{2}\rfloor$, where $\lfloor x\rfloor$ is the largest integer no greater than x. This means that one new L value is introduced each time M decreases by 2. The allowed L values for a given N are $L=N, N-2, N-4, \ldots, 1$ or 0, being even or odd according as N is even or odd. This is just the branching rule given above.