Phase transitions in wave turbulence

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We consider turbulence within the Gross–Pitaevsky model and look into the creation of a coherent condensate via an inverse cascade originating at small scales. The growth of the condensate leads to a spontaneous breakdown of statistical symmetries of overcondensate fluctuations: First, isotropy is broken, then a series of phase transitions marks the changing symmetry from twofold to threefold to fourfold. We describe respective anisotropic flux flows in the \( k \) space. At the highest level reached, we observe a short-range positional and long-range orientational order (as in a hexatic phase). In other words, the more one pumps the system, the more ordered the system becomes. The phase transitions happen when the system is pumped by an instability term and does not occur when pumped by a random force. We thus demonstrate nonuniversality of an inverse-cascade turbulence with respect to the nature of small-scale forcing.

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Probably the most unexpected discovery made in studying turbulence is an inverse cascade\,[1]. Flying in the face of an intuitive picture of turbulence as a process of fragmentation, an inverse cascade is self-organized, i.e., the appearance of large-scale motions out of a small-scale noise. An inverse cascade culminates in the creation of a spectral condensate, a mode that is spatially coherent across the whole system. Understanding the interaction of turbulence with a coherent flow is a central problem in turbulence studies in fluid mechanics and beyond, both from fundamental and practical perspectives. In fluids, condensates are system-sized vortices or zonal flows\,[2–4]. Turbulence with the condensate shares many properties with quantum systems, displaying both fluctuations and coherence. This closeness is shown perhaps most vividly within the framework of the nonlinear Schrödinger (Gross–Pitaevsky) equation (NSE):

\[
i\psi_t = -\Delta \psi + |\psi|^2 \psi.
\]

In classical physics, NSE describes a spectrally narrow distribution of nonlinear waves; respective nonequilibrium states are called optical turbulence\,[1,5]. NSE conserves the wave action \( N = |\psi|^2 \) and the Hamiltonian \( \mathcal{H} = |\nabla \psi|^2 + |\psi|^4/2 \), with the overline being the space average. For weak nonlinearity, when \( \mathcal{H} \approx |\nabla \psi|^2 \), i.e., quadratic in \( \psi \) as \( N \), one can rigorously show that pumping at some scale produces two cascades, a direct one of \( \mathcal{H} \) toward smaller scales and an inverse one of \( N \) toward larger scales\,[1]. Generally, one may question the very existence of the inverse cascade\,[6]. Here we show that the inverse cascade of \( N \) exists even for \( |\nabla \psi|^2 \ll |\psi|^4/2 \) and analyze the ways \( N \) flows from pumping toward the condensate in \( k \) space. The condensate solution of Eq. (1) is just a constant, \( \psi = \sqrt{N_0} \exp(-iN_0t) \), while turbulence can consist of weakly interacting waves. This simple system demonstrates unexpectedly rich behavior.

Small overcondensate fluctuations follow the Bogolyubov dispersion relation \( \Omega_k^2 = 2N_0 k^2 + k^4 \)[7]. As \( N_0 = |\psi|^2 \) grows, the dispersion relation approaches the acoustic one with the sound speed \( c = \sqrt{2N_0} \). Acoustic waves running in the same direction interact strongly, producing shocks. On the other hand, the matrix element of the three-wave interaction decreases as \( N_0^{1/2} \) (see, e.g., Ref. [5]). The nonlinearity parameter for overcondensate fluctuations can be estimated as \( (N - N_0)/N_0 \), using the weak-turbulence approximation\,[1,8]. That suggests that when the number of waves with zero momentum outgrows that with nonzero momentum, the nonlinear interaction must be effectively weak despite the fact that the nonlinear term in Eq. (1) is dominant. That explains the observation made in Ref. [9], and confirmed here for much higher \( N_0 \), that the statistics of the fluctuations approaches Gaussian as \( N_0 \) grows. Still, the nature of even weakly nonlinear turbulence coexisting with a strong condensate is very much a mystery. This Rapid Communication is intended to elucidate the most salient features of such turbulence: nonuniversality with respect to excitation mechanism and statistical symmetries.

To simulate turbulence, we add pumping and dissipation. Waves in fluids and plasma are usually excited by instability, while in quantum physics and optics they are excited by an external force. Therefore, we treat two types of excitation, adding to the right-hand side of the Fourier-transformed NSE either (i) an instability term \( \gamma_k \psi_k \) or (ii) a random complex force \( F_k(t) \) with a constant amplitude and phases uncorrelated both in time and \( k \) space. The (real) growth rate \( \gamma_k \) and \( |F_k(t)| \) are positive within the narrow shell at the middle of the spectral domain \( k_1 < k < k_2 \), negative at \( k > k_2 \), and zero at \( k < k_1 \). To study the steady state, we interrupt the simulation and restart it with additional friction, replacing \( \gamma_k \) with \( \gamma_k^* = \gamma_k - \alpha \) (we choose \( \alpha \) empirically to stabilize \( N_0 \) at a desirable level). The results presented below are obtained in \( 8\pi \times 8\pi \) domain at a spectral resolution of \( 512 \times 512 \) points (excluding dealiased modes) (more details can be found in Ref. [8]). Numerics run long past the time when all vortices disappear, so we are left only with a large condensate and small-amplitude waves running over it. The pumping wavelength \( \lambda = 4\pi/(k_1 + k_2) \) is effectively the smallest scale in the system since dissipation dominates at smaller scales, and we use \( \epsilon = N_0\lambda^2/4 \) as a dimensionless measure of nonlinearity. All simulations are done with \( \lambda \approx 2\pi/30 \), so \( \epsilon \approx 0.01N_0 \).
At the beginning, waves in the pumping shell are pumped isotropically. Let us now discuss what possible anisotropy may the turbulence spectra acquire after the condensate appears. The standing lowest modes of the box provide spatial modulation of the condensate intensity $N_0$ and respective modulation of the sound speed $c$ for shorter waves, according to the Bogolyubov dispersion relation. The regions of low $N_0$ (and $c$) must act as waveguides, with short waves moving predominantly along the minima of $N_0$. In other words, the angular maxima of $n_k = \langle |\psi_k|^2 \rangle$ at high $k$ would correspond to minima at low $k$. (Angular brackets denote time averaging.) That suggests a simple picture of possible anisotropy: The lowest modes will have the symmetry of the square box and impose that symmetry on the small-scale turbulence (turned by $\pi/4$). The main result of this paper is that this is not the case, as a result of nonlinear interaction unaccounted for in that simple picture.

We focus first on the instability-driven system. It undergoes a series of phase transitions between states of different symmetry. Initially, the condensate is fed by an inverse cascade carried by an almost isotropic spectrum of fluctuations. At this stage, the nonlinear (interaction) term in the Hamiltonian is less or comparable to the quadratic term (that describes diffraction and dispersion for waves or kinetic energy for particles). As the condensate grows, the first symmetry breaking appears at $\epsilon = \epsilon_1 \simeq 1$ when the system becomes anisotropic and the spectrum turns into an oval. A further transformation happens gradually, in the interval $2 \lesssim \epsilon \lesssim 5$, as the oval becomes thinner at the waist, turning into a dumbbell. As $N_0$ grows further, the symmetry changes from twofold to threefold, and then to fourfold, as seen in Fig. 1. Comparing slowly evolving runs with no friction with those stabilized at different values of $N_0$, we find the same symmetry states (which means that the evolution is slow enough so that the system is close to a steady state at any moment). The phase transitions thus happen between phases which are steady states. To avoid a slow evolution from initial thermal noise to higher levels of the condensate, we made separate runs, taking the initial conditions for $\psi$ as thermal noise plus a large constant (preset condensate). In these cases, the same states (with two, three, and four petals) appear, i.e., they are true attractors, independent of the way one reaches a given $N_0$. Further transitions are possible: We observe six-petal spectra with a triangular spatial pattern in the amplitude, and a transient eight-petal spectrum, with patches of square patterns oriented at a $45^\circ$ angle (see Ref. [8] for more details).
In evolving systems, the transitions occur sharply, yet the threshold values of $\epsilon$ fluctuate from run to run, being sensitive to the phases of initial noise. This shows that the phenomenon is not caused by a linear instability with a well-defined threshold. Indeed, the condensate is stable with respect to the infinitesimal perturbations. It is likely that our transitions are of a probabilistic nature, similar to transitions in a pipe flow [10] or fiber laser [11]: With the change of the control parameter (condensate amplitude in our case) one changes the probability that a finite-amplitude perturbation will lead the system away to a new state.

One may suggest a possible physical mechanism of the transitions as follows: Acoustic waves effectively interact only within the angle $k/\sqrt{N_0}$, which decreases as $N_0$ grows, so the turbulence tends to be broken into jets; the number of jets $j$ increases as an inverse of the interaction angle, i.e., as $N_0^{1/2} \propto \epsilon^{1/2}$. We observe that the transitions occur at $\epsilon_{1-3} \approx 10$, $\epsilon_{3-4} \approx 43$, which may roughly correspond to $\epsilon_j \sim j^2$. Even if this is indeed the basic mechanism of the transitions, we still lack any understanding of whether there is a unifying principle that can predict which turbulent state is realized the way the variational principle does for thermal equilibrium. One direction worth exploring is whether one can develop an approach similar to the weak crystallization theory [12] despite the fact that we have developed turbulence with power-law spectra carrying flux in $k$ space, as seen in Figs. 2 and 3.

With $\epsilon$ growth, two-petal spectra concentrate in a more narrow angle and broaden again just before the transition. The three-petal spectra broaden with $\epsilon$ growth, especially at high $k$. For $\epsilon$ just below the transition value, the spectra are almost uniform, with seeds of new symmetry visible at low $k$ in the coherent part of the averaged spectra. The population of the pumping shell increases between the transitions and falls during the transitions [8], which leads to a complicated evolution of $N(t)$, different from the simple laws of condensate growth suggested before, from $N(t) \propto \sqrt{t}$ and $N(t) \propto t$ [9] to $N \propto t^2$ [13].

For all states (except possibly very close to the transition events), and for all symmetries, the angular width of the spectra decreases toward larger wave numbers, as seen in Figs. 1 and 2. This may be analogous to spectrum anisotropization observed for acoustic turbulence [1,14]. It is worth stressing though that in the above references the direct energy cascade (realized by pumping-generated long waves turning into shocks) was studied, while we are dealing with the inverse cascade. One can therefore say that as the inverse cascade proceeds toward a larger wavelength, the spectra are getting wider as weak turbulence theory predicts [1]; yet presently we see no way it can predict the spontaneous appearance of anisotropy and changes in symmetry, particularly, significant anisotropy at the pumping scale.

For the twofold and threefold spectra, maxima and minima are distributed randomly in space. The most remarkable finding is seen in the last image of the top row in Fig. 1: On top of a long-range orientational order, a short-range positional order appears. The symmetry of this state is between that of a solid and an isotropic liquid, very much as a hexatic phase in two-dimensional (2D) melting [15]. It requires future studies (and larger domains) to establish whether this is indeed a turbulent analog of the Berezinski-Kosterlitz-Thouless transition; to avoid misunderstanding, we remind that there are no vortices (holes) in the condensate left at this stage (see Fig. 1), and the phase fluctuates weakly. It is the amplitude of the small condensate perturbations which is getting ordered into a lattice with defects. The spatial correlations are also seen in Fig. 2(e), which shows the correlation function $\psi(x)\overline{\psi(x+r)} - \overline{\psi^2}$, the overline being the average over $x$. The crystallization which we observe is very much different from that externally imposed by a cutoff in 2D incompressible turbulence [16]. Likewise, in Ref. [17] the symmetry change was caused by a change in the domain shape (from square to rectangle); in our case, the symmetry of the environment is not changed, and the control parameter is the overall excitation level (the number of waves) in the system.

The time-spectral filtering [18] shows that the mode coherently oscillating with the frequency $N$ is not a simple constant; it has an intricate spatial structure and multiscale correlations whose anisotropy is directly related to the anisotropy of over-condensate fluctuations (see Fig. 1). The spatial spectrum of the coherent mode has double the number of angular maxima and is generally more symmetric than the whole spectrum. For
nonzero $k$, the intensity of the temporally coherent mode is much smaller than the overall $r_k$, which decays approximately by power laws inside the petals, as seen in the fourth panel of Fig. 2. Note that $1/k^2$ is the spectrum of the (isotropically weakly turbulent) inverse cascade which coincides in this case, up to logarithmic factor, with thermal equilibrium, while $1/k^3$ is the spectrum due to shock waves [1]. Conservation of $N$ by Eq. (1) allows one to write NSE as a continuity equation and define the flux: $\text{div} \mathbf{J}_k \equiv \langle |\psi_k|^2 \rangle \gamma_k \equiv Q_k$. We compute $Q_k$ and solve the above equation for $\mathbf{J}_k$. Figure 3 shows the flux lines while positive (negative) source $Q_k$ is shown by dark (light) regions. We have also calculated the flatness which is slightly below the Gaussian value 3 for overcondensate fluctuations [8], i.e., the condensate effectively suppresses strong fluctuations, contrary to what condensate vortices do to 2D incompressible turbulence [2].

A central issue in nonequilibrium physics is that of universality: What properties of a state depend on the excitation mechanism? We thus described a phenomenon of spontaneous symmetry breaking (or different types of order) compete via different contributions to the thermodynamic potential (e.g., energy competes against entropy). In all cases of turbulence known so far, the symmetry of the forcing and boundary. To put it simply, one had no different turbulent states under similar conditions and turbulence: The former is known to realize an extremum of some thermodynamic potential (say, free energy), while for the latter, to the best of our knowledge, no variational principle has been discovered, despite much effort. Phase transitions provide such an important window into equilibrium statistical physics because they show how order and disorder (or different types of order) compete via different contributions to the thermodynamic potential (e.g., energy competes against entropy). In all cases of turbulence known so far, the symmetry of the state was completely determined by the symmetry of the forcing and boundary. To put it simply, one had no different turbulent states under similar conditions and now we have. We conclude by asking the following question: Is there any quantity that the system tries to optimize by undergoing the series of phase transitions discovered here?

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