L9

The last time we have studied the spectrum of mesons at small $h << m^{15/8}$ within the two-quark approximation. The mass spectrum is derived from the quantization condition

$$\Delta(p_0) = 0\,, \tag{9.1}$$

where [1]

$$\Delta(p) = \int_q \left[2p\,\frac{\Omega(p)-\Omega(q)}{p^2-q^2} - i\lambda\,G_{\rm reg}(p|q) - \lambda\,G_{\rm even}(p|q)\right] e^{i\,S(q)} \tag{9.2}$$

and

$$S(q) = \frac{1}{\lambda}\int \Omega(q)\,, \qquad \Omega(p) = \omega(P/2+p) + \omega(P/2-p) - E\,. \tag{9.3}$$

The point p_0 is determined by the equation

$$\Omega(\pm\,p_0) = 0\,. \tag{9.4}$$

In (9.2) and below I use the notation

$$\lambda = \frac{2\bar\sigma h}{m^2} << 1\,. \tag{9.5}$$

Evaluating the integral in (9.2) in the stationary-phase approximation leads to the semi-classical meson spectrum. The stationary phase approximation is good when λ is small and at the same time the level number n is sufficiently large,

$$\lambda << 1\,, \qquad n >> 1\,. \tag{9.6}$$

If however n is fixed, while $\lambda \to 0$, the stationary-phase points $p = \pm\,p_0$ approach each other and their contributions can not be treated independently; It is easy to see that

$$p_0/m \simeq \big(\lambda\,n\big)^{1/3}\,. \tag{9.7}$$

In this regime one can develop another systematic approximation for solving the Bethe-Saltpeter equation, which is valid at small λ and fixed n; I will call it the low-energy expansion.

---

[1]

$$G_{\rm even}(p|q) = \int_{C_-} \frac{2k\,G(p|k)}{k^2-q^2}\,\frac{dk}{2\pi}\,.$$

In this approximation we assume that all the quark momenta inside the meson are small as compared to m, and makes systematic expansion in powers of the momenta. For instance, assuming that p is small, we have

$$\Omega(p) = \omega(P/2+p) + \omega(P/2-p) - E = -E + E_0(P) + \frac{8\,p^2}{E_0^3(P)} + O(p^4) \tag{9.8}$$

where

$$E_0(P) = \sqrt{4m^2 + P^2}\,. \tag{9.9}$$

Correspondingly, for the "action"

$$S(q) = \frac{1}{\lambda}\int \Omega(q)$$

we have

$$S(q) = S_0(q) + \delta S(q)\,, \tag{9.10}$$

where

$$S_0(q) = -(E - E_0)\,q + \frac{8}{E_0^3}\,\frac{q^3}{3}\,, \qquad \delta S = O(q^5)\,. \tag{9.11}$$

Let us write the equation (9.2) as

$$\Delta(p) = \int_q \left\{ \left[2p\,\frac{\Omega(p)-\Omega(q)}{p^2-q^2} - i\lambda\,G_{\rm reg}(p|q) - \lambda\,G_{\rm even}(p|q) \right] e^{i\,\delta S(q)} \right\} e^{iS_0(q)} \tag{9.12}$$

and expand the expression in the curly brackets in double power series in p and q and evaluates the q-integral order by order in this power series. The zeroth order corresponds to the non-relativistic Schrodinger equation with the potential $\lambda\,|x|$ which we have already considered before. In general this expansion leads to the expansion of E in powers of the parameter t^2,

$$E = E_0 + t^2\,E_2 + t^4\,E_4 + \ldots\,, \qquad t^2 = \lambda^{2/3}\,. \tag{9.13}$$

It turns out that to the first few orders (to the order t^6 inclusive) this expansion is consistent with the relativistic dispersion law

$$E = \sqrt{M^2 + P^2} \tag{9.14}$$

with

$$M = 2m + \mu_2\,t^2 + \mu_4\,t^4 + \ldots\,, \tag{9.15}$$

for instance

$$E_2 = \frac{\mu_2\,m}{\sqrt{4m^2+P^2}}\,, \quad \text{etc}\,. \tag{9.16}$$

Explicit calculation yields for (9.15)

$$M = m\left(2 + z\,t^2 - \frac{z^2}{20}\,t^4 + \left(\frac{11\,z^3}{1400} - \frac{57}{280}\right)t^6 + O(t^8)\right), \tag{9.16}$$

where again $-z$ is any zero of the Airy function,

$$\mathrm{Ai}(-z) = 0\,. \tag{9.17}$$

The expansion can be developed further, but one has to be careful here since the last term written down in (9.16) already exceeds the accuracy of the two-quark approximation. To see this let us take a look at the multi-quark components of the mesons.

Beyond the two-quark approximation the meson state contains contributions from multi-quark sectors

$$\mid Meson(P)\,\rangle = \int_{p_1,p_2} \Psi(p_1,p_2)\ \mid p_1,p_2\,\rangle + \int_{p_1,\ldots,p_4} \Psi_4(p_1,...,p_4)\ \mid p_1,...,p_4\,\rangle + \ldots\,. \tag{9.18}$$

We are interested in the meson state with definite momentum P, hence all the amplitudes Ψ_{2n} contain the momentum delta-function. As usual, if the meson is well below the 2-meson threshold, i.e. $M - 2m << 2m$, the multi-quark components can be taken into account perturbatively. This leads to the corrections $O(\lambda^2)$ to the right hand side of the Bethe-Saltpeter equation,

$$\Omega(p)\,\Psi(p) - \lambda \int_q G(p|q)\,\Psi(q) - \lambda^2 \int_q G_2(p|q)\,\Psi(q) - O(\lambda^3) = 0\,, \tag{9.19}$$

Here the correction G_2 to the kernel G comes from the multi-quark contributions in the second order of the perturbation theory, $G_2(p|q) =$

$$= \sum_{n=2}^{\infty} \int_{k_1,\ldots,k_{2n}} \frac{\langle\, p_1,p_2 \mid \sigma(0) \mid k_1,...,k_{2n}\,\rangle\langle\, k_1,...,k_{2n} \mid \sigma(0) \mid q_1,q_2\,\rangle}{\omega(k_1)+\cdots+\omega(k_{2n}) - E}\,\delta(P - \sum k_i)\,, \tag{9.20}$$

where again $p_1 = P/2 + p, p_2 = P/2 - p, q_1 = P/2 + q, q_2 = P/2 - q$.

It is obvious from this definition that the correction term $G_2(p|q)$ has the same general symmetry properties as the main kernel $G(p|q)$,

$$G_2(p|q) = G_2(q|p) = -G_2(-p|q) = -G_2(p|-q)\,. \tag{9.21}$$

If this term was regular function of p and q, its effect would be very small. Indeed, from the symmetries it must expand as

$$G_2^{\mathrm{reg}}(p|q) \sim p\,q + \text{higher terms}\,. \tag{9.22}$$

Since in the low energy expansion we have $p, q \sim t$, the contribution to the meson mass from this term would be

$$\lambda^2\,t^2\,t = t^9\,. \tag{9.23}$$

(Note that it is odd power of t), and it has no effect on the terms written down in (9.16).

The above estimate ignores the fact that the matrix elements involved in the expression (9.20) have singularities (poles) when one of the intermediate momentum k approaches any of the external momenta p or q. These are principal-value poles, and integration over k_i in general produces terms proportional to the delta-functions supported by the configurations with coinciding initial and final momenta, like

$$\delta(p-q) \quad \text{or} \quad \delta(p+q)\,. \tag{9.24}$$

By the same power counting, such terms would bring contributions

$$\text{delta} - \text{terms}: \quad \sim \lambda^2\, t/t = t^6\,, \tag{9.25}$$

where t in the denominator is due to the delta-function. Such terms would compete with the last term in (9.20).

Natural mechanism of generating such delta-function terms is from the "disconnected configurations". Suppose only one of the two particles experiences direct interaction with the spin operator insertions in (9.20), while another does not come close to the interaction domain **Fig.1**

This configurations would affect the "self-energies" of the individual quarks, in particular give rise to the quark mass renormalization. As the result, the effective quark mass is not exactly m but has some perturbative corrections,

$$m_q = m + a_q\,\frac{h^2}{m^{\frac{11}{4}}} + O(h^4)\,, \tag{9.26}$$

where a_q is dimensionless number. How to find the expansion (9.26)?

Let me stress that since the quarks do not appear as asymptotic states, there is no definition of the quark mass independent of the perturbation theory. Nonetheless it seems natural to define the perturbative corrections to the quark mass in the usual manner, in terms of the self-energy part: for the h^2 term in (9.26) we have [2]

$$\delta m_q^2 = -h^2 \int \langle\, A(\theta) \mid \sigma(x)\sigma(0) \mid A(\theta)\,\rangle_{\rm irred}\, d^2x\,, \tag{9.27}$$

where $\langle ... \rangle_{\rm irred}$ denotes connected and *one-particle irreducible* matrix element.

Let me demonstrate how the correction (9.27) is computed. The equation (9.27) involves the matrix element of two operators σ between one-particle states. As we know already, such matrix elements are expressed in terms of the functions $\Psi_\pm(x,\theta)$, solutions of the linear problem associated with the sinh-Gordon system. The expressions for the connected parts

$$G(\theta_1|\theta_2) = \langle\, A(\theta_1) \mid \sigma(x)\sigma(x') \mid A(\theta_2)\,\rangle - 2\pi\,\delta(\theta_1-\theta_2)\,G\,, \quad G = \langle\, \sigma(x)\sigma(x')\,\rangle\,, \tag{9.28}$$

$$\tilde G(\theta_1|\theta_2) = \langle\, A(\theta_1) \mid \mu(x)\mu(x') \mid A(\theta_2)\,\rangle - 2\pi\,\delta(\theta_1-\theta_2)\,\tilde G\,, \quad \tilde G = \langle\, \mu(x)\mu(x')\,\rangle\,, \tag{9.29}$$

was found already using the ward identities of the doubled IFT (in (9.28) and (9.29), and below, the dependence of x, x' is suppressed). Here I exhibit particular case $\theta_1 = \theta_2 = \theta$ which is only needed for the calculation in (9.27),

$$G(\theta|\theta) = \Psi_+(\theta)\Psi_-(\theta)\left[\tilde G - G\,\frac{d}{d\,\theta}\ln\left(\frac{\Psi_+(\theta)}{\Psi_-(\theta)}\right)\right], \tag{9.30}$$

$$\tilde G(\theta|\theta) = \Psi_+(\theta)\Psi_-(\theta)\left[G - \tilde G\,\frac{d}{d\,\theta}\ln\left(\frac{\Psi_+(\theta)}{\Psi_-(\theta)}\right)\right], \tag{9.31}$$

The expression (9.30) is almost ready to be substituted into the equation (9.27); it only remains to subtract the one-particle reducible parts. The one-particle reducible parts correspond to diagrams in **Fig.2**

where the lines represent propagation of the particles. The diagrams Fig.2a and Fig2.b

[2] About normalizations: In standard perturbation theory with $f.t.\langle\varphi(x)\varphi(0)\rangle = \frac{1}{p^2+m^2}$ we have $\langle 0 \mid \varphi(0) \mid a(p)\rangle = 1$, with $\langle a(p) \mid a(p')\rangle = 2\pi\, 2\omega_p\,\delta(p-p')$. Then $\Sigma = \frac{1}{2}\,\langle\varphi(Y)O(x)O(y)\varphi(X)\rangle \to \frac{1}{2}\int\langle a(p) \mid O(x)O(0) \mid a(p)\rangle$. Since $\mid a(p)\rangle = \sqrt{2} \mid A(\theta)\rangle$ we have (9.27).

have one and three particles in the intermediate state, respectively. We have

$$S(\theta|\theta) \equiv \text{Fig.2a} + \text{Fig.2b} =$$

$$\int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \Big[\langle\, A(\theta) \mid \sigma(x) \mid A(\theta')\, \rangle \, \langle\, A(\theta') \mid \sigma(x') \mid A(\theta)\, \rangle +$$

$$+ \langle\, 0 \mid \sigma(x) \mid A(\theta) A(\theta')\, \rangle \, \langle\, A(\theta) A(\theta') \mid \sigma(x') \mid 0\, \rangle \Big] \,. \qquad (9.32)$$

The matrix elements appearing here are the form-factors

$$\langle\, 0 \mid \sigma(0) \mid A(\theta_1) A(\theta_2)\, \rangle = i\bar{\sigma}\, \tanh \frac{\theta_1 - \theta_2}{2} \,,$$

$$\langle\, A(\theta_1) \mid \sigma(0) \mid A(\theta_2)\, \rangle = i\bar{\sigma}\, \coth \frac{\theta_1 - \theta_2}{2} \,, \qquad (9.33)$$

After subtracting the one-particle parts, we have for (9.27)

$$\delta m_q^2 = -h^2 \int \big[G(\theta|\theta) - S(\theta|\theta) \big]\, d^2x \,, \qquad (9.34)$$

where the integration is over Euclidean coordinates of x. Since the functions in the integrand here and depend only on r and $\alpha - i\theta$

$$G(\theta|\theta), \quad S(\theta|\theta) \qquad \text{depend on} \quad r,\ \beta = \alpha - i\theta \,, \qquad (9.35)$$

(where r, α are the polar coordinates associated with $x - x'$), the result of integration in (9.34) does not depend on θ. After the one-particle part S is subtracted, the integrand in (9.34) decays exponentially at $r \to \infty$, i.e. it converges fast.

The integration in (9.34) can be performed numerically with arbitrary accuracy. In fact, in this calculation some nice properties of the sinh-Gordon system are useful. Just like the two-point functions G and $\tilde{G}$ are conveniently parameterized by φ and χ,

$$G = e^{\chi/2} \cosh \varphi/2 \,, \qquad \tilde{G} = e^{\chi/2} \sinh \varphi/2 \,, \qquad (9.36)$$

where φ and χ satisfy the sinh-Gordon system, it is convenient to parameterize the functions $\Psi_\pm$ in terms of two auxiliary functions $\phi(r, \beta)$ and $\rho(r, \beta)$

$$e^{\chi/2}\, \Psi_\pm = e^{(\rho \pm i\phi)/2} \,. \qquad (9.37)$$

Substitution of this form into the linear system that $\Psi_\pm$ satisfy leads to the equations

$$\partial(\varphi + i\phi) = \frac{m\, e^{\theta}}{2} \cosh\big(\varphi - i\phi\big) \,, \qquad \bar{\partial}(\varphi - i\phi) = \frac{m\, e^{-\theta}}{2} \cosh\big(\varphi + i\phi\big) \,, \qquad (9.38)$$

and

$$\partial(\rho-\chi)=\frac{m\,e^{\theta}}{2}\,\sinh\big(\varphi-i\phi\big)\,,\qquad \bar\partial(\rho-\chi)=\frac{m\,e^{-\theta}}{2}\,\sinh\big(\varphi+i\phi\big)\,. \tag{9.39}$$

The equation (9.38),(9.39) is a version of the so-Called Backlund transformation which allows one to generate new solutions of the sine-Gordon equation starting from known ones. In our case (9.38), (9.39) relate solution φ, χ of the sinh-Gordon system to the solution ϕ, ρ of the sine-Gordon system. Namely, it is straightforward to check that if φ and χ satisfy the sinh-Gordon system [3] than it follows from (9.38),(9.39) that

$$\partial\bar\partial\phi=-\frac{m^2}{8}\sin(2\phi)\,,\quad \partial\bar\partial\rho=\frac{m^2}{8}\Big[1+\cos(2\phi)\Big]\,,$$
$$\partial^2\rho-\big(\partial\phi\big)^2=0\,,\qquad \bar\partial^2\rho-\big(\bar\partial\phi\big)^2=0\,. \tag{9.40}$$

The solution of (9.40) which appears through (9.38),(9.39) is rather interesting. The phase ϕ is not a single-valued function of the coordinates. Instead, when written in terms of the polar coordinates (r,β), it is quasiperiodic function of the angle, $\phi(r,\beta+2\pi)=\phi(r,\beta)+2\pi$, as demanded by the monodromy properties of the functions $\Psi_\pm(r|\beta)$. Qualitatively, it can be described as the juxtaposition of two sine-Gordon domain walls (i.e. the sine-Gordon soliton solutions, in the Euclidean nomenclature) of opposite sign, extending along the some axis in opposite directions, **Fig.3a**; the solution is singular at $r=0$, and its shape in the "junction" region is shown in **Fig.3b**.

In numerical evaluation of the integrand in (9.34) one first finds numerically the rotationally invariant solution $\varphi(r)$, $\chi(r)$ of the sinh-Gordon system; this amounts to solving ordinary differential equations. Then the first-order differential equations (9.38), (9.39) are used to compute numerics for ϕ and ρ, and hence $\Psi_\pm$. Finally, the integrand in (9.34) is found using (9.30). Then numerical integration over d^2x in (9.34) yields

$$m_q=m\,\big(1+a_q\,\xi^2+...\big)\,,\qquad a_q=\bar s^2\;0.142021619(1)\;. \tag{9.41}$$

[3] Namely $\partial\bar\partial\varphi=\frac{m^2}{8}\sinh 2\varphi$, $\partial^2\chi-\big(\partial\varphi\big)^2=0$, $\bar\partial^2\chi-\big(\bar\partial\varphi\big)^2=0$, and $\partial\bar\partial\chi=\frac{m^2}{8}\big[1-\cosh 2\varphi\big]$.

With the quark mass correction taken into account the t^2 expansion (9.16) of the meson masses becomes exact to the order t^6 inclusive.

Multi-quark contributions to the higher-order terms are more subtle. Besides the correction to the quark mass described above, the delta-function contribution $\sim \lambda^2$ to $G_2(p|q)$ also modify the dispersion law of the quarks inside the meson. Up to the terms λ^2 the BS equation corrected by the multi-quark contributions can be written as

$$\big[\epsilon(p_1)+\epsilon(p_2)-E\big]\,\Psi(p_1,p_2) = \int_{q_1,q_2} \mathcal{G}(p_1,p_2|q_1,q_2)\,(2\pi)\,\delta(p_1+p_2-q_1-q_2)\,\Psi(q_1,q_2)\,, \quad (9.42)$$

where

$$\epsilon(p) = \sqrt{m_q^2+p^2} - \frac{\lambda^2}{8}\,\frac{m_q^3\,p^2}{(m_q^2+p^2)^{5/2}} + O(\lambda^4)\,, \quad (9.43)$$

and

$$\mathcal{G}(p-1,p_2|q_1,q_2) = \lambda\,G(p_1,p_2|q_1,q_2) + \lambda^2\,G_2(p_1,p_2|q_1,q_2) + O(\lambda^3)\,, \quad (9.44)$$

where in the kernel the G_2 term is regular at $p_1 = q_1$ and $p_1 = q_2$.

The correction term $\sim \frac{p^2}{\omega^5(p)}$ in the dispersion law (9.43) has simple origin. When we have considered the quark mass correction, we have subtracted the one-particle reducible parts from the self energy. These parts are depicted in Fig.2.

In a given frame the distinction between the contributions in Fig.2a and Fig.2b can be understood in terms of the time order of events: in Fig.2a the incoming particle is absorbed by the first sigma-insertion, and the intermediate particle $A(\theta')$ is simultaneously emitted; similar process occurs at another insertion. There is only one particle at any given instant of time. Therefore, the diagram in Fig.2a (more precisely, the 2-particle diagram with the one-particle part as in Fig.2a) is generated by iteration of the original BS equation with no multi-quark terms, and its subtraction was necessary to avoid over-counting. The diagram in Fig2b has more then one particle in the intermediate state; it is not a part of the two-quark approximation, hence it has to be added back.

Let

$$V(\tau) = h\int_{-\infty}^{\infty} \sigma(x,\tau)\,dx\,. \quad (9.45)$$

The "zigzag" diagram in Fig.2b yields the following contribution to the single-quark energy

$$
- \int_0^\infty \langle\, A(\theta_1) \mid V(\tau)\, V(0) \mid A(\theta_2)\,\rangle \to
$$

$$
- \int (2\pi)^2\, \delta(p(\theta_1) - p(\theta')\delta(p(\theta_2) - p(\theta'))\, \frac{F(\theta_1,\theta')F^*(\theta_2,\theta')}{\omega(\theta)+\omega(\theta')}\, \frac{d\theta'}{2\pi}\,, \tag{9.45}
$$

where the momentum delta-functions are due to the x-integrations, the energy denominator comes from the τ-integral, and

$$
F(\theta,\theta') = i\bar\sigma\, \tanh\frac{\theta-\theta'}{2} \tag{9.46}
$$

are the form-factors associated with the vertices in Fig.2b. The momentum delta-functions fix $\theta_1 = \theta_2 = -\theta$, i.e.

$$
(9.45) = -\frac{(2\pi)\,\delta(\theta_1-\theta_2)}{\cosh^2\theta_1}\, \frac{h^2\,|F(\theta_1,-\theta_1)|^2}{2\,m^2\,\cosh\theta_1} = -(2\pi)\,\delta(\theta_1-\theta_2)\, \frac{(\bar\sigma h)^2}{2\,m^2}\, \frac{\tanh^2\theta_1}{\cosh^3\theta_1} = \tag{9.47}
$$

$$
= \langle\, A(\theta_1) \mid A(\theta_2)\,\rangle \left[-\frac{\lambda^2}{8}\,\frac{p^2}{\omega^5}\right]. \tag{9.48}
$$

Let me also mention that adding this term

$$
\omega(p) \to \omega(p) - \frac{\lambda^2}{8}\,\frac{p^2}{\omega^5} \tag{9.49}
$$

restores the Lorentz invariance at the order t^8. It is also satisfying to note that in the "infinite momentum frame" $P \to \infty$ the correction terms in (9.49) disappear.

Another important effect of multi-quark contributions is the renormalization of the parameter λ in the BS equation. When λ is not too small, the "string tension" is not exactly $2\bar\sigma h$ but receives corrections $\sim h^3$ and higher. I will not discuss this effect in details here.

Some remarks are in order. Although the two-quark approximation is designed to describe the spectrum at small λ, and also only the part of the spectrum with $M_n - M_1 << M_1$, it turns out to be remarkably accurate in much wider domain. It accurately describes masses of "heavy" mesons with $n >> 1$ all the way up to the their decay thresholds

$$
M_n = 2\,M_1\,. \tag{9.50}
$$

When the mass M_n crosses this stability threshold, the meson M_n generally becomes a resonance state. It seems these resonances are very narrow at small λ (there are arguments their width are exponentially small when $\lambda \to 0$, although this is still not fully understood). The two-quark approximation ignores the possibility of decay, and for instance the semiclassical mesons pretend to be stable particles at any n. This of course is only approximation, but it seems the two-quark approximation reproduces the masses of the resonant mesons with good accuracy as well.

The two-quark approximation reproduces the masses of mesons with remarkable accuracy even for relatively large λ (i.e. small η). Thus, the leading semiclassical approximation reproduces the masses of all stable mesons with the accuracy better then 1% even for $\eta \geq 0.8$. Loop correction to the leading semiclassical approximation further improves the accuracy.

Naturally, the behavior of the spectrum which follows from the two-quark approximation is in full agreement with the Wu-McCoy scenario. Thus, the fifth particle M_5 disappears from the spectrum at $\eta \approx 1.80$, and M_4 at $\eta \approx 1.04$

$$M_5 : \quad \eta > 1.80\,, \qquad M_4 : \quad \eta > 1.04\,. \tag{9.51}$$

At $\eta = 0$ the two-quark approximation predicts three stable particles.

The case $\eta = 0$ (i.e. $m = 0$ with $h \neq 0$) is special. It is possible to show that the corresponding field theory

$$\mathcal{A} = \mathcal{A}_{c=1/2\ \mathrm{CFT}} + h \int \sigma(x)\, d^2x \tag{9.52}$$

is integrable. It has infinitely many local integrals of motion and factorizable S-matrix. Originally, I planned to include more or less detailed discussion of this integrable field theory in this cycle of lectures. As usual, everything in this life takes too long (which is not necessarily bad, by the way), and presently detailed discussion seems to be out of question. There are many published papers on this integrable theory, for instance review by Giuseppe Mussardo in Physics Reports.

I only mention here that this integrable theory involves eight stable particles M_1, M_2, ..., M_8, of which only three are below the two-particle threshold,

$$M_1, M_2, M_3 < 2\,M_1\,, \qquad 2\,M_1 < M_4, \ldots, M_8\,. \tag{9.53}$$

Since integrability breaks down at arbitrarily small non-zero η, the particles $M_4, ..., M_8$ become unstable at $\eta \neq 0$. Thus, the two-quark approximation correctly predicts the qualitative structure of the spectrum even at $\eta \to 0$. Is there any way to refine the two-quark approximation in such a way that it would show something like integrable structure at $\eta = 0$? I do not know. Anyway, I hope I will say some words about the S-matrix a little later.

Now let me consider another week coupling domain $h << |m|^{15/8}$, but this time in the high-T regime $m < 0$; this corresponds to large negative η. The situation is much simper here then at $\eta \to +\infty$. We have one particle, whose mass M_1 equals m at $\eta = -\infty$,

$$M_1(h = 0) = m\,. \tag{9.54}$$

Turning on nonzero h we just add week interaction of these particles. Perturbation theory in h can be developed, and not only everything expands in integer powers of h, but also there are all reasons to expect the perturbation theory expansions to be convergent in a finite domain around $\xi \equiv h/|m|^{15/8} = 0$.

As an example, let us consider the mass M_1. Since at $h = 0$

$$\langle\, A(\theta) \mid \sigma(0) \mid A(\theta)\,\rangle_{h=0,m<0} = 0 \tag{9.55}$$

the correction to the mass M_1 is $\sim h^2$. We have

$$\begin{aligned}\delta M_1^2 = -h^2 \int \langle A(\theta) \mid \sigma(x)\sigma(0) \mid A(\theta)\,\rangle_{h=0,m<0} = \\ = -h^2 \int \langle A(\theta) \mid \mu(x)\mu(0) \mid A(\theta)\,\rangle_{h=0,m>0}\,,\end{aligned} \tag{9.56}$$

where in writing (9.56) the duality property was used. According to our analysis of the matrix elements (which was done for $m > 0$), the correction involves the integral of the diagonal matrix element $\tilde{G}(\theta|\theta)$. One has to subtract disconnected parts **Fig.4**

whose contribution is

$$\begin{aligned}\tilde{S}(\theta|\theta) = \langle\, A(\theta) \mid \mu(x) \mid 0\,\rangle\langle\, 0 \mid \mu(x') \mid A(\theta)\,\rangle + \langle\, 0 \mid \mu(x) \mid A(\theta)\,\rangle\langle\, A(\theta) \mid \mu(x') \mid 0\,\rangle = \\ = 2\bar{\sigma}\, \cosh\big(mr\, \sin\beta\big)\,,\end{aligned} \tag{9.57}$$

where again $\beta = \alpha - i\theta$. The integral

$$-h^2 \int \big[\tilde{G}(\theta|\theta) - \tilde{S}(\theta|\theta)\big]\, d^2x \tag{9.58}$$

is evaluated numerically, as in the case if the quark mass:

$$M_1 = |m| \left(1 + a\, \frac{h^2}{|m|^{15/4}} + O(h^4)\right), \tag{9.59}$$

$$a = 10.7619899\,. \tag{9.60}$$