L8

The last time we have started to the IFT in the presence of the external field. This theory is described by the formal action

$$\mathcal{A}_{rmIFT} = \mathcal{A}_{c=1/2\ \text{CFT}} + \frac{m}{2\pi}\int \varepsilon(x)\, d^2x + h\int \sigma(x)\, d^2x\,. \tag{8.1}$$

Here $\varepsilon(x)$ and $\sigma(x)$ are the spin-less primary fields of the scale dimensions $D = \Delta + \bar{\Delta}$ equal 1 and 1/8, respectively.

Specifically, we have considered the situation in the weak-field domain $h << m^{8/15}$ in the low-T domain $m > 0$. We have found that the free fermions of the zero-h theory become at non-zero h the "quarks' which are attracted to each other by confining interaction, so that only the two-quark bound states - the "mesons" with the masses $M_n > 2m$ - appear in the particle spectrum of the theory. This cannot be particularly surprising- after all, we have seen at the beginning of this course that the lattice Ising model with nonzero external field is equivalent, by duality transformation, to the Ising model coupled to the Z_2 gauge field.

By treating the quarks as relativistic particles which are attracted by a linear potential but otherwise free, we have developed a "naive" semiclassical approximation for the masses

$$M_n = 2m\, \cosh\theta_n\,, \tag{8.2}$$

where θ_n are solutions of the equation

$$\sinh 2\theta_n - 2\theta_n = 2\pi\lambda\left(n - 1/4\right), \quad n = 0,\ 1,\ 2,\ \dots\,, \tag{8.3}$$

with

$$\lambda = \frac{2\bar{\sigma}\, h}{m^2} = \frac{2\,\bar{s}}{\eta^{15/8}}\,, \qquad \lambda << 1\,. \tag{8.4}$$

This approximation is expected to be valid at small λ (which plays the role of the Planck's constant) and sufficiently large n.

Also, if n is fixed and $\lambda \to 0$ the non-relativistic approximation for the quarks is valid, and one can reduce the problem to solving the Schroedinger equation for the two-quark system interacting through the linear potential $\sim |x_1 - x_2$. This yields

$$M_n = 2m + m\,t^2\, z_n\,, \tag{8.5}$$

where $-z_n$ are successive zeros of the Airy function, and I have used the notation

$$t = \lambda^{1/3}\,. \tag{8.6}$$

Now I want to refine the above approximations and also estimate their accuracy. This will be done by the analysis of the integral equation describing the relativistic two-quark system - the Bethe-Saltpeter equation.

The mesons are isolated states with finite energy. At $h = 0$ we have free particles, and the free Hamiltonian of course is

$$H^{(0)} - E_0^{(0)} = H_{\mathrm{FF}} = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, \omega(p)\, a_p^\dagger a_p\,, \tag{8.7}$$

where $E_0^{(0)}$ is the ground-state energy of the system at $h = 0$, again $\omega(p) = \sqrt{m^2 + p^2}$, and $a_p^\dagger$, a_p are creation and annihilation operators. Here I have found it advantageous to change normalization of the creation and annihilation operators as compared with the operators $A^\dagger(\theta)$, $A(\theta)$ I have used before, namely

$$A^\dagger(\theta) = \sqrt{\omega(p)}\; a_p^\dagger\,, \qquad A(\theta) = \sqrt{\omega(p)}\; a_p\,, \qquad \big(p = m\,\sinh\theta\big) \tag{8.8}$$

where p and θ are related as usual

$$p = m\,\sinh\theta\,. \tag{8.9}$$

They have canonical anti-commutators in terms of the momenta

$$\{a_p^\dagger, a_{p'}\} = 2\pi\,\delta(p - p')\,. \tag{8.10}$$

At $h \neq 0$ we have to add the term involving the spin operator

$$H - E_0 = H_{\mathrm{FF}} + h\int_{-\infty}^{\infty} \sigma(x)\,dx\,, \tag{8.11}$$

where now x is the *spatial* coordinate, and $\sigma(x) \equiv \sigma(x, t = 0)$. As we know, σ is nonlocal with respect to ψ, and it has no simple or useful closed expression in terms of the creation-annihilation operators. However, we know all the matrix elements of the of the field $\sigma(x)$ between any multi-particle states involving arbitrary numbers of the fermions, and hence we know all matrix elements of the Hamiltonian (8.11) between any eigenstates of the free theory.

Having in mind the idea of meson as a two-quark bound state, let us look for the eigenstates of the hamiltonian (8.11) in the form

$$\mid \text{Meson} \,\rangle = \frac{1}{2} \int \frac{dp_1}{2\pi}\frac{dp_2}{2\pi}\, \Psi(p_1, p_2)\, a^\dagger_{p_1} a^\dagger_{p_2} \,\mid 0\,\rangle + \text{multi} - \text{quark terms}\,, \tag{8.12}$$

where the "multi-quark terms" denotes possible contributions of the states with four, six, or more quarks,

$$\text{m} - \text{q terms} = \int_{p_1,p_2,p_3,p_4} \Psi_4(p_i)\, a^\dagger_{p_1} a^\dagger_{p_2} a^\dagger_{p_3} a^\dagger_{p_4} \,\mid 0\,\rangle + \big(\text{six quark term}\big) + \cdots\,, \tag{8.13}$$

and assume that the two-quark states constitute the dominating part. Formal justification is that if we are interested in the eigenvalues which are close to $2m$, the multi-quark states are separated by a wide gap $> 2m$. Thus, we can try to solve the problem in the two-quark sector, and then take into account the multi-quark contributions using perturbation theory. I'll be more specific below.

Thus, we will start by looking at the meson states in the two-quark form, i.e. the form explicitly written down in (8.12). Let us apply the Hamiltonian (8.11) to such two-quark state. The first free-fermion term H_{FF} is already diagonal, but for the "interaction term" involving σ we have

$$\begin{aligned}\int dx\, \sigma(x) \,\mid q_1, q_2\,\rangle = \int dx \int_{p_1,p_2} \mid p_1, p_2\,\rangle\, \langle\, p_1, p_2 \mid \sigma(x) \mid q_1, q_2\,\rangle + \\ + \text{multi} - \text{quark states}\,,\end{aligned} \tag{8.14}$$

where

$$\int_{p_1,p_2} \cdots = \frac{1}{2}\int \frac{dp_1\, dp_2}{(2\pi)^2}\, \cdots\,, \tag{8.15}$$

and I have used the short-hand

$$\mid q_1, q_2\,\rangle = a^\dagger_{q_1}\, a^\dagger_{q_2} \,\mid 0\,\rangle \tag{8.16}$$

for the two-quark states. The "multi-quark states" in (8.15) denote the 4-quark, 6-quark, etc, terms in the decomposition of unity in (8.14). In the two-quark approximation we neglect the multi-quark terms, then

$$\big(H - E_0\big) \,\mid \text{Meson}\,\rangle = \int_{p_1,p_2} \big[\omega(p_1) + \omega(p_2)\big]\, \Psi(p_1, p_2) \,\mid p_1, p_2\,\rangle +$$

$$+\int_{p_1,p_2}\int_{q_1,q_2} V(p_1,p_2|q_1,q_2)\,(2\pi)\,\delta(p_1+p_2-q_1-q_2)\,\Psi(q_1,q_2)\ \mid p_1,p_2\,\rangle\,, \quad (8.17)$$

where

$$V(p_1,p_2|q_1,q_2) = \langle\, p_1,p_2 \mid h\,\sigma(0) \mid q_1,q_2\,\rangle\,, \quad (8.18)$$

and the momentum delta-function in the second term in (8.17) is the result of the integration over x in $\int h\,\sigma(x)\,dx$. The eigenvalue equation

$$(H-E_0)\ \mid \text{Meson}\,\rangle = E\ \mid \text{Meson}\,\rangle \quad (8.19)$$

then reads

$$\big[\omega(p_1)+\omega(p_2)\big]\,\Psi(p_1,p_2)+\int_{q_1,q_2} V(p_1,p_2|q_1,q_2)\,(2\pi)\delta(p_1+p_2-q_1-q_2)\,\Psi(q_1,q_2) = 0\,. \quad (8.20)$$

Since obviously the Hamiltonian conserves the momentum, it is consistent to look for the meson states with definite momentum P. With some abuse of notations, we write

$$\Psi(p_1,p_2) = 2\pi\,\delta(p_1+p_2-P)\,\Psi(P|p)\,, \quad p=\frac{p_1-p_2}{2}\,, \quad (8.21)$$

so that in the two-quark approximation

$$\mid \text{Meson}(P)\,\rangle = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,\Psi(P|p)\ \mid\ P/2+p,\ P/2-p\ \rangle\,. \quad (8.22)$$

The equation (8.20) becomes integral equation for the function $\Psi(p)=\Psi(P|p)$ of single variable p, the difference of the quark moments, with the total momentum P playing the role of a parameter.

The matrix element V, Eq.(8.18), is essentially the matrix element of σ between two two-particle states. We have found such matrix elements when studying the IFT with $h=0$:

$$\langle\, A(\eta_1)A(\eta_2) \mid \sigma(0) \mid A(\theta_1)A(\theta_2)\,\rangle =$$

$$= -\bar\sigma\,\tanh\frac{\eta_1-\eta_2}{2}\,\tanh\frac{\theta_1-\theta_2}{2}\,\prod_{i,j=1}^{2}\coth\frac{\eta_i-\theta_j}{2} \quad (8.23)$$

At this point it is more convenient to work in terms of momenta instead of the rapidities; one can convert using

$$\tanh\frac{\theta_1-\theta_2}{2} = \frac{p_1-p_2}{\omega_1+\omega_2} = \frac{\omega_1-\omega_2}{p_1-p_2}\,, \qquad p_{1,2}=m\,\sinh\theta_{1,2}\,, \quad \omega_{1,2}=m\,\cosh\theta_{1,2}\,. \quad (8.24)$$

As the result, one gets the following integral equation

$$\big[\omega(P/2+p)+\omega(P/2-p)-E_P\big]\,\Psi(P|p)=\lambda\ \mathrm{VP}\int_{-\infty}^{\infty}G_P(p|q)\,\Psi(P|q)\,\frac{dq}{2\pi}\,, \quad (8.25)$$

where again $\lambda = 2\bar{\sigma}h/m^2$. Here VP signifies the principal value of the singular integral, and the kernel $G_P(p|q)$ was obtained from the matrix element (8.23) by changing to the momentum notations (and also adding factors $\sqrt{m}/\sqrt{\omega(p_1)}$ for each leg, because of different normalization of the states we are presently using). The resulting expression is rather cumbersome [1], we just mention its most important properties:

i. The kernel is anti-symmetric in p and q, and symmetric w.r.t. the interchange of these variables,

$$G_P(p|q) = -G_P(p|-q) = -G(-p|q)\,, \qquad G(p|q) = G(p|q)\,. \quad (8.26)$$

ii. As the function of q, the kernel $G_P(p|q)$ is regular at the real axis everywhere except for two second-order poles at $q = \pm p$; the poles have residues ± 1 at these poles, respectively,

$$G_P(p|q) \to \frac{1}{(q-p)^2} \quad \text{as } q \to p\,, \qquad G_P(p|q) \to -\frac{1}{(q-p)^2} \quad \text{as } q \to -p\,. \quad (8.27)$$

Moreover, the "regular part" of the kernel $G_P(p|q)$ is indeed regular, i.e.

$$G_P^{\mathrm{reg}} = G_P(p|q) - \frac{1}{(p-q)^2} + \frac{1}{(p+q)^2} \quad (8.28)$$

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[1] It is

$$G_P(p|q) = \frac{1}{4}\,\frac{1}{\sqrt{\omega(p_1)\omega(p_2)\omega(p_3)\omega(p_4)}}\left[\frac{\big(\omega(p_1)+\omega(q_1)\big)\big(\omega(p_1)+\omega(q_1)\big)}{(p-q)^2} - \right.$$

$$\left. -\frac{\big(\omega(p_1)+\omega(q_2)\big)\big(\omega(p_2)+\omega(q_1)\big)}{(p+q)^2} + \frac{4\,p\,q}{\big(\omega(p_1)+\omega(p_2)\big)\big(\omega(q_1)+\omega(q_2)\big)}\right]$$

where

$$p_1 = P/2+p\,, \quad p_2 = P/2-p\,, \quad q_1 = P/2+q\,, \quad q_2 = P/2-q\,.$$

is analytic in both p and q in some vicinity of the whole real axis.

The equation (8.25) is understood as the eigenvalue problem. Namely, we are interested in special values of E_P such that the corresponding solution $\Psi(P|p)$ is normalizable. The scalar products of the meson states

$$| \, \text{Meson} \, \rangle = \int_{p_1,p_2} \Psi(p_1,p_2) \, a^\dagger_{p_1} a^\dagger_{p_2} \, | \, 0 \, \rangle \tag{8.29}$$

follow from the normalization of the two-quark states; we have

$$\langle \, \text{Meson}_1 \mid \text{meson}_2 \, \rangle = \int_{p_1,p_2} \Psi^*_1(p_1,p_2)\Psi_2(p_1,p_2) \, . \tag{8.30}$$

More precisely, since the mesons have continuous momentum, they must be delta-normalizable in terms of the momentum P. For given P

$$\Psi(p_1,p_2) = 2\pi \, \delta(P - p_1 - p_2) \, \Psi(P|p)$$

we have

$$\langle \, \text{Meson}(P') \mid \text{Meson}(P) \, \rangle = 2\pi \, \delta(P - P') \, \|\Psi\|^2 \, , \tag{8.31}$$

where

$$\|\Psi\|^2 = \frac{1}{2} \int_{-\infty}^{\infty} |\Psi(P|p)|^2 \, \frac{dp}{2\pi} \, . \tag{8.32}$$

The mesons are the states for which

$$\|\Psi\|^2 < \infty \, . \tag{8.33}$$

The states normalizable in this sense correspond to single-particle states with momentum P. From Lorentz invariance, the energies of such states must have the form

$$E_P = \sqrt{M^2 + P^2} \, . \tag{8.34}$$

However, approximations made in deriving the equation (8.25), namely ignoring the multi-quark contributions to the state, in principle break Lorentz invariance. This indeed happens, but proper treatment of the multi-quark contributions allows for restoration of the symmetry. I will discuss this a little later.

The equation (8.25) admits solution symmetric and anti-symmetric with respect to the interchange of two quarks; since $a^\dagger$'s in (8.28) are fermions, we are interested in anti-symmetric solutions

$$\Psi(p_1,p_2) = -\Psi(p_2,p_1) \, , \qquad \Psi(P|p) = -\Psi(P|-p) \, . \tag{8.35}$$

First let us observe how the equation (8.25) reproduces the "naive" picture. Consider the case of small λ and low bound states, for which we expect the quark motion inside the meson to be non-relativistic, $p, q << m$. To simplify the following equation let me take the center-of mass frame, i.e. set $P = 0$. Then for

$$p/m << 1\,, \quad q/m << 1$$

we can make non-relativistic expansion in (8.25)

$$\left[-E+2m+p^2/m+O(p^4)\right]\Psi(p) = \lambda \int_{-\infty}^{\infty} \left[\frac{2}{(p-q)^2} + G^{\rm reg}(p|q)\right] \Psi(q)\,\frac{dq}{2\pi}\,. \quad (8.36)$$

(In writing this equation I have used the anti-symmetry of $\Psi(p)$ to combine $1/(p-q)^2$ and $1/(p+q)^2$ in the kernel.) At small p, q the singular term in the kernel in the r.h.s in (8.36) dominates. In the leading week-coupling approximation we ignore both $G^{\rm reg}$ and the terms $O(p^4)$. Then (8.36) becomes just the momentum-space version of the Schrodinger equation with the linear potential

$$-\lambda\ {\rm VP} \int \frac{2}{p^2}\, e^{ipx}\, \frac{dp}{2\pi} = \lambda\,|x|\,. \quad (8.37)$$

Hence the leading correction to the meson masses is expressed in terms of the zeros of the Airy function, in agreement with our earlier suggestion.

It is somewhat instructive to re-derive this result directly in the momentum space. The equation to solve is

$$[p^2 - p_0^2]\Psi(p) = \lambda \int_{-\infty}^{\infty} \frac{2}{(p-q)^2}\, \Psi(q)\, \frac{dq}{2\pi}\,. \quad (8.38)$$

where p_0 is used to parameterize the eigenvalue (the meson mass)

$$E = 2m + p_0^2$$

and I have set $m = 1$ for simplicity. Let us split the wavefunction $\Psi(p)$ into the sum

$$\Psi(p) = \Psi_+(p) + \Psi_-(p)\,, \quad (8.39)$$

of two pieces, one analytic in the lower half-plane (Ψ_+), and another analytic in the upper half-plane (Ψ_-) of p,

$$\Psi_+(p) = \int_{-\infty}^{\infty} \frac{A(q)}{p-q-i0}\,\frac{dq}{2\pi}\,, \qquad \Psi_-(p) = \int_{-\infty}^{\infty} \frac{A(q)}{p+q+i0}\,\frac{dq}{2\pi}\,. \quad (8.40)$$

The fact that these two integrals involve the same $A(q)$ follows from the anti-symmetry condition $\Psi(-p) = -\Psi(p)$, so that

$$\Psi_+(p) = -\Psi_-(p)\,. \tag{8.41}$$

Also, it will be convenient to write $A(q)$ in the form

$$A(q) = e^{iS(q)}\,. \tag{8.42}$$

Let us plug this form into the r.h.s. of (8.38). For instance

$$\lambda \int_{p'} \frac{2}{(p-p')^2}\,\Psi_+(p') = \lambda \int_{p'} \int_q \frac{2}{(p-p')^2}\,\frac{e^{iS(q)}}{p'-q-i0}\,, \tag{8.43}$$

where the singularity at $p = p'$ is understood as the principal value. Here and below I use notation

$$\int_p = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\,.$$

It is not difficult now to perform the integration over p'. The integrand has two poles - double pole at $p' = p$, and single pole at $p' = q + i0$. By definition of the principle value integral, the p'-integral in (8.43) is the sum of two terms corresponding to two contours in **Fig.1**

$$\frac{1}{2}\quad \text{---------------} \quad + \quad \frac{1}{2}\quad \text{---------------} \tag{8.44}$$

The second integral vanishes since the contour can be removed far into the lower half-plane. The first term yields

$$(8.43) = i\lambda \int_q \frac{e^{iS(q)}}{(p-q)^2}\,, \tag{8.45}$$

which can be further transformed as

$$(8.45) = i\lambda \int_q e^{iS(q)}\,\frac{d}{dq}\,\frac{1}{p-q} = \lambda \int_q \frac{S'(q)}{(p-q)}\,e^{iS(q)}\,. \tag{8.46}$$

Now, the difference between the l.h.s and the r.h.s of (8.38) can be written as

$$\int_q \frac{p^2 - p_0^2 - \lambda\,S'(q)}{p-q}\,e^{iS(q)} - \big(p \to -p\big)\,, \tag{8.47}$$

where the $(p \to -p)$ term comes from the Ψ_- term in the wavefunction. Now it is clear that if $\lambda S'(q)$ coincides with the "free part" $p^2 - p_0^2$ with p replaced by q, i.e.

$$\lambda\, S'(q) = q^2 - p_0^2 \qquad \text{i.e.} \quad S(q) = \frac{1}{\lambda}\left(q^3/3 - p_0^2\, q\right), \tag{8.48}$$

then the integrand in (8.47) is nonsingular function of q, and moreover

$$(8.47) = 2p \int_q e^{iS(q)}\,. \tag{8.49}$$

The spectrum is determined from the condition that the expression (8.49) vanishes. Since

$$\mathrm{Ai}(x) \sim \int_p e^{ip^3/3 - ipx} \tag{8.50}$$

it follows

$$p_0^2 = t^2\, z\,, \qquad t = \lambda^{1/3} \tag{8.51}$$

where $-z$ is any zero of the Airy function, $\mathrm{Ai}(-z) = 0$.

I did demonstrate this long computation because the solution of the full Bethe-Saltpeter equation (8.25) can be studied with similar technique. The general equation has the form

$$\Omega(p)\,\Psi(p) - \lambda \int_q G(p|q)\,\Psi(q) = 0\,, \tag{8.52}$$

where I did not indicate explicitly the total momentum P which plays the role of parameter. In (8.52)

$$\Omega(p) = \omega(P/2 + p) + \omega(P/2 - p) - E(P)\,, \tag{8.53}$$

and $G(p|q) = G_P(p|q)$ is the kernel related to the $2-2$ particle matrix element of σ; I have discussed its properties before, and I'll repeat them now. The equation (8.53) is the eigenvalue problem for the energy $E(P)$.

We would like to develop a technique of solving the equation (8.52) at small λ. When λ is not small the corrections to the two-quark approximation become significant; more about those corrections below. The most important properties of (8.53) we will use are the following:

i. $\Omega(p) = \Omega(-p)$, and at any E the function $\Omega(p)$ has only two real zeros $p = \pm\, p_0$, i.e.

$$\Omega(p) = 0 \quad \Rightarrow p = \pm\, p_0 \quad (\text{real } p)\,. \tag{8.54}$$

ii. $G(p|q)$ has all the symmetries and singularities which I mentioned before, i.e.

$$G(p|q) = G(q|p) = -G(-p|q)\,, \qquad G(p|q) = \frac{1}{(p-q)^2} - \frac{1}{(p+q)^2} + G_{\rm reg}(p|q)\,, \tag{8.55}$$

where $G_{\rm reg}(p|q)$ has no singularities at real p, q.

We look for the solution in the form similar to that of the non-relativistic problem,

$$\Psi(p) = \Psi_+(p) + \Psi_-(p) + \delta\Psi(p)\,, \tag{8.56}$$

where again

$$\Psi_+(p) = \int_q \frac{e^{iS(q)}}{p-q-i0}\,, \qquad \Psi_-(p) = \Psi_+(-p)\,, \tag{8.57}$$

with

$$S(q) = \frac{1}{\lambda}\int^p \Omega(q)\,, \qquad S'(p) = \frac{1}{\lambda}\,\Omega(p)\,. \tag{8.58}$$

The term $\delta\Psi$ is in some sense small; its role will become clear below.

Let me denote $\hat{O}$ the operator defined by the left-hand side of (8.52),

$$\hat{O}\Psi = \Omega\Psi - \lambda\int G\,\Psi\,. \tag{8.59}$$

First I want to find

$$\Delta(p) = \hat{O}\left(\Psi_+ + \Psi_-\right). \tag{8.60}$$

This can be done by writing $\Psi_\pm$ in the form (8.57) and then repeating the transformations we have done in the non-relativistic case. The most important difference is that now the integral over the contour going below all the real singularities,

$$- - - - - - - \diamond - - - - - - - - - - - \bullet - - - - - - - \tag{8.61}$$

can not be ignored, because the part $G_{\rm reg}(p|p')$ can and does have complex singularities in the lower half-plane. I will skip the detailed computations (doing them can be taken as an **Exercise**); the result is

$$\Delta(p) = \int_q \left[2p\,\frac{\Omega(p)-\Omega(q)}{p^2-q^2} - i\lambda\,G_{\rm reg}(p|q) - \lambda\,G_{\rm even}(p|q)\right] e^{iS(q)}\,, \tag{8.62}$$

where

$$G_{\rm even}(p|q) = \int_{C_-} \frac{2k\, G(p|k)}{k^2 - q^2}\, \frac{dk}{2\pi}\,, \tag{8.63}$$

with the contour C_- going just below all real singularities of the integrand, i.e. it is determined by the complex singularities of the kernel $G(p|q)$. Note that the integrand in (8.62) has no real singularities. Note also that $G_{\rm even}(p|q)$ is an even function of q.

To satisfy the original equation (8.58), i.e. $\hat{O}\Psi = 0$, we have to have

$$\hat{O}\delta\Psi(p) = -\Delta(p)\,,$$

or

$$\Omega(p)\delta\Psi(p) = -\Delta(p) + \lambda \int_q G(p|q)\, \delta\Psi(q)\,. \tag{8.64}$$

The equation (8.58) is solved if we find $\delta\Psi(p)$ which is non-singular at all real p and decays at $p \to \infty$ (otherwise the wavefunction $\Psi(p)$ cannot be normalizable). At small λ I can ignore the second term in the r.h.s., and this leads to the condition

$$\Delta(p_0) = 0\,. \qquad \big(\text{recall} \quad \Omega(\pm\, p_0) = 0\big) \tag{8.65}$$

This equation determines the spectrum of $E(P)$ in the Bethe-Saltpeter equation (8.52).

There are two systematic ways to handle the equation (8.65) at small λ. One is evaluation of the integral (8.62) by the stationary-phase method. Let us see how the leading order in the stationary-phase analysis leads to the Bohr-Zommerfeld quantization rule for the meson masses we have obtained by "naive" semiclassical analysis.

The points of the stationary phase in the integral (8.62) are determined by the usual equation

$$S'(q) = 0\,,$$

i.e. by construction they are $q = \pm\, p_0$. In the leading order the terms $\sim \lambda$ with $G_{\rm reg}$ and $G_{\rm even}$ in the pre-exponential factor have to be neglected, and the remaining part is evaluated at the corresponding stationary phase points. Since the first term in (8.62) is an even function of q, this leads to an overall factor,

$$\Delta(p) \approx 2p\, \frac{\Omega(p) - \Omega(p_0)}{p^2 - p_0^2} \int_q e^{iS(q)}\,. \tag{8.66}$$

Hence in this approximation we are interested in the values of parameters which lead to vanishing of the integral in (8.66). Stationary-phase evaluation of this integral includes contributions of the two points $\pm p_0$; assuming that p_0 is positive, this yields [2]

$$\Delta(p) \sim \cos\big(S(p_0) + \frac{\pi}{4}\big)\,. \qquad (p_0 > 0) \tag{8.67}$$

We need to find $S(p_0)$. We have

$$\Omega(p) = \omega(P/2 + p) + \omega(P/2 - p) - E(P)\,. \tag{8.68}$$

It is useful to parameterize $E(P)$ using new parameter p_0: by definition

$$E(P) = \omega(P/2 + p_0) + \omega(P/2 - p_0)\,, \tag{8.69}$$

so that $\Omega(\pm p_0) = 0$. Also useful is to introduce new variables β and θ instead of P and p_0,

$$P/2 + p_0 = m\,\sinh(\beta + \theta)\,, \qquad P/2 - p_0 = m\,\sinh(\beta - \theta)\,, \tag{8.70}$$

so that

$$E(P) = \omega(P/2 + p_0) + \omega(P/2 - p_0) = 2\,\cosh\theta\;\cosh\beta\,. \tag{8.71}$$

From (8.70)

$$P = 2\,\cosh\theta\;\sinh\beta\,, \qquad p_0 = \sinh\theta\,\cosh\beta\,. \tag{8.72}$$

We see that

$$M = 2\,\cosh\theta \tag{8.73}$$

is the meson mass, and β is its rapidity,

$$E = M\,\cosh\beta\,, \qquad P = M\,\sinh\beta\,. \tag{8.74}$$

Now let us evaluate the action

$$S(p_0) = \frac{1}{\lambda}\int_0^{p_0} \big[\omega(P/2 + p) + \omega(P/2 - p) - M\,\cosh\beta\big]\,dp =$$

$$= \frac{1}{4\lambda}\big[2(\beta+\theta) + \sinh 2(\beta+\theta) - 2(\beta-\theta) - \sinh 2(\beta-\theta) - 2M\,p_0\,\cosh\beta\big]\,. \tag{8.75}$$

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[2] derivation

With $p_0 = \sinh\theta\,\cosh\beta$ this is transformed to

$$S(p_0) = \frac{1}{2\lambda}\left(2\theta - \sinh 2\theta\right). \tag{8.76}$$

Note that β has dropped out of this expression. This means that the Lorentz invariance is preserved in this approximation.

With this the condition

$$\Delta(p_0) = 0 \quad \Rightarrow \cos\left(S(p_0) + \pi/4\right) = 0 \tag{8.77}$$

leads to the quantization condition (8.3), i.e

$$\sinh 2\theta - 2\theta = 2\pi\,\lambda\,(n - 1/4)\,, \qquad n > 0\,. \tag{8.78}$$

(The last condition $n > 0$ comes from the fact that we have assumed $p_0 > 0$, and this was used in derivation of (8.67).)

It is interesting to note that when quantization condition is satisfied not only $\Delta(p_0) = 0$ but the whole function $\Delta(p)$ turns to zero altogether.

The leading WKB approximation can be systematically improved by evaluation corrections to the saddle-point approximation. We will come back to this subject in few minutes.

The stationary phase approximation described above is good when λ is small and at the same time the level number n is sufficiently large. At fixed n and $\lambda \to 0$ the stationary-phase points $\pm p_0$ approach each other and their contributions are not completely independent; It is easy to see that

$$p_0/m \simeq \left(\lambda\, n\right)^{1/3}. \tag{8.79}$$

But in this regime one can develop another systematic approximation for solving the Bethe-Saltpeter equation (8.52) which is valid at small λ and fixed n; I will call it the low-energy expansion.

In this approximation we assume that all the quark momenta inside the meson are small as compared to m, and makes systematic expansion in powers of the momenta. For instance, assuming that p is small, we have

$$\Omega(p) = \omega(P/2 + p) + \omega(P/2 - p) - E = -E + E_0(P) + \frac{8\,p^2}{E_0^3(P)} + O(p^4) \tag{8.80}$$

where

$$E_0(P) = \sqrt{4m^2 + P^2}\,. \tag{8.81}$$

Correspondingly, for the "action"

$$S(q) = \frac{1}{\lambda} \int \Omega(q)$$

we have

$$S(q) = S_0(q) + \delta S(q)\,, \tag{8.82}$$

where

$$S_0(q) = -(E - E_0)\,q + \frac{8}{E_0^3}\,\frac{q^3}{3}\,, \qquad \delta S = O(q^5)\,. \tag{8.83}$$

Then in the equation

$$\Delta(p) = \int_q \left\{ \left[2p\,\frac{\Omega(p) - \Omega(q)}{p^2 - q^2} - i\lambda\, G_{\mathrm{reg}}(p|q) - \lambda\, G_{\mathrm{even}}(p|q) \right] e^{i\,\delta S(q)} \right\} e^{iS_0(q)} \tag{8.84}$$

one expands the expression in the curly brackets in double power series in p and q and evaluates the q-integral order by order in this power series. The zeroth order corresponds to the non-relativistic calculation we have done before as the warm-up exercise. In general this expansion leads to the expansion of E in powers of the parameter t^2,

$$E = E_0 + t^2\,E_2 + t^4\,E_4 + \ldots\,, \qquad t^2 = \lambda^{2/3}\,. \tag{8.85}$$

To the first few orders this expansion is consistent with the relativistic dispersion law

$$E = \sqrt{M^2 + P^2}$$

with

$$M = 2m + \mu_2\,t^2 + \mu_4\,t^4 + \ldots\,, \tag{8.86}$$

for instance

$$E_2 = \frac{\mu_2\,m}{\sqrt{4m^2 + P^2}}\,. \tag{8.87}$$

For (8.86) one obtains explicitly

$$M = m\left(2 + z\,t^2 - \frac{z^2}{20}\,t^4 + \left(\frac{11\,z^3}{1400} - \frac{57}{280}\right) t^6 + O(t^8)\right), \tag{8.88}$$

where again $-z$ is any zero of the Airy function.

This expansion can be developed further, but with the last term written down in (8.88) it already exceeds the accuracy of the two-quark approximation. It is important at this point to understand the role of the multi-quark components of the mesons.

Beyond the two-quark approximation the meson state contains contributions from multi-quark sectors

$$\mid Meson(P)\,\rangle = \int_{p_1,p_2} \Psi(p_1,p_2)\ \mid p_1,p_2\,\rangle + \int_{p_1,\ldots,p_4} \Psi_4(p_1,...,p_4)\ \mid p_1,...,p_4\,\rangle + \ldots . \tag{8.89}$$

We are interested in the meson state with definite momentum P, hence all the amplitudes Ψ_{2n} contain the momentum delta-function. As usual, if the meson is well below the 2-meson threshold, i.e. $M - 2m << 2m$, the multi-quark components can be taken into account perturbatively. This leads to the corrections $O(\lambda^2)$ to the right hand side of the equation (8.52),

$$\Omega(p)\,\Psi(p) - \lambda \int_q G(p|q)\,\Psi(q) - \lambda^2 \int_q G_2(p|q)\,\Psi(q) - O(\lambda^3) = 0\,, \tag{8.90}$$

Here the correction G_2 to the kernel G has the form $G_2(p|q) =$

$$= \sum_{n=2}^{\infty} \int_{k_1,\ldots,k_{2n}} \frac{\langle\, p_1,p_2 \mid \sigma(0) \mid k_1,...,k_{2n}\,\rangle\langle\, k_1,...,k_{2n} \mid \sigma(0) \mid q_1,q_2\,\rangle}{\omega(k_1)+\cdots+\omega(k_{2n}) - E}\,\delta(P - \sum k_i)\,, \tag{8.91}$$

where again $p_1 = P/2 + p, p_2 = P/2 - p, q_1 = P/2 + q, q_2 = P/2 - q$.

It is obvious from this definition that the correction term $G_2(p|q)$ has the same general symmetry properties as the main kernel $G(p|q)$,

$$G_2(p|q) = G_2(q|p) = -G_2(-p|q) = -G_2(p|-q)\,. \tag{8.92}$$

If this term was regular function of p and q, its effect would be very small. Indeed, from the symmetries it must expand as

$$G_2^{\rm reg}(p|q) \sim p\,q + \text{higher terms}\,. \tag{8.93}$$

Since in the low energy expansion we have $p, q \sim t$, the contribution to the meson mass from this term would be

$$\lambda^2\,t^2\,t = t^9\,. \tag{8.94}$$

(Note that it is odd power of t), and it has no effect on the terms written down in (8.88).

But the above estimate ignores the fact that the matrix elements involved in the expression (8.91) have singularities (poles) every time one of the intermediate momentum k approaches any of the external momenta p or q. These are principal-value poles, and integration over k_i in general produces terms proportional to the delta-functions supported by the configurations with coinciding initial and final momenta, like

$$\delta(p-q) \quad \text{or} \quad \delta(p+q)\,. \tag{8.95}$$

By the same power counting, such terms would bring contributions

$$\text{delta}-\text{terms}: \quad \sim \lambda^2\, t/t = t^6\,, \tag{8.96}$$

where t in the denominator is due to the delta-function. Such terms could interfere with the last term in (8.88).

Natural mechanism of generating such delta-function terms is from the "disconnected configurations". Suppose only one of the two particles experiences direct interaction with the spin operator insertions in (8.91), while another does not come close to the interaction domain **Fig.2**

This configurations would affect the "self-energies" of the individual quarks, in particular give rise to the quark mass renormalization. As the result, the effective quark mass is not exactly m entering the formal action (8.1) but differs from it by perturbative corrections,

$$m_q = m + a_q\,\frac{h^2}{m^{\frac{11}{4}}} + O(h^4)\,, \tag{8.97}$$

where a_q is dimensionless number. This and other corrections will be considered next time.