L7

As we have seen the last time, it is useful to consider the matrix elements involving the one-particle states,

$$F(x,\theta) = \langle\, 0 \mid \sigma(x/2)\mu(-x/2) \mid A(\theta)\,\rangle\,. \tag{7.1}$$

From this, and similar matrix element $\tilde{F}(x,\theta)$, one can construct two functions $\Psi_\pm(x,\theta)$ which satisfy the equations

$$\partial \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\partial\varphi & \frac{m\, e^{\theta}}{4}\, e^{\ \varphi} \\ \frac{1}{2}\partial\varphi & -\,\frac{m\, e^{\theta}}{4}\, e^{-\varphi} \end{pmatrix} \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}, \tag{7.2a}$$

$$\bar\partial \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\bar\partial\varphi & -\,\frac{m\, e^{-\theta}}{4}\, e^{-\varphi} \\ -\frac{1}{2}\bar\partial\varphi & \frac{m\, e^{-\theta}}{4}\, e^{\ \varphi} \end{pmatrix} \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}, \tag{7.2b}$$

These equations are exactly the linear problem associated with the integrable sinh-Gordon equation, i.e. the equation

$$\partial\bar\partial\varphi = \frac{m^2}{8}\,\sinh 2\varphi\,. \tag{7.3}$$

is is the condition of compatibility of (7.2a) and (7.2b).

The matrix elements containing more then one particle can be obtained from the Ward identities in a similar manner. They all are expressed as appropriately (anti-) symmetrized products of the functions $\Psi_\pm(x,\theta)$. Here I only quote the result for the matrix elements

$$\langle\, A(\theta_1) \mid \sigma(x/2)\sigma(-x/2) \mid A(\theta_2)\,\rangle = (2\pi)\,\delta(\theta_1-\theta_2) + G(\theta_1|x|\theta_2) \tag{7.4}$$

and

$$\langle\, A(\theta_1) \mid \mu(x/2)\mu(-x/2) \mid A(\theta_2)\,\rangle = (2\pi)\,\delta(\theta_1-\theta_2) + \tilde{G}(\theta_1|x|\theta_2) \tag{7.5}$$

where the last terms are the connected matrix elements. For them we have

$$\frac{G_+(\theta_1|\theta_2)}{G_+} = \frac{e^{\theta_1}\,\Psi_-(\theta_1)\Psi_+(\theta_2) - e^{\theta_2}\,\Psi_+(\theta_1)\Psi_-(\theta_2)}{e^{\theta_1}-e^{\theta_2}}\,, \tag{7.6a}$$

$$\frac{G_-(\theta_1|\theta_2)}{G_-} = -\frac{e^{\theta_1}\,\Psi_+(\theta_1)\Psi_-(\theta_2) - e^{\theta_2}\,\Psi_-(\theta_1)\Psi_+(\theta_2)}{e^{\theta_1} - e^{\theta_2}}\,, \tag{7.6b}$$

where

$$G_\pm(\theta_1|x|\theta_2) = G(\theta_1|x|\theta_2) \pm \tilde{G}(\theta_1|x|\theta_2)\,, \qquad G_\pm(x) = G(x) \pm \tilde{G}(x)\,, \tag{7.7}$$

and in writing (7.6) I have suppressed the argument x.

Let us come back to the two-point functions

$$G(x) = \langle\, \sigma(x)\sigma(0)\,\rangle \quad \text{and} \quad \tilde{G}(x) = \langle\, \mu(x)\mu(0)\,\rangle\,. \tag{7.8}$$

As we have learned already, they are expressed in terms of $\chi(x)$ and $\varphi(x)$

$$G \simeq e^{\chi/2}\,\cosh\varphi/2\,, \qquad \tilde{G} \simeq e^{\chi/2}\,\sinh\varphi/2\,, \tag{7.9}$$

which satisfy the sinh-Gordon system, in particular φ satisfies the sinh-Gordon equation (7.3)

What kind of solution we are interested in? First, since the correlation functions are rotationally invariant, i.e. they depend on r only, we are interested in rotationally invariant $\varphi = \varphi(r)$; the Eq.(7.3) then becomes an ordinary differential equation

$$\varphi_{rr} + \frac{1}{r}\,\varphi_r = \frac{m^2}{2}\,\sinh\big(2\varphi\big)\,. \tag{7.10}$$

Next, we do not want singularities at real positive r, except at $r = 0$ and $r = \infty$. And we need this function to decay at large positive r,

$$\varphi(r) \to 0 \quad \text{as} \quad r \to +\infty\,. \tag{7.11}$$

This last property would guarantee that

$$G(r) = e^{\chi(r)/2}\,\cosh(\varphi(r)/2) \to \text{Const} = e^{\chi(\infty)/2}\,,$$

and

$$\tilde{G}(r) = e^{\chi(r)/2}\,\sinh(\varphi(r)/2) \to 0 \quad \text{as} \quad r \to \infty\,. \tag{7.12}$$

The condition (7.11), i.e. that $\varphi(r)$ decays at the infinity, does not fix the solution uniquely. In addition, one has to specify, say, the character of the singularity at $r \to 0$. Consistent assumption is that φ diverges at 0, i.e. $\varphi(r) \to \infty$ at $r \to 0$.

When φ is large, one can neglect one of the exponentials in the $\sinh 2\varphi$ in the r.h.s. of (7.10); then it becomes the radial Liouville equation

$$\partial\bar{\partial}\varphi = \frac{m^2}{16}\, e^{2\varphi}\,, \quad \text{or} \quad \varphi_{rr} + \frac{1}{r}\,\varphi_r = \frac{m^2}{4}\, e^{2\varphi}\,, \quad (mr << 1)\,. \tag{7.13}$$

Suitable class of singular solutions of the Liouville equation is well-known. The function $\varphi(r)$ has logarithmic behavior

$$\varphi(r) = -a\,\log r + b + \frac{e^{2b}}{16\,(a-1)^2}\, r^{2-2a} + \cdots\,, \quad 0 < a \leq 1\,, \tag{7.14}$$

where a and b are parameters, and a is taken between 0 and 1.

Now, let us take this such singular solution and find corresponding $r \to 0$ behavior of the functions $\Psi_\pm$. In this analysis only the leading logarithmic term in (7.14) is important,

$$\varphi \to -\frac{a}{2}\,\log z\bar{z}\,, \qquad \partial\varphi \to -\frac{a}{2z}\,, \quad \bar{\partial}\varphi \to -\frac{a}{2\bar{z}}\,, \quad (r \to 0)\,. \tag{7.15}$$

In the linear system (7.2) the terms with the derivatives of φ dominate at $r \to 0$,

$$\partial\Psi_+ \approx \frac{a}{4\,z}\,\Psi_+\,, \qquad \bar{\partial}\Psi_+ \approx -\frac{a}{4\,\bar{z}}\,\Psi_+\,, \tag{7.16}$$

and similarly, with $a \to -a$, for Ψ_-. One finds that at small r

$$\Psi_+(r|\beta) \sim \left(\frac{z}{\bar{z}}\right)^{\frac{a}{4}} \sim e^{\frac{ia\,\beta}{2}}\,, \qquad \Psi_-(r|\beta) \sim \left(\frac{\bar{z}}{z}\right)^{\frac{a}{4}} \sim e^{-\frac{ia\,\beta}{2}}\,. \tag{7.17}$$

In writing these equations I have used the polar coordinates $z = r\,e^{i\alpha}$, and also the fact that by rotational/Lorentz symmetry the functions $\Psi_\pm$ depend only on the combination

$$\beta = \alpha - i\theta\,. \tag{7.18}$$

As follows from their definition in terms of the matrix elements (7.1) and the local properties of the spin fields σ and μ, the functions $\Psi_\pm(r|\beta)$ have certain periodicity property,

$$\Psi_+(\beta + i\pi) = i\,\Psi_+(\beta)\,, \qquad \Psi_-(\beta + i\pi) = -i\,\Psi_-(\beta)\,. \tag{7.19}$$

It is easy to see that it is consistent with the asymptotic form (7.17) only if $a = 1$. So we have to choose this value of a.

It is evident from the form of the expansion (7.14) that $a = 1$ is a special case (in the Liouville theory it is known as "parabolic singularity"). In this case the power-like terms get dressed by logarithms. Simple analysis shows that in this case the $r \to 0$ expansion of φ has the form

$$\varphi = -\log\big(m\,r/2\big) - \log\big(-\log(mr) - c\big) + O(r^4)\,, \tag{7.20}$$

where c is adjustable parameter. But once c is fixed, the solution is specified uniquely. At generic c the solution develops singularity at finite positive r, but there is a unique value of this parameter such that the corresponding solution is regular and decays at large r. Obviously, we are interested in this very solution. Finding this special value of c is rather involved mathematical problem, which can be solved with finite but substantial amount of computations. At this point I will skip this computation (may be we will have time to do it later in somewhat more general context). I just quote the result: the desired value of c is $\log\big(e^{\gamma_E}/8\big)$, where $\gamma_E = 0.577...$ is the Euler's constant. Thus the right solution has the following $r \to 0$ behavior

$$\varphi(r) = -\log\big(m\,r/2\big) - \log\big(-\Omega\big) + O(r^4)\,. \tag{7.21}$$

Here and in the future the notation

$$\Omega = \log\left(\frac{e^{\gamma_E}}{8}\,m\,r\right) \tag{7.22}$$

is used [1] .

On the other hand, the solution $\varphi(r)$ decays at $r \to \infty$, and at sufficiently large r it must approach the decaying solution of the linearized equation

$$\frac{1}{r}\,\partial_r\big(r\,\partial_r\varphi\big) = m^2\,\varphi \tag{7.23}$$

[1] As was stated, ones c is fixed, the solution is uniquely specified. Thus, one can iterate the equation (7.10) at small r starting with the leading terms (7.21). This yields

$$\varphi(r) = -\log\big(mr/2\big) - \log\big(-\Omega\big) + \frac{(2\Omega-1)(4\Omega^2-2\Omega+1)}{2^{11}\,\Omega}\,(mr)^4 + ...$$

.

which is the MacDonald function $K_0(mr)$, up to a constant factor. We will see that

$$\varphi(r) \to \frac{2}{\pi}\, K_0(mr) \qquad \text{as} \quad r \to \infty\,. \tag{7.24}$$

We also need the function $\chi = \chi(r)$. It satisfies the equations

$$\partial^2\chi + \big(\partial\varphi\big)^2 = 0\,, \qquad \bar\partial^2\chi + \big(\bar\partial\varphi\big)^2 = 0\,, \tag{7.25}$$

and also

$$\partial\bar\partial\chi = -\frac{m^2}{8}\,\big(\cosh(2\varphi) - 1\big)\,. \tag{7.26}$$

Again, since we consider the rotationally-invariant solutions, these equations are ordinary differential equations. One can exclude the second derivative,

$$\frac{2}{r}\,\chi_r = \big(\varphi_r\big)^2 + \frac{m^2}{2}\,\big(1 - \cosh 2\varphi\big)\,, \tag{7.27}$$

which allows to find $\chi(r)$ up to a constant. According to (7.8), more precisely

$$G(r) = m^{1/4}\, e^{\chi(r)/2}\, \cosh\varphi(r)/2\,, \qquad \tilde G(r) = m^{1/4}\, e^{\chi(r)/2}\, \sinh\varphi(r)/2\,, \tag{7.28}$$

the constant affects only the overall normalization of the two-point correlation functions. Our convention about normalization is such that

$$G(r) \to r^{-1/4}\,, \qquad \tilde G(r) \to r^{-1/4} \quad r \to 0 \tag{7.29}$$

with the coefficients 1. This fixes the constant term in $\chi(r)$, so that

$$\chi(r) = \frac{1}{2}\,\log\big(4m\,r\big) + \log\big(-\Omega\big) + \frac{(mr)^2}{8} + \dots\,. \tag{7.30}$$

It is possible to show that with this normalization the solution of (7.27) at large r approaches a constant which is exactly $4\log\bar s$,

$$\chi(r) \to 4\,\log\bar s \quad as \quad r \to \infty\,, \tag{7.31}$$

where $\bar s = 1.357838342$ is the spontaneous magnetization we have determined earlier. With this, one finds that at large r

$$\tilde G(r) = \frac{\bar\sigma^2}{\pi}\, K_0(mr) + O\big(e^{-3mr}\big)\,, \tag{7.32}$$

and

$$G(r) \simeq \bar{\sigma}^2 + O\big(e^{-2mr}\big)\,. \tag{7.33}$$

Of course these functions also can be written in terms of the spectral sums, for example

$$G(r) = \sum_{n=2k+1=1,3,...} \frac{1}{n!}\int |F^\sigma(\theta_1,\cdots,\theta_n)|^2\, e^{-mr\,(\cosh\theta_1+\cdots+\cosh\theta_n)}\,\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_n}{2\pi}\,; \tag{7.34}$$

similar representation with $n = 2k = 0, 2, ...$ is valid for $\tilde{G}(r)$. These are obtained by putting a complete sets of the particle states between the spin operators in

$$G(r) = \langle\, 0 \mid \sigma(x)\,\sigma(0) \mid 0\,\rangle\,, \qquad \tilde{G}(r) = \langle\, 0 \mid \mu(x)\,\mu(0) \mid 0\,\rangle\,. \tag{7.35}$$

So far we have studied the IFT with in zero external field $H = 0$. This was the (generally massive) free fermion theory. In another language it could have been described as the $c = 1/2$ CFT with the mass term added as the "perturbation",

$$\mathcal{A}_{\rm FF} = \mathcal{A}_{c=1/2\ \rm CFT} + \frac{m}{2\pi}\int \varepsilon(x)\, d^2x\,, \tag{7.36}$$

where

$$\varepsilon(x) = i\bar{\psi}\psi(x) \tag{7.37}$$

is the "energy density" field coupled to $m \sim K - K_c$. To study the general case of the Ising model with $H \neq 0$ in the scaling domain $K_c - K \to 0$ we have to add a term with the spin field,

$$\mathcal{A}_{\rm IFT} = \mathcal{A}_{c=1/2\ \rm CFT} + \frac{m}{2\pi}\int \varepsilon(x)\, d^2x + h\int \sigma(x)\, d^2x\,, \tag{7.38}$$

where $h \sim H$ with the coefficient which depend on details of the microscopic model, and anyway proportional to certain power of the microscopic scale (say the lattice spacing a),

$$H = A_H\, a^{15/8}\, h\,; \tag{7.39}$$

The value of the exponent here comes from the dimensional analysis [2], and A_H is dimensionless constant which depend on the microscopic model [3].

[2] Since $\sigma \sim M^{1/8}$ we have $h \sim M^{15/8}$, where M is the mass. Hence (7.39).

[3] For instance for the square-lattice Ising model $A_H = 1.192353447$.

Precise meaning of the "perturbed CFT" like (7.36) or (7.38), i.e. how to understand these formal actions, will be discussed later. At this point I would only like to discuss expected qualitative properties of the IFT (7.38) as the particle theory. Note that the IFT (7.38) has two dimensional parameters, m and h. Since the mass dimensions of ε and σ are 1 and 1/8, respectively, we have

$$m \sim [\text{mass}]\,, \qquad h \sim [\text{mass}]^{15/8}\,. \tag{7.40}$$

Therefore the theory essentially depends on a single dimensionless parameter, the so-called scaling parameter. It can be defined many ways but I will usually use two forms

$$\xi = \frac{h}{|m|^{15/8}} \quad \text{or} \quad \eta = \frac{m}{|h|^{8/15}}\,; \tag{7.41}$$

Of course ξ and η are dependent parameters, for instance if m and h are both positive

$$\eta = \xi^{-8/15} \qquad (m > 0\,,\ h > 0)\,. \tag{7.42}$$

The theories which have the same scaling parameter may differ only by the overall re-scaling of the masses.

The basic qualitative picture of the mass spectrum of the IFT (7.38) was understood long ago by Wu and McCoy, and I'll refer to it as the *Wu-McCoy scenario*.

In this discussion it is useful to think in terms of the parameter η. When $h \to 0$ this parameter goes to plus or minus infiniy

$$h \to 0\ : \qquad \eta \to \pm\infty$$

depending on the sign of m. When η changes from $-\infty$ to $+\infty$ we expect to have some kind of interpolation between the high-T $h = 0$ theory and the low-T $h = 0$ theory, both are the theories of free particles of mass m.

Let us start with the limit $\eta = +\infty$, i.e. $h = 0$ and $m > 0$, the low-T theory. In this case we have two exactly degenerate ground states $\mid 0_\pm \rangle$ corresponding to the opposite values of the spontaneous magnetization

$$\langle\, 0_\pm \mid \sigma(x) \mid 0_\pm \,\rangle = \pm\,\bar\sigma\,, \tag{7.43}$$

while the only particles in existence are the kinks separating domains of these two vacua; these are the free fermions of the theory $\mathcal{A}_{FF}$ having the mass m. If we add a small but nonzero h exact degeneracy of the two vacua is lifted; the energy densities associated with the two phases now differ by the amount $\sim 2h\,\bar\sigma$,

$$\Delta F = F_+ - F_- \simeq 2\bar\sigma\, h\,, \qquad (7.44)$$

so that

$$\text{for} \quad h > 0 \quad |\, 0_- \,\rangle \ \rightarrow \ \text{"true vacuum"} \quad |\, 0_+ \,\rangle \ \rightarrow \ \text{"false vacuum"} \qquad (7.45)$$

(and it is the other way round for $h < 0$), since $|\, 0_- \,\rangle$ has lower energy. Here by energy density F I understand the coefficient in

$$\log Z \rightarrow -F\,V \qquad \text{as} \quad V \rightarrow \infty\,, \qquad V = 2 - \text{volume} \qquad (7.46)$$

where V is the volume of the Euclidean space-time. It is interpreted as the specific free energy if we interpret the IFT as classical statistical mechanics in 2D, hence the notation. But if interpretation is in terms of the 1+1 D QFT, it is indeed the energy density

$$E_0 \rightarrow F\,R \qquad \text{as} \quad R \rightarrow \infty\,, \qquad R = \text{spatial size} \qquad (7.47)$$

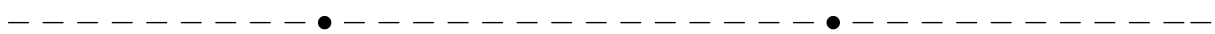
where R is the *spatial size* of the system.

Anyway, at nonzero h (I will assume $h > 0$) the true vacuum is $|\, 0_- \,\rangle \rightarrow |\, 0\,\rangle$, while $|\, 0_+ \,\rangle$ becomes unstable-it gives rise to a global resonance state usually called the "false vacuum". We will come back to this state soon.

Ones the degeneracy of the two vacua is lifted the "kinks" can no longer exist as separated particles. Indeed, if we have two kinks in the vacuum $(-)$, which are separated by large distance $|x|$, the energy of the state has large positive term

$$\text{Energy} \simeq \Delta F\,|x|\;, \qquad \Delta F \simeq 2\bar\sigma\, h\,, \qquad (7.48)$$

because the spatial domain between the kinks is filled with the "false" vacuum $(+)$, **Fig.1**.



This is typical picture of quark confinement in 2D: the particles are attracted to each other by a linear potential, and single particle cannot exist as the asymptotic state, i.e. as a single particle infinitely separated from the rest of the universe. In our case localized states of finite energy involve even number of kinks, so that we always have "true' vacuum at both spatial infinities. In particular, the attractive linear potential (7.48) gives rise to a tower of two-kink bound states, which are particles in usual sense, i.e. they can escape to infinity. I will refer to these bound states as the "mesons":

$$-\,-\,-\,-\,-\,-\bullet-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\bullet-\,-\,-\,-\,-\,- \qquad \rightarrow \quad \text{tower of "mesons"} \tag{7.49}$$

In the following discussion I often refer to the kinks (which rise from the original free fermions at $h = 0$) as the "quarks". Thus, at large positive η we expect to have rather dense set of mesons which are the two-kink bound states.

Wu and McCoy have suggested to visualize this phenomenon as follows. Consider the two-point correlation function of the spin operators in the momentum space

$$\mathcal{G}(k^2) = \int G(x)\, e^{-ikx}\, d^2x\,, \qquad G(x) = \langle\, \sigma(x)\sigma(0)\,\rangle\,, \tag{7.50}$$

where k is the (Euclidean) 2-momentum. As is well known, this function is analytic in the whole complex k^2 plane with the exception of the negative part of the real axis. The negative k^2 correspond to the real time-like Minkowski space-time momenta, where possible eigenvalues of the energy-momentm of the excited states reside. Possible intermediate states in the spectral decomposition of the correlation function show up as the singularities at negative real k^2: stable particles lead to poles while multi-particle states give rise to branch cuts.

Consider the analytic structure of $G(k^2)$ in the plane of

$$-k^2 = \left(p^0\right)^2 - \left(p^1\right)^2\,.$$

At $\eta = +\infty$ we have free quarks, and as we know only even number of the quarks can be created by the spin operator. Hence we have branch cuts from $4\,m^2$, $16\,m^2$, ... associated with two-particle, four-particle, etc. states

$$-\,-\,-\,-\,-\,-\,-\,x\,-\,-\,-\,-\,-\,-\,-\,-\,-\,*{*} \qquad \eta = +\infty \tag{7.51a}$$

Now, when h is nonzero but small, i.e. η is large but finite, the 2-quark branch cut breaks into many poles corresponding to the mesons, with some masses

$$M_i = M_1,\ M_2,\ M_3,\ \ldots < 2\,M_1\,. \tag{7.52}$$

The mesons whose masses are greater then two masses of the lightest meson are in general expected to be unstable against decays into pairs of the lightest mesons, and do not appear as poles (they may give rise to resonance poles away from the principal sheet of the complex $-k^2$ plane). The 4-quark cut in (7.51a) transforms into a sequence of cuts associated with the 2-meson, 3-meson, etc states:

$$-------x----------\bullet-\bullet-\bullet-\bullet-\bullet-\bullet-\bullet-\times\times\times\times\times\times\times\times\times \qquad \eta >> +1 \tag{7.51b}$$

As η decreases, the strength of the quark attraction (the "string tension") increases, and the spectrum of the mesons becomes less dense; one by one the heavier mesons disappear in the continuum, most likely becoming resonance states.

$$-------x----------\bullet---\bullet---\bullet---\bullet-\times\times\times\times\times\times\times\times\times\times \qquad \eta \sim 1 \tag{7.51c}$$

This process continues when η becomes negative. Eventually, at some sufficiently large negative η all but one meson states disappear

$$-------x------\bullet-----------\times\times\times\times\times\times\times\times\times\times\times \qquad -\eta >> 1 \tag{7.51d}$$

Finally, at $\eta = -\infty$ we have $h = 0$ again, but now we are at the high-T domain. The Z_2 symmetry $\sigma \to -\sigma$ is restored, and all branch cuts with even number of particles disappear,

$$-------x----\bullet----------****************\times\times\times\times\times \qquad \eta = -\infty \tag{7.51e}$$

Note that the last picture is exactly the analytic structure of the correlation function $\langle\, \mu(x)\mu(0)\,\rangle$ in the momentum space, at $m > 0$

$$\tilde{\mathcal{G}}(k^2) = \int \tilde{G}(x)\, e^{-ikx}\, d^2x\,. \tag{7.53}$$

Indeed, we have seen that at $m > 0$ (and $h = 0!$) this correlation function involves only odd-particle intermediate states. In fact, it is simple consequence of duality that $G(x)$ at $\eta = -\infty$ (i.e. at $m < 0$) coincides with $\tilde{G}(x)$ at $\eta = +\infty$.

Thus, the lightest meson M_1 at $\eta = -\infty$ becomes the fundamental particle of the free fermion theory which is what we have at $h = 0$ and $m < 0$ [4].

This is beautiful scenario, but it would be nice to throw in a bit of quantitative information. The difficulty is that at $|\eta| < \infty$ the theory is no longer free and it is not clear how to solve it exactly. Also, except for an important special special case $\eta = 0$, it is not integrable neither. In general we have to rely on approximate methods.

The most obvious idea is to consider the cases of small h,

$$|h| << |m|^{8/15},$$

and try to develop some kind of expansion in the small parameter $\xi = h/|m|^{8/15}$. In the remaining part of this lecture, and perhaps in the next one, I am going to discuss such expansions.

Let me start with very simple analysis of the low-T domain with small h. When h is small, the confining potential between two quarks

$$V(x) = 2\bar{\sigma}\, h\; |x_1 - x_2| \tag{7.54}$$

leads to small constant force acting on each quark. Although the quarks are confined, most of the time they move as almost free particles, subject to this constant force. In this situation the semi-classical approximation is expected to be meaningful.

For the first run, let me try to develop "naive" semi-classical approximation, just based on the above picture. I have two quarks, with the coordinates x_1 and x_2, and momenta p_1 and p_2. The energy is

$$E = \omega(p_1) + \omega(p_2) + \mu^2\, |x_1 - x_2|\,, \tag{7.55}$$

where

$$\omega(p) = \sqrt{m^2 + p^2}\,, \qquad \mu = 2\bar{\sigma}\, h\,, \tag{7.56}$$

[4] Fermion=boson

and m is the quark mass. There is a potential, but nevertheless the problem is Lorentz invariant. Indeed, the potential term in (7.55) transforms as the time component of the 2-vector

$$W_\mu = \epsilon_{\mu\nu}\,(x_1 - x_2)^\mu\,. \tag{7.57}$$

It is straightforward to write down and solve the classical equations of motion. When the quarks are separated by finite distance, each moves with constant acceleration directed towards another. This means the trajectory of each quark between the collisions is a hyperbola

$$(x - X)^2 - (t - T)^2 = R^2\,, \tag{7.58}$$

where X and T are parameters showing the space-time position of the focus, and

$$R = \frac{m}{\mu^2} = \frac{m}{2\bar{\sigma}\,h} \tag{7.59}$$

is large when h is small. The momentum p depends on time linearly,

$$p = \pm\,\mu^2\,(t - T)\,, \qquad \left\{ \begin{matrix} + & \text{if} & x - X > 0 \\ - & \text{if} & x - X < 0 \end{matrix} \right\}. \tag{7.60}$$

When quark trajectories cross each other, their positions are interchanged and the direction of the acceleration is reversed. Classical trajectory of two quarks is therefore a combination of the segments of such parabolas; Typical trajectory is depicted in **Fig.2**

To simplify equations, I will put myself in the center-of-mass frame, so that

$$COM: \quad x_1 = -x_2 = x\,, \qquad p_1 = -p_2 = p\,, \tag{7.61}$$

and

$$E = 2\,\omega(p) + 2\mu\,|x|\,. \tag{7.62}$$

At the segment between two collisions where $x > 0$ one can parameterize the trajectory in terms of τ as

$$x = X - R\,\cosh\tau\,, \qquad t - T = R\,\sinh\tau\,, \tag{7.63}$$

and

$$p = -\mu^2\,R\,\sinh\tau\,, \tag{7.64}$$

where $X \geq R$ is the position of the focus, and T is an appropriate shift of the time. It is convenient to introduce θ according to

$$X = R\,\cosh\theta \geq R\,, \tag{7.65}$$

so that

$$x = R\left(\cosh\theta - \cosh\tau\right)\,, \qquad p = -m\,\sinh\tau\,. \tag{7.66}$$

These equations describe the motion between two collisions which occur at $\tau = \pm\,\theta$, so these equations apply at

$$-\theta < \tau < \theta\,. \tag{7.67}$$

The energy can be found from (7.62), but easier is to consider the collision point $\tau = \theta$ where the potential energy vanishes and the two quarks have momenta $p_{1,2} = \pm\,m\,\sinh\theta$; either way

$$E = 2m\,\cosh\theta\,. \tag{7.68}$$

Note also that for given solution the maximal separation between the quarks is

$$l = 2R\left(\cosh\theta - 1\right)\,. \tag{7.68$'$}$$

The WKB quantization requires evaluation of the reduced action over the period. Assuming first that the quarks are distinguishable we find the period to be twice the segment described by the (7.66),(7.67). Therefore [5]

$$S/2 = 2\int_{\rm period} p\,dx = 2\,\frac{m^2}{\mu^2}\int_{-\theta}^{\theta}\sinh^2\tau\,d\tau = \frac{1}{\lambda}\left(\sinh 2\theta - 2\theta\right)\,, \tag{7.69}$$

[5] The factor of 2 is because we have two quarks. Said differently, one can separate the motion into the COM motion and the relative motion described by the momentum $p = (p_1 - p_2)/2$ and conjugated coordinate $x = x_1 - x_2$.

where

$$\lambda = \frac{\mu^2}{m^2} = \frac{2\bar{\sigma}\, h}{m^2} = 2\, \bar{s}\, \xi\,, \tag{7.70}$$

where $\bar{s} = 1.357838342...$, and $\xi = h/m^{8/15}$.

The Bohr-Zommerfield quantization rule states

$$S = 2\pi\,(N + 1/2)\,, \quad N = 0, 1, 2, ... \tag{7.71}$$

Recall now that the quarks are identical fermions. Therefore only odd levels with $N = 2n - 1\,,\; n = 1, 2, ...$ correspond to correct anti-symmetric wave functions. We have then $S/2 = 2\pi\,(n - 1/4)$ and hence the quantization rule for the parameter θ

$$\sinh 2\theta_n - 2\theta_n = 2\pi\,\lambda\,\big(n - 1/4\big)\,, \quad n = 1, 2, 3, ... \,. \tag{7.72}$$

The corresponding energies coincide with the meson masses

$$M_n = 2m\,\cosh\theta_n\,. \tag{7.73}$$

Here

$$\lambda = 2\bar{s}\,\xi = \frac{2\bar{s}}{\eta^{15/8}}\,. \tag{7.74}$$

is the small parameter playing the role of the Planck's constant. From the maximal separation we find the "size" of the meson M_n

$$l_n = \frac{2\,\lambda}{m}\,\big(\cosh\theta_n - 1\big)\,.eqno(7.74')$$

The solutions correspond to stable particles as long as $M_n < 2\,M_1$.

The semi-classical spectrum derived above conforms qualitatively with the Wu-McCoy scenario. To get qualitative understanding of the semiclassical spectrum one can plot the function

$$\bar{S}(\theta) = \sinh 2\theta - 2\theta \tag{7.75}$$

appearing in the l.h.s of (7.72). The second term cancels the linear part of the first term, so at small θ it has cubic behavior,

$$\bar{S}(\theta) \simeq \frac{4}{3}\,\theta^3 \qquad \text{for} \quad \theta << 1\,. \tag{7.76}$$

But it grows as $\exp(2\theta)/2$ at large θ, see **Fig.3**.

The solutions for θ_n are related to the intersections with the horizontal lines $2\pi\lambda\,(n-1/4)$. In fact, it turns out that this semiclassical spectrum gives very good approximation at all positive η not too close to 0 ($\eta > 1$), as I will discuss later.

It is interesting to note that the masses M_n computed through the equations (7.72) and (7.73) show some sort of non-perturbative behavior: at fixed n and small λ they do not have expansion in integer powers of λ; instead they expands in some fractional power of this parameter, i.e. in fractional powers of h. Indeed, at small λ (and fixed n) the r.h.s of (7.72) becomes small. Then one can use the cubic approximation (7.76) for θ_n,

$$\frac{4}{3}\,\theta_n^3 = 2\pi\,\lambda\,(n-1/4)\,, \qquad \theta_n = \left[\frac{3\pi}{2}\,\lambda\,(n-1/4)\right]^{1/3}. \tag{7.77}$$

Hence

$$M_n = 2m\,\cosh\theta_n \simeq 2m\,\big(1+\theta_n^2/2\big) \simeq 2m + m\,\lambda^{2/3}\left[\frac{3\pi}{2}\,(n-1/4)\right]^{2/3}. \tag{7.78}$$

It is clear from this analysis that M_n expand in powers of $\lambda^{2/3}$.

It is also useful to note that the expression in the square brackets in (7.78) is known approximation of the zeros of the Airy function,

$$\mathrm{Ai}\big(-z_n\big) = 0\,, \qquad z_n \approx \left[\frac{3\pi}{2}\,(n-1/4)\right]^{2/3}. \tag{7.79}$$

I remind you that the Airy function is the solution of the Schrodinger equation

$$\left[-\frac{d^2}{dx^2} + x\right]\mathrm{Ai}(x) = 0\,, \qquad \mathrm{Ai}(x) \to 0 \quad \text{as} \quad x \to +\infty\,, \tag{7.80}$$

which decays at large positive x (i.e. in the classically unaccessible region).

Of course this is well expected. When $n \sim 1$ is fixed and $\lambda << 1$ the momenta of the quarks inside the meson are small as compared with m, $p < m\,\sinh\theta_n$. The motion is non-relativistic and we can describe it by the non-relativistic Schrodinger equation. The non-relativistic limit of (7.62) is

$$E = 2m + \frac{p^2}{m} + \mu^2\,|x|\,, \quad |x| = |x_1 - x_2|\,, \tag{7.81}$$

where now $|x| = |x_1 - x_2|$ is the separation between the quarks which is canonically conjugated to the COM momentum $p = (p_1 - p_2)/2$. The eigenvalues M of (7.81) are

$$M = 2m + \Delta M \tag{7.82}$$

where ΔM is the eigenvalue of the Hamiltonian

$$\left[-\frac{1}{m}\frac{d^2}{dx^2} + m^2\,\lambda\,|x| \right] \Psi(x) = \Delta M\,\Psi(x)\,. \tag{7.83}$$

One must assume

$$\Psi(x) = -\Psi(-x)\,, \qquad \Psi(x) \to 0 \quad \text{as} \quad |x| \to \infty\,. \tag{7.84}$$

The first condition is because we have two fermions, and the second is to ensure that $\Psi(x)$ is normalizable.

Making re-scaling of x

$$x = \frac{y}{m\,\lambda^{1/3}}\,, \qquad \Delta M = m\,\lambda^{2/3}\,\epsilon \tag{7.85}$$

we gets rid of all parameters,

$$\left[-\frac{d^2}{dy^2} + |y| \right] \Psi(y) = \epsilon\,\Psi(y)\,. \tag{7.86}$$

It is easy to check that this equation is solved by

$$\Psi(y) = \text{sign}(y)\,A\big(|y| - \epsilon\big)\,, \tag{7.87}$$

where $A(x)$ is any solution of the Airy equation (7.80). Then decay condition requires that $A(x) = \text{Ai}(x)$, and then the smoothness at $y = 0$ requires $\text{Ai}(-\epsilon) = 0$, which leads to $\epsilon = z_n$ where z_n are minus Airy zeros,

$$\text{Ai}(-z_n) = 0\,. \tag{7.88}$$

Thus we have at small λ but fixed n

$$M_n = m\left(2 + z_n\,\lambda^{2/3} + ...\right). \tag{7.89}$$

Further corrections-next time.