L6

As we have seen the last time, the Ward identities of the doubled IFT lead to the recurrent relations for the form-factors. Solving these relations one can express the form-factors in terms of two constants (I assume that we are working with the low-T regime $m > 0$),

$$\langle\, 0 \mid \sigma(0) \mid 0\,\rangle = \bar{\sigma}\,, \qquad \langle\, 0 \mid \mu(0) \mid A(\theta)\,\rangle = \bar{\mu}\,. \tag{6.1}$$

We have explicitly elaborated only the matrix elements

$$\langle\, 0 \mid O(0) \mid A(\theta_1)\cdots A(\theta_N)\,\rangle\,, \tag{6.2}$$

(the "form-factors"), where O is σ or μ. Of course, the analysis can be repeated for general matrix elements. One can take again, say, the commutation relation

$$[\mathbf{Y}_1, \mu_a\mu_b] = -\partial\sigma_a\,\sigma_b + \sigma_a\,\partial\sigma_b \tag{6.3}$$

and sandwich it between arbitrary states involving, as before, the A-particles only, say between

$$\langle\, A(\theta'_1)\cdots A(\theta'_M) \mid ... \mid A(\theta_1)\cdots A(\theta_N)\,\rangle\,. \tag{6.4}$$

Again, the action of $\mathbf{Y}_1$ converts of of the A-particles to B-particle, and after factorization into the correlation functions of the individual copies one finds expression for the matrix element (6.4) in terms of the matrix element involving one particle less, and the one-particle matrix element with the B-particle. Solving this relation one finds (again for $m > 0$)

$$\langle A(\theta'_1)\cdots A(\theta'_M) \mid O(0) \mid A(\theta_1)\cdots A(\theta_N)\rangle = 0 \text{ if } \begin{cases} O = \mu & \text{and} \quad N+M = \text{even}\,, \\ O = \sigma & \text{and} \quad N+M = \text{odd}\,, \end{cases} \tag{6.5}$$

otherwise

$$\langle\, A(\theta'_1)\cdots A(\theta'_M) \mid O(0) \mid A(\theta_1)\cdots A(\theta_N)\,\rangle =$$

$$= i^{[\frac{N+M}{2}]}\, g^{N+M}\,\bar{\sigma} \prod_{i<j}^{N} \tanh\frac{\theta_i-\theta_j}{2} \prod_{p<q}^{M} \tanh\frac{\theta'_p-\theta'_q}{2} \prod_{s,t} \coth\frac{\theta'_s-\theta_t}{2}\,, \tag{6.6}$$

where $g = \bar{\mu}/\bar{\sigma}$. For instance,

$$\langle\, 0 \mid \sigma(0) \mid A(\theta_1)A(\theta_2)\,\rangle = i\, g^2\, \bar{\sigma}\ \tanh\frac{\theta_1-\theta_2}{2}\,, \tag{6.7}$$

$$\langle\, A(\theta_1) \mid \sigma(0) \mid A(\theta_2)\,\rangle = i\, g^2\, \bar{\sigma}\ \coth\frac{\theta_1-\theta_2}{2}\,. \tag{6.8}$$

In fact, we are going to find that in the infinite system

$$\bar{\mu} = \bar{\sigma}\,, \tag{6.9}$$

i.e. $g = 1$. There are many ways to derive this result [1]. One can take a closer look at the matrix element (6.8) (which better deserves the name "form-factor" since its structure reflects the "form" of the particle A). Important feature of this expression is the pole at $\theta_1 = \theta_2$. Such pole is the manifestation of the fact that in the low-T phase the particle A is the "kink" separation domains "filled" by the vacua with opposite values of the spontaneous magnetization $\pm\bar{\sigma}$. Let me give a "pictorial" argument, which is not rigorous but simple and clear, and exposes the idea which can be used to develop rigorous proof .

Let us assume that we have a particle A which is such kink, so that if A sits at some spatial point x_0 we the the expectation value of $\sigma(x)$ is

$$\langle\sigma\rangle \to \begin{cases} -\bar{\sigma} & \text{for} \quad x >> x_0 \\ +\bar{\sigma} & \text{for} \quad x << x_0 \end{cases}, \tag{6.10}$$

(or the other way round, it does not much matter), see **Fig.1**.

Let us also assume that the initial state of this particle at $t \to -\infty$ is the state with definite momentum

$$p_1 = p(\theta_1) = m\,\sinh\theta_1\,. \tag{6.11}$$

[1] Relatively easy way to relate $\bar{\mu}$ and $\bar{\sigma}$ is to study the correlation function $\langle\,\psi(x)\mu(0)\,\rangle$. When $x \to 0$ the OPE relates the normalization of this correlation function to the expectation value $\langle\,\sigma(0)\,\rangle$; at $x \to \infty$ its asymptotic form is related to the matrix element $\langle\, A(\theta) \mid \mu(0) \mid 0\,\rangle$

The state is described by a plane wave

$$\frac{1}{\sqrt{\omega_2}}\, e^{ip_1 x} \tag{6.12}$$

(the factor $1/\sqrt{\omega_1}$ is added for relativistic normalization. Also, let us assume that we have the operator $\sigma(0)$ sitting at $(x,t) = (0,0)$. This operator can absorb this particle and emit instead a particle with the momentum p_2; such process of "direct interaction" cannot be responsible for the singularity at $p_1 - p_2 = 0$. On the other hand, the particle in the plane-wave state (6.12) has infinite uncertainty in x, and it almost never comes close to the point $(0,0)$ to allow for the possibility of direct absorbtion-emission. In usual situation in $D > 2$ (say, in perturbation theory calculation of the matrix element of a scalar field φ^2 in $D = 2$, and always in $D > 2$) this would lead to the disconnected part of the matrix element (6.8) proportional to

$$2\pi\, \delta(\theta_1 - \theta_2)\, \langle\, 0 \mid \sigma(0,0) \mid 0 \rangle\, . \tag{6.12}$$

In our case the situation is different, since the particle A can "miss" the insertion point $(0,0)$ passing from the right or from the left. In the first case the insertion $\sigma(0,0)$ measures the magnetization to the left from the position of the "kink", i.e. $+\bar{\sigma}$, while in the second case it measures the magnetization $-\bar{\sigma}$ to the right, see **Fig.2**.

Correspondingly, the "bulk" part of the wave (6.12) gives rise, at $t \to +\infty$, to the superposition

$$\frac{1}{\sqrt{\omega_1}} \Big[+ \bar{\sigma}\, \theta(x)\, e^{ip_1 x} - \bar{\sigma}\, \theta(-x)\, e^{ip_1 x} \Big] \qquad |x| \to \infty\, . \tag{6.13}$$

(of course this equation shows only the large-x form of the outgoing wave; at finite $x \sim m^{-1}$ it is more complicated). The singular part of the matrix element (6.8) appears from this part of the wave; expanding (6.13) in usual plane waves (6.12) one finds instead of the delta-function (6.12)

$$\text{singular part of (6.8)} = \frac{\bar{\sigma}}{\sqrt{\omega_1\, \omega_2}} \int_{-\infty}^{\infty} dx\, e^{i(p_1 - p_2)x} \big[- \theta(x) + \theta(-x) \big] =$$

$$= \frac{\bar\sigma}{\sqrt{\omega_1\omega_2}}\,\mathrm{VP}\,\frac{2i}{p_1-p_2} \sim \mathrm{VP}\,\frac{2i\,\bar\sigma}{\theta_1-\theta_2}\,, \tag{6.14}$$

where VP stands for the principal value. There are also contributions of the processes with direct absorbtion-emission, but they cannot be singular at $\theta_1 = \theta_2$. Thus, the singularity of (6.8) indicates the that the "form" of the particle A in terms of local magnetization is a kink, with

$$\sigma(x=+\infty) - \sigma(x=-\infty) = i\,\mathrm{res}_{\theta_1=\theta_2}\,\langle A(\theta_1)\mid\sigma\mid A(\theta_2)\rangle\,. \tag{6.15}$$

Consistency requires that $g=1$. This analysis also reveals the nature of the singularity in (6.8) in terms of the distribution, as the principal value. The Eq.(6.14) is the simplest case of the so-called "annihilation poles" of the form-factors.

It remains to find the constant $\bar\sigma$, i.e. the spontaneous magnetization of the IFT in the low-T domain. Again, there are many ways to do that. I will do the calculation using the idea of so-called "angular quantization", to demonstrate this method.

Since by definition of $\sigma(0)$ the fermion $\psi(x), \bar\psi(x)$ changes sign when x is brought around the point 0, it is natural to try to relate the expectation value $\langle\,\sigma(0)\,\rangle$ to the functional integral

$$\langle\,\sigma\,\rangle \sim Z^{-1}\int D[\psi,\bar\psi]_{(-)}\,e^{-\mathcal{A}_{FF}[\psi,\bar\psi]}\,, \tag{6.16}$$

where we assume that the integration is over the space of fields $\psi(x), \bar\psi(x)$ having this property,

$$\psi(x) \to -\psi(x) \quad \text{when} \qquad x\ \bullet \qquad\qquad \bullet\ 0 \qquad ; \tag{6.17}$$

this is indicated by the subscript in $D[\psi,\bar\psi]_{(-)}$. In (6.16) $\mathcal{A}[\psi,\bar\psi]$ is the free-fermion action which we have written down before, and Z is the partition function, which is defined as the functional integral over the *single-valued* fields $(\psi,\bar\psi)$,

$$Z = \int D[\psi,\bar\psi]_{(+)}\,e^{-\mathcal{A}_{FF}[\psi,\bar\psi]}\,, \tag{6.16a}$$

I will specify exact relation a little later, but first let write the action using the polar coordinates (r,α),

$$z = r\,e^{i\alpha}\,, \qquad \bar z = r\,e^{-i\alpha}\,;$$

in fact it is useful to write the radial coordinate r as

$$r = e^{\rho}\,,$$

then

$$z = e^{u}\,, \quad \bar{z} = e^{\bar{u}}\,, \qquad \text{where} \quad u = \rho + i\alpha\,, \quad \bar{u} = \rho - i\alpha\,. \tag{6.18}$$

The transformation $z \to u$ is a conformal transformation, therefore the massless part of the action retains the original form if one transforms the fermi fields as prescribed,

$$\psi(z,\bar{z}) = e^{-u/2}\,\Psi(u,\bar{u})\,, \qquad \bar{\psi}(z,\bar{z}) = e^{-\bar{u}/2}\,\Psi(u,\bar{u})\,. \tag{6.19}$$

The mass term of the action is not conformally invariant and it acquires certain factor. As the result we have

$$\mathcal{A}_{FF} = \frac{1}{2\pi}\int_0^{2\pi} d\alpha \int_{-\infty}^{\infty} d\rho \left[\Psi\partial_{\bar{u}}\Psi + \bar{\Psi}\partial_u\bar{\Psi} + im\,e^{\frac{u+\bar{u}}{2}}\,\bar{\Psi}\Psi\right]. \tag{6.20}$$

Note that since $(\psi,\bar{\psi})$ in the integral (6.16) has the monodromy property (6.17), the field $(\Psi,\bar{\Psi})$ in the transformed integral (6.16) are *periodic* functions of the angular variable α,

$$P: \qquad \big(\Psi(\rho,\alpha+2\pi),\bar{\Psi}(\rho,\alpha+2\pi)\big) = \big(\Psi(\rho,\alpha),\bar{\Psi}(\rho,\alpha)\big)\,. \tag{6.21}$$

At the same time, when computing the partition function Z in these coordinates one has to integrate over anti-periodic fields Ψ, $\bar{\Psi}$. Of course this is exactly analogous to the phenomenon well-known in CFT and in string theory, i.e. that in the operator-state correspondence the R-field correspond to the states with periodic fermions, while the NS-fields correspond to the anti-periodic fermions.

Thus we would like to represent the above expectation values as the ratio

$$\langle\,\sigma(0)\,\rangle \sim \frac{\int D[\Psi,\bar{\Psi}]_P\, e^{-\mathcal{A}[\Psi,\bar{\Psi}]}}{\int D[\Psi,\bar{\Psi}]_A\, e^{-\mathcal{A}[\Psi,\bar{\Psi}]}}\,, \tag{6.22}$$

where P denotes the integration over the periodic fields, as in (6.21), and A stands for the integral over the anti-periodic fields

$$A: \qquad \big(\Psi(\rho,\alpha+2\pi),\bar{\Psi}(\rho,\alpha+2\pi)\big) = -\big(\Psi(\rho,\alpha),\bar{\Psi}(\rho,\alpha)\big)\,. \tag{6.23}$$

Evaluation of the functional integrals can be done using Hamiltonian formalism. We will use the Hamiltonian picture in which the angular variable

$$\alpha = \text{Euclidean time}\,,$$

and the radii $\alpha = const$, $r \in [0:\infty]$ are taken to be the equal-time slices **Fig.3**.

Note that this picture is different both from the picture of radial quantization, and from the "textbook" quantization. We expect to have a different space of states here.

Canonical quantization of the action (6.20) is performed in a standard way. One finds that the fields $\Psi, \bar\Psi$ obey canonical equal-time anti-commutation relations,

$$\big\{\Psi(\rho), \Psi(\rho')\big\} = 2\pi\, \delta(\rho-\rho')\,, \qquad \big\{\bar\Psi(\rho), \bar\Psi(\rho')\big\} = -2\pi\, \delta(\rho-\rho')\,, \tag{6.24}$$

while Ψ and $\bar\Psi$ anti-commute. To get rid of the irritating minus sign in the last of the equations (6.24) I will use

$$\Psi_L(\rho) = \Psi(\rho)\,, \qquad \Psi_R(\rho) = i\,\bar\Psi(\rho)\,. \tag{6.25}$$

Then the "angular Hamiltonian" is (if $\alpha = it$ the time evolution operator is $e^{-i\mathbf{K}\,t}$)

$$\mathbf{K} = \frac{i}{2\pi}\int_{-\infty}^{\infty} d\rho\,\Big[-\Psi_L\partial_\rho\Psi_L + \Psi_R\partial_\rho\Psi_R + m\,e^{\rho}\,\Psi_R\Psi_L\Big]\,. \tag{6.26}$$

It has the form of usual free-fermion Hamiltonian, with the mass term having the coordinate dependence: the mass falls to zero in the limit $\rho \to -\infty$ (i.e. close to the insertion point) but grows exponentially at large positive ρ. This growth of the mass prevents the particles from penetrating too far to the right; I'll call this phenomenon the "mass barrier".

Anyway, the Hamiltonian (6.26) is diagonalized by using the decompositions

$$\Psi_L(\rho,\alpha) = \int_{-\infty}^{\infty}\frac{d\nu}{\sqrt{2\pi}}\,C_\nu\,U_\nu(\rho)\,e^{-\nu\alpha}\,, \qquad \Psi_R(\rho,\alpha) = \int_{-\infty}^{\infty}\frac{d\nu}{\sqrt{2\pi}}\,C_\nu\,V_\nu(\rho)\,e^{-\nu\alpha}\,, \tag{6.27}$$

where $\big(U_\nu(\rho)\, e^{-\nu\alpha}, V_\nu(\rho)\, e^{-\nu\alpha}\big)$ are the solutions of the Dirac equation which decay at large positive ρ,

$$\begin{pmatrix} U_\nu(\rho) \\ V_\nu(\rho) \end{pmatrix} = \frac{\sqrt{2m}\ e^{\rho/2}}{\Gamma\big(1/2 - i\nu\big)} \left(\frac{m}{2}\right)^{-i\nu} \begin{pmatrix} K_{\frac{1}{2}-i\nu}(m\, e^{\rho}) \\ K_{\frac{1}{2}+i\nu}(m\, e^{\rho}) \end{pmatrix}, \tag{6.28}$$

and C_ν are operators which satisfy their canonical anti-commutation relations

$$\big\{C_\nu, C_{\nu'}\big\} = 2\pi\, \delta(\nu+\nu')\,. \tag{6.29}$$

At $\rho \to -\infty$ the solutions (6.28) describe massless fermions with the momenta ν,

$$\begin{pmatrix} U_\nu(\rho) \\ V_\nu(\rho) \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\nu\,\rho} + S_F(\nu) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\nu\,\rho} \qquad \text{as} \quad \rho \to \infty \tag{6.30}$$

which are reflected off the mass barrier with the reflection amplitude

$$S_F(\nu) = \left(\frac{m}{2}\right)^{-2i\nu} \frac{\Gamma\big(\frac{1}{2}+i\nu\big)}{\Gamma\big(\frac{1}{2}-i\nu\big)}\,. \tag{6.31}$$

The space of states of the angular quantization is again the space of free fermions with continuous momenta ν, but this time those are the scattering states describing this reflection.

The angular Hamiltonian $\mathbf{K}$ is

$$\mathbf{K} = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi}\ \frac{\nu}{2}\ C_{-\nu} C_\nu\,. \tag{6.32}$$

Now, recall that our problem is to compute the ratio of the functional integrals (6.22), i.e.

$$\frac{\int_P\ D[\Psi, \bar\Psi]\, e^{-\mathcal{A}[\Psi,\bar\Psi]}}{\int_A\ D[\Psi, \bar\Psi]\, e^{-\mathcal{A}[\Psi,\bar\Psi]}}, \tag{6.33}$$

over the fields Ψ which are either periodic or anti-periodic in the angular coordinate α with the period 2π. Since we treat α as the Euclidean time, and we have to impose (anti-) periodicity, the problem is formally equivalent to finding the partition function of the quantum system with the Hamiltonian $\mathbf{K}$ and the temperature 2π, i.e. with the thermal density matrix

$$e^{-2\pi\,\mathbf{K}}\,;$$

while the angle plays the role of the Matsubara time. By standard arguments

$$(6.33) \quad = \quad \frac{\mathrm{tr}\big((-)^F\, e^{-2\pi\, \mathbf{K}}\big)}{\mathrm{tr}\big(e^{-2\pi\, \mathbf{K}}\big)}\,, \tag{6.34}$$

where F is the fermion number, i.e. $(-)^F = 1$ for the states with even number of fermions, and $(-)^F = -1$ for odd fermion states.

The suitable space of states is the Fock space with the vacuum

$$C_\nu \mid vac\,\rangle = 0 \quad \text{for} \quad \nu > 0\,. \tag{6.35}$$

Since any constant term in the Hamiltonian cancels between the numerator and the denominator, we can write it in the normal ordered form

$$\mathbf{K} = \int_0^\infty \frac{d\nu}{2\pi}\; \nu\, C_{-\nu} C_\nu\,. \tag{6.36}$$

The traces in (6.34) are computed in a straightforward manner,

$$(6.34) = \prod_n \frac{1 - e^{-2\pi\,\nu_n}}{1 + e^{-2\pi\,\nu_n}}\,, \tag{6.37}$$

where the product is over all eigenvalues ν_n of the angular Hamiltonian (6.36).

Here we face a difficulty, since by construction the coordinate ρ (which is $\log r$) ranges from $-\infty$ to $+\infty$. While the mass barrier makes the $+\infty$ unaccessible for the fermions, they can freely move as far towards the $-\infty$ as they want. As the result the spectrum of ν is continuous. This by itself is not a problem: the product (6.37) has to be replaced by expression which involves integral over the continuous spectrum,

$$(6.37) \to \exp\left\{ \int R(\nu)\, d\nu\, \log\left(\frac{1 - e^{-2\pi\,\nu}}{1 + e^{-2\pi\,\nu}}\right)\right\},$$

with some density of states $R(\nu)$. Worse is that the density of states diverges when the ρ-size of the system is infinite. If we put a cutoff forbidding the particles to move further then $-L$ to the left, i.e. put the limit $-L < \rho$, then for large L the density of states $R(\nu)$ contains the "bulk" part proportional to L. Since $\rho = \log r$, introducing such cutoff corresponds to drilling small hole of the size

$$r_0 = e^{-L}$$

around the insertion point. The "bulk" part proportional to L brings a factor of the form $e^{-aL} = r_0^a$, the manifestation of the fact that the insertion $\sigma(0)$ has an anomalous dimension. Let us make this statement more precise.

I would like to introduce explicit cutoff: I limit $-L < \rho$, and impose the boundary conditions

$$\big[\Psi_L(\rho) + \Psi_R(\rho)\big]_{\rho=-L} = 0\,. \tag{6.38}$$

This particular boundary condition has important physical interpretation.

If L is large, such that

$$m\,r_0 << 1\,, \tag{6.39}$$

the mass can be ignored at the scales $\sim r_0$. Ignoring the mass, one can check that the boundary condition (6.38) is conformally invariant. More generally, one can study conformally invariant boundary conditions in generic CFT, this is subject to the "boundary conformal field theory". Unfortunately, I do not have time to discuss this interesting theory in any details. Let me just say this. If I take CFT on the plane with a hole drilled around the point zero, and impose some boundary conditions on the boundary of the hole, the natural way to describe the effect of such hole is in terms of so-called boundary state. The boundary is a circle around zero, i.e. it is an "equal-time" slice of the radial quantization. The boundary state is certain vector in the space of the radial quantization,

$$\odot \to \mathcal{F}\,, \qquad |\, B\,\rangle \in \mathcal{F}\,, \tag{6.40}$$

which represents the effect of the hole for the outside world. It generally has the form

$$|\, B\,\rangle = \sum_i\, (r_0)^{2\Delta_i}\, A_i\ |\, I_i\,\rangle\,, \qquad i: \quad O_i\ -\ \text{spinless primary fields} \tag{6.41}$$

where i runs the set of all spinless primary fields O_i of the CFT, and $|\, I_i\,\rangle$ denote the so-called Ishibashi states

$$|\, I_i\,\rangle = \left(1 + r_0^2\, \frac{L_{-1}\bar{L}_{-1}}{2\,\Delta_i} + \cdots\right) |\, O_i\,\rangle \tag{6.42}$$

whose structure is uniquely determined by the conformal invariance, and r_0 is the radius of the hole. The dimensionless coefficients A_i are are determined from the requirement that the boundary condition is local. It is important that every term in (6.41) comes with the factor r_0 to the power equal to the mass dimension of the field in this term. The equation (6.41) can be thought of as some sort of short-distance operator expansion which expresses the insertion of the hole in terms of the sum of local insertions,

$$\underset{|B\rangle}{\odot} \to \sum_i \bullet \quad O_i(0)$$

In all minimal CFT the conformal boundary conditions are classified and associated boundary states are explicitly constructed. The CFT of the Ising model admits three different conformal boundary conditions. From the microscopic viewpoint these three correspond to the scaling limits of the "fixed" boundary conditions, where the microscopic spins are fixed to the values ± 1 at the boundary, and the "free" boundary, where no restrictions are imposed on the boundary spins. The boundary condition (6.38) corresponds to the "fixed" boundary condition. Associated boundary state(s) has the form

$$\underset{\sigma=+1}{\odot} \qquad |\, B\,\rangle_{\rm fixed} = \frac{1}{\sqrt{2}}\ |\, I_I\,\rangle \pm \frac{(r_0)^{\frac{1}{8}}}{2^{\frac{1}{4}}}\ |\, I_\sigma\,\rangle + \frac{r_0}{\sqrt{2}}\ |\, I_\epsilon\,\rangle\,. \tag{6.43}$$

With the hole of small but finite size, imposing the condition that $\psi(x)$ changes the sign when x is brought around the hole has the effect of projecting out all the states but those from the Ishibashi state $|\, I_\sigma\,\rangle$. To the contrary, integration over the single-valued fields projects onto the NS components of the boundary state (6.43). It follows that in the presence of the hole with the boundary conditions (6.38), and when r_0 is small

$$\frac{\int D[\psi]_{(-)}\, e^{-\mathcal{A}_{FF}}}{\int D[\psi]_{(+)}\, e^{-\mathcal{A}_{FF}}} = \frac{{\rm tr}\big[(-)^F\, e^{-2\pi\,\mathbf{K}}\big]}{{\rm tr}\big[e^{-2\pi\,\mathbf{K}}\big]} \to 2^{\frac{1}{4}}\,(r_0)^{\frac{1}{8}}\,\langle\,\sigma(0)\,\rangle \quad \text{as} \quad r_0\to 0\,. \tag{6.44}$$

The ratio in (6.44) is exactly the ratio of the traces (6.34) taken over the states with the boundary condition (6.38) imposed. This boundary condition breaks the continuous spectrum of states into discrete levels. Assuming that $L = -\log r_0$ is very large, I can use the asymptotic form (6.30) of the waves associated with each ν,

$$\begin{pmatrix} U_\nu(\rho) \\ V_\nu(\rho)\end{pmatrix} \to \begin{pmatrix} e^{\,i\nu\,\rho} \\ S_F(\nu)\, e^{-i\nu\,\rho}\end{pmatrix}, \tag{6.45}$$

then (6.38) leads to quantization condition

$$1 + e^{2i\nu\,L}\, S_F(\nu) = 0\,, \quad \text{or} \quad 2i\nu\, L + \log S_F(\nu) = 2\pi i\,(n+1/2)\,. \tag{6.46}$$

Then, at large but finite L the product (6.37) is over positive solutions of the equation (6.46), or

$$(6.34) = \exp\left\{ \sum_{\nu_n > 0} \log\left(\frac{1 - e^{-2\pi\,\nu_n}}{1 + e^{-2\pi\,\nu_n}} \right) \right\}. \tag{6.47}$$

In the limit $L \to \infty$ the sum here can be replaced by integral with the density derived from (6.46),

$$R(\nu) = \frac{L}{\pi} + \frac{1}{2\pi i}\frac{d}{d\nu}\log S_F(\nu)\,. \tag{6.48}$$

The only subtlety is the singularity of the expression in the sum (6.47) at $\nu = 0$; more careful analysis (using the technique of the counting functions) yields for (6.47) at $L \to \infty$ [2]

$$\exp\left\{ \frac{1}{2}\log 2 + \int_0^\infty d\nu\, R(\nu)\,\log\left(\frac{1 - e^{-2\pi\nu}}{1 + e^{-2\pi\nu}} \right) \right\} = 2^{\frac{1}{2}}\,(r_0)^{\frac{1}{8}}\,e^c\,, \tag{6.49}$$

[2] This is done as follows. Write the exponential in (6.47) as

$$\int_C \frac{d\nu}{2\pi i}\,\frac{X'(\nu)}{1 + X(\nu)}\,\log Y(\nu)\,,$$

where $Y(\nu) = \frac{1-e^{-2\pi\,\nu}}{1+e^{-2\pi\,\nu}}$ and $X(\nu)$ is

$$X(\nu) = e^{2i\nu\,L}\,S_F(\nu)\,;$$

levels ν_n are solutions of $X(\nu)+1 = 0$. The contour C encloses the positive solutions. One can break $C = C_+ + C_-$ where C_+ goes above and C_- goes below the poles.

* * * * * * → * * * * * *

Let us transform the C_- integral as follows

$$\int_{C_-} \frac{d\nu}{2\pi i}\,\frac{X'}{X}\,\log Y - \int_{C_-} \frac{d\nu}{2\pi i}\,\frac{X'}{X^2\,(1 + X^{-1})}\,\log Y$$

. The first term becomes the $\int_0^\infty R(\nu)\,d\nu\,\log Y$, while in the second part one changes $\nu \to -\nu$ and uses $X(-\nu) = X^{-1}(\nu)$ and $Y(-\nu) = -Y(-\nu)$ to combine it with the C_+ integral to the integral from $-\infty$ to ∞ above the real axis; this integral is exponentially small in L, it is controlled by the complex singularities of $\log Y$. The term $\log\sqrt{2}$ appears because of the singularity of $\log Y$ at $\nu = 0$, since $\log Y(-\nu) \to i\pi + \log Y(\nu)$.

where the factor $(r_0)^{\frac{1}{8}}$ comes from the part of the density (6.48) proportional to $L = -\log r_0$, and c stands for the convergent and finite integral

$$c = \int_0^\infty \frac{d\nu}{2\pi i} \partial_\nu S_F(\nu) \log\left(\frac{1-e^{-2\pi\nu}}{1+e^{-2\pi\nu}}\right). \tag{6.50}$$

Since $S_F(\nu)$ is known

$$S_F(\nu) = \left(\frac{m}{2}\right)^{-2i\nu} \frac{\Gamma(1/2+i\nu)}{\Gamma(1/2-i\nu)}, \tag{6.51}$$

the integral can be evaluated in a closed form [3]. Then, using (6.44) and taking the limit $r_0 \to 0$, one finds

$$\bar{\sigma} = m^{\frac{1}{8}}\,\bar{s}\,, \qquad \bar{s} = 2^{\frac{1}{12}}\, e^{-\frac{3}{2}\zeta'(-1)} = 1.35783834170660... \tag{6.52}$$

Let us come back to the matrix elements of fields between the particle states. As another application of the Ward identities of the doubled Ising field theory let us derive the matrix elements which involve two spin insertions, like

$$\langle\, 0 \mid \sigma(x)\sigma(x')\ \mid A(\theta_1)\cdots A(\theta_N)\,\rangle\,, \tag{6.53}$$

or similar matrix elements with $\mu(x)\mu(x')$ and such.

The simplest (and most basic) matrix elements of this kind are the one-particle matrix elements of the product $\sigma(x)\mu(x')$. This product can create a one-particle state in both regimes of $m > 0$ and $m < 0$. We consider

$$\begin{aligned} \langle\, 0 \mid \sigma(x)\mu(x') \mid A(\theta)\,\rangle &= E(x+x',\theta)\, F(x-x',\theta)\,, \\ \langle\, 0 \mid \mu(x)\sigma(x') \mid A(\theta)\,\rangle &= E(x+x',\theta)\, \tilde{F}(x-x',\theta)\,. \end{aligned} \tag{6.54}$$

[3] It is convenient to transform this integral using the integral representation of the $\log\Gamma(z)$, so that

$$\frac{1}{i}\log S_F(\nu) = -2\nu \log\frac{m}{2} - \int_0^\infty \frac{dt}{t}\left[\frac{\sinh 2\nu t}{\sinh t} - 2\nu\, e^{-2t}\right].$$

Here I have isolated the dependence of the overall "center of mass" shift which is controlled by the total energy-momentum of the one-particle state,

$$E(x+x';\theta) = e^{-\frac{x_2+x_2'}{2}\, m\,\cosh\theta + i\,\frac{x_1+x_2}{2}\, m\,\sinh\theta}\,, \tag{6.55}$$

the remaining factors F and $\tilde{F}$ depend on the separation only. Here (x_1, x_2) are the cartesian coordinates of the point x, and likewise for x'. Although the two expressions above are related by the interchange $x \leftrightarrow x'$ only, I find it convenient to use both notations.

The IFT has rotational symmetry (=Lorentz symmetry in Minkowski theory); this has simple consequences on the structure of the functions F and $\tilde{F}$. It is useful to introduce the polar coordinates associated with the difference $x - x'$, i.e.

$$z - z' = r\,e^{\alpha}\,, \qquad \bar{z} - \bar{z}' = r\,e^{-i\alpha}\,. \tag{6.56}$$

Since rotations shift the angle α and also shift the particle rapidity θ (by pure imaginary value), the functions F and $\tilde{F}$ in fact depend on α and θ separately but on the Lorentz-invariant combination $\beta = \alpha - i\theta$ only. With some abuse of notations I will sometimes write

$$F(x,\theta) = F(r|\beta)\,, \qquad \tilde{F}(x,\theta) = \tilde{F}(r|\beta)\,. \tag{6.57}$$

According to our definition, the product $\sigma(x)\mu(x')$ changes the sign when x is brought around x', so that

$$F(r|\beta + 2\pi) = -F(r|\beta)\,, \tag{6.58}$$

and the same for $\tilde{F}$. Also, $\tilde{F}$ is related to F by the interchange of the positions of σ and μ, and one can be obtained from another by 180^o rotation. There is a sign ambiguity; I will chose the relative sign of the functions (6.54) by defining F as the result of 180^o rotation of $\tilde{F}$ in the positive (counterclockwise) direction,

$$\tilde{F}(r|\beta+\pi) = F(r|\beta)\,, \qquad F(r|\beta+\pi) = -\tilde{F}(r|\beta)\,. \tag{6.59}$$

Using technique we have already developed in the last lecture it is easy to show that these matrix elements satisfy *linear* differential equations with the coefficients which expressed through the two-point correlation functions

$$G(x-x') = \langle\, \sigma(x)\sigma(x')\,\rangle\,, \qquad \tilde{G}(x-x') = \langle\, \mu(x)\mu(x')\,\rangle\,. \tag{6.60}$$

To see this consider for instance the identities in the doubled IFT

$$\langle\, 0 \mid \mathbf{Y}_1\, \sigma_a(x)\sigma_b(x)\, \sigma_a(x')\sigma_b(x') \mid A(\theta)\,\rangle = 0\,,$$

$$\langle\, 0 \mid \mathbf{Y}_1\, \sigma_a(x)\sigma_b(x)\, \sigma_a(x')\sigma_b(x') \mid A(\theta)\,\rangle = 0\,. \tag{6.61}$$

Using the commutation relations

$$[\mathbf{Y}_1, \sigma_a\sigma_b] = -\partial\mu_a\, \mu_b + \mu_a\partial\mu_b\,, \tag{6.62}$$

and the relation identical to this with σ's replaced by μ's and vice versa, one can move the operator $\mathbf{Y}_1$ to the right, where it acts by converting the A-particle to the B-particle and also multiplying the state by the factor $m\, e^{\theta}/2$,

$$\mathbf{Y}_1 \mid A(\theta)\,\rangle = \frac{m}{2}\, e^{\theta} \mid B(\theta)\,\rangle\,. \tag{6.63}$$

As the result, after factorization into the a and b copies, the equations (6.61) yield two differential equations for the functions $F(x-x',\theta)$ and $\tilde{F}(x-x',\theta)$, which are linear in these functions, and also involve the correlation functions (6.60). I am not going to go through the details here since these manipulations are straightforward.

The equations take nicer form if one uses again the parametrization

$$G(x) = m^{1/4}\, e^{\chi(x)/2}\, \cosh\big(\varphi(x)/2\big)\,, \qquad \tilde{G}(x) = m^{1/4}\, e^{\chi(x)/2}\, \sinh\big(\varphi(x)/2\big)\,, \tag{6.64}$$

and also introduces the combinations

$$F(x,\theta) + i\tilde{F}(x,\theta) = \bar{\omega}\, m^{1/4}\, e^{\chi(x)/2}\; \Psi_+(x,\theta)\,,$$

$$F(x,\theta) - i\tilde{F}(x,\theta) = \omega\, m^{1/4}\, e^{\chi(x)/2}\; \Psi_-(x,\theta)\,, \tag{6.65}$$

where again $\omega = e^{\frac{i\pi}{4}}$. With these notations one finds

$$\partial\Psi_+ = -\frac{1}{2}\,\partial\varphi\,\Psi_+ + \frac{m\, e^{\theta}}{4}\, e^{\varphi}\,\Psi_-\,,$$

$$\partial\Psi_- = \frac{1}{2}\,\partial\varphi\,\Psi_- - \frac{m\, e^{\theta}}{4}\, e^{-\varphi}\,\Psi_+\,. \tag{6.66}$$

Using $\mathbf{Y}_{-1}$ instead of $\mathbf{Y}_1$ one obtains the second pair of linear equations,

$$\bar{\partial}\Psi_+ = \frac{1}{2}\,\bar{\partial}\varphi\,\Psi_+ - \frac{m\, e^{-\theta}}{4}\, e^{-\varphi}\,\Psi_-\,,$$

$$\bar{\partial}\Psi_- = -\frac{1}{2}\,\bar{\partial}\varphi\,\Psi_- + \frac{m\,e^{-\theta}}{4}\,e^{\varphi}\,\Psi_+\,. \tag{6.67}$$

One recognizes the linear system associated with the integrable sinh-Gordon equation,

$$\partial\bar{\partial}\varphi = \frac{m^2}{8}\,\sinh\big(2\varphi\big)\,, \tag{6.68}$$

with the quantity

$$\lambda = \frac{m}{4}\,e^{\theta}\,, \qquad \bar{\lambda} = \frac{m}{4}\,e^{-\theta} \tag{6.69}$$

playing the role of the spectral parameter. As we have seen, the correlation functions are expressed in terms of solutions of the integrable sinh-Gordon system. Now we observe that the functions $\Psi_\pm$ involved in the associated linear problem are interpreted as the matrix elements (6.54) of $\sigma(x)\mu(x')$ in with the one-particle state.

Let me first list some simple properties of $\Psi_\pm$. Just like F and $\tilde{F}$ they depend on r and $\beta = \alpha - i\theta$, and from (6.59)

$$\Psi_+(r|\beta+\pi) = i\Psi_+(r|\beta)\,, \qquad \Psi_-(r|\beta+\pi) = -i\Psi_-(r|\beta)\,. \tag{6.70}$$

Also, it is possible to show that

$$\Psi_+(r|\beta) = \Psi_-(r|-\beta)\,, \tag{6.71}$$

and that if both r and β are real (the last condition requires that the rapidity θ is taken pure imaginary), then

$$\Psi_+(r|\beta) = \Psi_-^*(r|\beta) \quad \text{for real} \quad (r,\beta)\,. \tag{6.72}$$

In the CIFT, i.e. at $m = 0$, the fields σ and μ obey the OPE

$$\sigma(x)\mu(x') = \frac{\omega}{\sqrt{2}}\,(z-z')^{3/8}\,(\bar{z}-\bar{z}')^{-1/8}\,\psi(x') + \frac{\bar{\omega}}{\sqrt{2}}\,(z-z')^{-1/8}\,(\bar{z}-\bar{z}')^{3/8}\,\bar{\psi}(x') + \ldots\,, \tag{6.73}$$

where dots represent contributions of the conformal descendants of the primary fields ψ and $\bar{\psi}$. The leading short-distance behavior is unchanged when one adds non-zero mass, hence from (6.73) one can infer the leading $r \to 0$ asymptotic of the functions F and $\tilde{F}$. For the $\Psi_\pm$ this leads to

$$e^{\chi(r)/2}\,\Psi_\pm(r|\beta) \to \sqrt{2\pi}\,(m\,r)^{1/4}\,e^{\pm i\beta/2} \quad \text{as} \quad r \to 0\,. \tag{6.74}$$

Also, it follows from (6.70) that $\Psi_{\pm}(r|\beta)$ are anti-periodic functions of β, and hence admit discrete Fourier decompositions

$$\Psi_{\pm}(r|\beta) = \sum_{n=-\infty}^{\infty} \Psi_{2n}(r)\, e^{\pm i\left(\frac{1}{2}+2n\right)\,\beta}\,, \tag{6.75}$$

where $\Psi_{2n}(r)$ are real at real r.

It is worth mentioning that the Fourier coefficients $\Psi_{2n}(r)$ in (6.75) can be interpreted in terms of the coefficients of the finite-distance and finite m version of the OPE (6.73)

$$\sigma(x)\mu(x') = \sum_{n=0}^{\infty} \left[C_n(x-x')\,\partial^n\psi\left(\frac{x+x'}{2}\right) + \bar{C}_n(x-x')\,\partial^n\bar{\psi}\left(\frac{x+x'}{2}\right)\right] +$$

$$+\text{"multi} - \text{fermion" terms}\,: \qquad \langle\, 0\,|\,\text{"}m.f.t\text{"}\,|\,A(\theta)\,\rangle = 0\,, \tag{6.76}$$

where the "multi-fermion terms" denote contributions of operators whose matrix elements between vacuum and one-particle state vanishes.

Exercise: Find the coefficients $C_n(x)$ and $\bar{C}_n(x)$ in the above OPE in terms of the Fourier coefficients $\Psi_{2n}(r)$ of $\Psi_{\pm}(r|\beta)$.