L5.

The last time we have discussed the IFT at $H = 0$ but $m \neq 0$. It is a free field theory, but the correlation functions of the spin fields are complicated objects because σ and μ are not local with respect to the free fields $\psi, \bar{\psi}$.

We have observed that due to special properties of the spin fields their correlation functions satisfy differential equations, and that the coefficients in these differential equations can be realized in terms of another copy of the IFT; we thus came to considering a system that consists of two identical copies of the IFT - the "doubled" Ising field theory.

We now will study the "doubled" IFT in some details. We will observe that this system has rich symmetry, and that closed differential equations for the IFT spin correlation functions appear simply as the Ward identities associated with this symmetry.

Thus, we consider the "doubled" Ising field theory which consists of two identical copies of the IFT, with the same mass parameter m, and with no interaction between the copies. I will distinguish the copies by giving the label a to one copy, and b to another. Thus we have two copies of the Majorana fermion field,

$$\left(\psi_a, \bar{\psi}_a\right) \quad \text{and} \quad \left(\psi_b, \bar{\psi}_b\right), \tag{5.1}$$

two copies of spin fields

$$\sigma_a\,,\ \mu_a\ \in \mathcal{R}_a\,, \quad \text{and} \quad \sigma_b\,,\ \mu_b\ \in \mathcal{R}_b\,, \tag{5.2}$$

which are the Fock vacua in respective copies of the Ramond space $\mathcal{R}_a$ and $\mathcal{R}_b$, and so on. The correlation functions of the doubled theory factorize in terms of the correlation functions of each separate copy.

The above straightforward definition has inconvenient feature that the fermion fields ψ_a and ψ_b belonging to different copies commute. To make it possible to handle fermions in a conventional way, we fix this problem by introducing, in addition to the two Ising copies, two auxiliary variables, the so called "Klein factors" η_a and η_b. These variables are assumed to commute with all observables from both the Ising copies, and to satisfy the following defining relations

$$\eta_a^1 = 1\,, \qquad \eta_b^2 = 1\,, \qquad \eta_a\eta_b = -\eta_b\eta_a\,. \tag{5.3}$$

We then modify the definitions of the fields of the doubled Ising field theory as follows,

$$\psi_a(x) \to \eta_a\, \psi_a(x)\,, \qquad \psi_b(x) \to \eta_b\, \psi_b(x)\,, \tag{5.4a}$$

$$\bar\psi_a(x) \to \eta_a\, \bar\psi_a(x)\,, \qquad \bar\psi_b(x) \to \eta_b\, \bar\psi_b(x)\,, \tag{5.4b}$$

and

$$\sigma_a(x) \to \sigma_a(x)\,, \qquad \sigma_b(x) \to \sigma_b(x)\,, \tag{5.5a}$$

$$\mu_a(x) \to \eta_a\, \mu_a(x)\,, \qquad \mu_b(x) \to \eta_b\, \mu_b(x)\,. \tag{5.5b}$$

This modification has no effect on any of the relations concerning each single copy of the IFT which we derived in the last Lecture, and hence it does not bring any change to the correlation functions involving only the fields from a single copy, either a or b. On the other hand, the Klein factors give rise to additional sign factors when factorizing generic correlation functions with entries from both copies present. For example, $\langle\, \sigma_a(x)\sigma_b(x)\, \sigma_a(x')\sigma_b(x')\, \rangle\ =\ \langle\, \sigma(x)\, \sigma(x')\, \rangle^2_{\text{single}}$, but $\langle\, \mu_a(x)\mu_b(x)\, \mu_a(x')\mu_b(x')\, \rangle\ =\ -\langle\, \mu(x)\, \mu(x')\, \rangle^2_{\text{single}}$, where the expectation values in the right-hand sides are that of a single Ising field theory. The desirable effect of this modification is that fermi fields from different copies now anti-commute inside the correlation functions,

$$\psi_a(x)\psi_b(x') = -\psi_b(x')\psi_a(x)\,, \qquad \text{etc.} \tag{5.6}$$

Of course, with this definition the doubled Ising field theory is identical to the theory of a free Dirac fermi field

$$\begin{aligned} \big(\Psi(x), \bar\Psi(x)\big) &= \big(\psi_a(x) + i\,\psi_b(x),\ \bar\psi_a(x) + i\,\bar\psi_b(x)\big)\ , \\ \big(\Psi^\dagger(x), \bar\Psi^\dagger(x)\big) &= \big(\psi_a(x) - i\,\psi_b(x),\ \bar\psi_a(x) - i\,\bar\psi_b(x)\big)\ . \end{aligned} \tag{5.7}$$

Like a single copy of the IFT, the doubled IFT is the free field theory. Each copy of the fermion fields satisfy free Dirac equation

$$\begin{aligned} \bar\partial\psi_a &= \frac{im}{2}\,\bar\psi_a\,, \qquad \bar\partial\psi_b = \frac{im}{2}\,\bar\psi_b\,, \\ \partial\bar\psi_a &= -\frac{im}{2}\,\psi_a\,, \qquad \partial\bar\psi_b = -\frac{im}{2}\,\bar\psi_b\,, \end{aligned} \tag{5.8}$$

It also has infinitely many integrals of motion quadratic in the fermion fields ψ_a and ψ_b. These IM form a nontrivial, noncommutative algebra. Important subset of these IM forms the affine algebra $\hat{sl(2)}$ (of level zero). Let me display the simplest IM which are the fundamental generating elements of the algebra.

The simplest IM is the $U(1)$ charge associated with phase rotations of the above Dirac field (5.7)

$$\mathbf{Z}_0 = \frac{1}{2\pi}\int [\psi_a\psi_b\, dz - \bar{\psi}_a\bar{\psi}_b\, d\bar{z}]\,. \tag{5.9}$$

Also, the energy-momenta $(\mathbf{P},\bar{\mathbf{P}})$ associated with each copy are conserved separately, hence not only the total momentum

$$\big(\mathbf{P}_a + \mathbf{P}_b, \bar{\mathbf{P}}_a + \bar{\mathbf{P}}_b\big)$$

but any linear combination of the momenta $\mathbf{P}_a$ and $\mathbf{P}_b$ conserve. Here, as before, I define the light-cone components of the energy-momentum operators for each copy as

$$\mathbf{P}_a = -\frac{1}{2\pi}\int \big[T_a\, dz + \Theta_a\, d\bar{z}\big] = \frac{1}{4\pi}\int \big[\psi_a\partial\psi_a\, dz + \frac{im}{2}\,\bar{\psi}_a\psi_a\, d\bar{z}\big]\,, \tag{5.10}$$

and similarly for the copy b.

Let us define the linear combinations

$$\mathbf{X}_1 = \mathbf{P}_a - \mathbf{P}_b = \tag{5.11a}$$

$$= \frac{1}{4\pi}\int \Big[\big(\psi_a\partial\psi_a - \psi_b\partial\psi_b\big)\, dz + \big(\frac{im}{2}\bar{\psi}_a\psi_a - \frac{im}{2}\,\bar{\psi}_b\psi_b\big)\, d\bar{z}\Big]$$

$$\mathbf{X}_{-1} = \bar{\mathbf{P}}_a - \bar{\mathbf{P}}_b\,. \tag{5.11b}$$

Less trivially, the continuity equations hold for certain currents which involve both copies of the fermion field. The simplest equations of this type are

$$\bar{\partial}\big(\psi_a\partial\psi_b\big) = \frac{im}{2}\,\partial\big(\bar{\psi}_a\psi_b\big)\,,$$

$$\partial\big(\bar{\psi}_a\bar{\partial}\bar{\psi}_b\big) = -\frac{im}{2}\,\bar{\partial}\big(\psi_a\bar{\psi}_b\big)\,; \tag{5.12}$$

Hence the following integrals

$$\mathbf{Y}_1 = \frac{1}{2\pi}\int \Big[\psi_a\partial\psi_b\, dz + i\frac{m}{2}\bar{\psi}_a\psi_b\, d\bar{z}\Big]\,, \tag{5.13a}$$

$$\mathbf{Y}_{-1} = \frac{1}{2\pi}\int \Big[\bar\psi_a \bar\partial \bar\psi_b \, d\bar z - i\frac{m}{2}\psi_a\bar\psi_b \, dz\Big] \,, \tag{5.13b}$$

are conserved.

We will need commutation relations between the above generators. Some of them can be written down right away. Thus, one observes that the IM

$$\mathbf{E}_+ = \frac{1}{m}\,\frac{1}{4\pi}\int \Big[\Psi\partial\Psi \, dz + \frac{im}{2}\,\bar\Psi\Psi \, d\bar z\Big] = \big(\mathbf{X}_1 + i\,\mathbf{Y}_1\big)/m \,, \tag{5.14}$$

where $\Psi, \bar\Psi$ are the components of the above Dirac fermion (5.7), has the $U(1)$ charge $+2$ (since the fermions Dirac fermions Ψ and $\Psi^\dagger$ have the charges $+1$ and -1, resp.). Therefore

$$\big[\mathbf{Z_0}, \mathbf{E}_+\big] = 2i\,\mathbf{E}_+ \,, \tag{5.15}$$

and hence

$$\big[\mathbf{Z}_0, \mathbf{X}_1\big] = 2i\,\mathbf{Y}_1 \,, \qquad \big[\mathbf{Z}_0, \mathbf{Y}_1\big] = -2i\,\mathbf{X}_1 \,, \tag{5.16a}$$

$$\big[\mathbf{Z}_0, \mathbf{X}_{-1}\big] = 2i\,\mathbf{Y}_{-1} \,, \qquad \big[\mathbf{Z}_0, \mathbf{Y}_{-1}\big] = -2i\,\mathbf{X}_{-1} \,, \tag{5.16b}$$

where the last line follows from identical argument with the combination $\mathbf{X}_{-1} + i\,\mathbf{Y}_{-1}$ which has $U(1)$ charge -2. The commutators between $\mathbf{X}_{\pm 1}$ and $\mathbf{Y}_{\pm 1}$ can be evaluated directly

$$\big[\mathbf{X}_1, \mathbf{Y}_{-1}\big] = 2i\Big(\frac{m}{2}\Big)^2\mathbf{Z}_0 \,, \qquad \big[\mathbf{X}_{-1}, \mathbf{Y}_1\big] = 2i\Big(\frac{m}{2}\Big)^2\mathbf{Z}_0 \,, \tag{5.16c}$$

$$\big[\mathbf{X}_1, \mathbf{X}_{-1}\big] = 0 \,, \qquad \big[\mathbf{Y}_1, \mathbf{Y}_{-1}\big] = 0 \,. \tag{5.16d}$$

Let us derive, for instance, the first of the relations (5.15c). First, let us note that for any field O_a from the copy a we have [1]

$$\big[\mathbf{P}_a, O_a(x)\big] = \frac{1}{2\pi}\oint_{C_x}\Big[T_a \, dz + \Theta_a \, d\bar z\Big]\, O_a(x) = i\,\partial O_a(x) \,, \tag{5.17a}$$

$$\big[\bar{\mathbf{P}}_a, O_a(x)\big] = \frac{1}{2\pi}\oint_{C_x}\Big[\bar T_a \, d\bar z + \Theta_a \, dz\Big]\, O_a(x) = -i\,\bar\partial O_a(x) \,, \tag{5.17b}$$

but

$$\big[\mathbf{P}_b, O_a(x)\big] = \big[\bar{\mathbf{P}}_a, O_b(x)\big] = 0 \,. \tag{5.18}$$

[1] Recall that $\partial O = \big(1/2\pi i\big) \oint \big[T dz + \Theta d\bar z\big]$ and similarly (with $i \to -i$) for $\bar\partial$.

Similarly, for any O_b operators $\mathbf{P}_b$ and $\bar{\mathbf{P}}_b$ act as the derivatives, but $\mathbf{P}_a$ and $\bar{\mathbf{P}}_a$ have no action. Therefore, say

$$\big[\bar{\mathbf{P}}_a, \mathbf{Y}_1\big] = \frac{1}{2\pi}\,\big[\bar{\mathbf{P}}_a, \int\, \big(\psi_a\partial\psi_b dz + (im/2)\,\bar{\psi}_a\psi_b\,d\bar{z}\big)\big] =$$

$$= \frac{-i}{2\pi}\int\,\big(\bar{\partial}\psi_a\partial\psi_b\,dz + (im/2)\,\bar{\partial}\bar{\psi}_a\psi_b\,d\bar{z}\big) =$$

$$= \Big(-\frac{i}{2\pi}\Big)\Big(\frac{im}{2}\Big)\int\,\big[\bar{\psi}_a\,\partial\psi_b\,dz + \bar{\partial}\bar{\psi}_a\,\psi_b\,d\bar{z}\big)\,. \qquad (5.19)$$

In the last line the Dirac equation is used.

Now, when IM are defined as the integrals of local densities, it is always assumed that the integration by parts is possible (i.e. either the integration contour is closed, as in the case of operators acting on fields, or the densities decay fast at the boundaries of the integration domain, like in the case of conventional local IM which are the integrals over the equal time slice). Then (5.16) is transformed to

$$-\frac{i}{2\pi}\,\frac{im}{2}\int\,\big[-\partial\bar{\psi}_a\,\psi_b\,dz - \bar{\psi}_a\bar{\partial}\psi_b\,d\bar{z}\big] = i\left(\frac{m}{2}\right)^2\frac{1}{2\pi}\int\,\big(\psi_a\psi_b\,dz - \bar{\psi}_a\bar{\psi}_b\,d\bar{z}\big) \quad (5.20)$$

The last factor in (5.20) is $\mathbf{Z}_0$. Commutator

$$\big[\mathbf{P}_b, \mathbf{Y}_1\big]$$

yields the same result with the opposite sign (this is clear from the fact that the total momentum $\bar{\mathbf{P}}_a + \bar{\mathbf{P}}_b$ commutes with all local IM of the form we are considering (why?)). Thus we arrive at (5.16c). The rest of the commutators is obtained in similar way.

Remark: The commutator algebra generated by (5.16) is isomorphic to the $\hat{sl}(2)$ Kac-Moody algebra with the level (i.e. the Kac-Moody central charge) $k = 0$. The standard Chevalley generators $\{\mathbf{E}_\pm, \mathbf{F}_\pm, \mathbf{H}_\pm\}$ are related to the above IM as

$$\mathbf{E}_\pm = (\mathbf{X}_1 \pm i\,\mathbf{Y}_1)/m\;, \qquad \mathbf{F}_\pm = (\mathbf{X}_{-1} \mp i\,\mathbf{Y}_{-1})/m\;, \qquad \mathbf{H}_\pm = \pm\mathbf{Z}_0\;. \quad (5.21)$$

They obey the defining relations

$$\big[\mathbf{H}_\pm, \mathbf{E}_\pm\big] = 2\,\mathbf{E}_\pm\,, \qquad \big[\mathbf{H}_\pm, \mathbf{F}_\pm\big] = -2\,\mathbf{F}_\pm\,, \qquad \big[\mathbf{E}_\pm, \mathbf{F}_\pm\big] = \mathbf{H}_\pm\,, \qquad (5.22a)$$

and

$$\big[\mathbf{E}_+,\mathbf{F}_-\big] = \big[\mathbf{E}_-,\mathbf{F}_+\big] = 0\,, \qquad \mathbf{H}_+ + \mathbf{H}_- = 0\,. \tag{5.22b}$$

Also, one can check the validity of the Serre relations

$$[\mathbf{E}_\pm,[\mathbf{E}_\pm,[\mathbf{E}_\pm,\mathbf{E}_\mp]]] = 0\,, \qquad [\mathbf{F}_\pm,[\mathbf{F}_\pm,[\mathbf{F}_\pm,\mathbf{F}_\mp]]] = 0\,. \tag{5.23}$$

End of Remark

The IM we have introduced are the basic generating elements of the algebra. Further commutators of these basic elements give rise to new IM, which all are integrals of local densities, quadratic in the fermion fields. For instance, the elements defined as the commutators

$$\mathbf{Z}_2 = \big[\mathbf{X}_1,\mathbf{Y}_1\big]\,, \qquad \mathbf{Z}_{-2} = \big[\mathbf{X}_{-1},\mathbf{Y}_{-1}\big] \tag{5.24}$$

are the integrals

$$\mathbf{Z}_2 = \frac{1}{2\pi}\int \left[\partial\psi_a\partial\psi_b\,dz + \left(\frac{m}{2}\right)^2 \psi_a\psi_b\,d\bar{z}\right], \tag{5.25a}$$

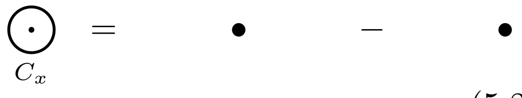
$$\mathbf{Z}_{-2} = \frac{1}{2\pi}\int \left[\bar{\partial}\bar{\psi}_a\bar{\partial}\bar{\psi}_b\,d\bar{z} + \left(\frac{m}{2}\right)^2 \bar{\psi}_a\bar{\psi}_b\,dz\right]. \tag{5.25b}$$

The symmetry of the doubled IFT generated by these IM leads to relation between the correlation functions of the theory - the Ward identities. To derive these relations we have to know how the symmetry generators act on local fields. In other words, we need to know the commutators of the generators with the local fields. If the field O is local with respect to the current $\big(A,\bar{A}\big)$ associated with the IM

$$IM = \int \left[A\,dz + \bar{A}\,d\bar{z}\right]$$

then the commutator has a local form; it is given by the integral

$$[IM, O(x)] = -\oint_{C_x} \big(A\,dz + \bar{A}\,d\bar{z}\big)\,O(x)\,.$$



(5.26)

We are interested in the correlation functions of the spin fields. Since the currents of all our IM are quadratic in the fermions, it is natural to consider the fields of the doubled theory which are products of the spin fields from different copies,

$$\sigma_a(x)\sigma_b(x)\,, \quad \sigma_a(x)\mu_b(x)\,, \quad \mu_a(x)\sigma_b(x)\,, \quad \mu_a(x)\mu_b(x)\,. \tag{5.27}$$

These fields are local with respect to the currents, and the contour in the integrals of the type (5.26) is closed for O of this form.

The commutators with the generators $\mathbf{X}_\pm$ can be written down immediately. These generators are differences of the momenta of the two copies, i.e. they generate infinitesimal shifts of all coordinates in the directions which are opposite in the two copies. For any two fields O_a and O_b from the copies a and b, respectively, we have

$$\begin{aligned}\big[\mathbf{X}_1, O_a(x)O_b(x)\big] &= i\,\partial O_a(x)O_b(x) - i\,\mathcal{O}_a(x)\partial\mathcal{O}_b(x)\,,\\ \big[\mathbf{X}_{-1}, O_a(x)O_b(x)\big] &= -i\,\bar\partial O_a(x)O_b(x) + i\,\mathcal{O}_a(x)\bar\partial\mathcal{O}_b(x)\,.\end{aligned} \tag{5.28}$$

Derivations of the other commutators are less elementary but still straightforward. Again, for any $O_a \in \mathcal{R}_a$ and $O_b \in \mathcal{R}_b$ one can write as we did before for a single IFT:

$$\begin{aligned}\begin{pmatrix}\psi_a(x)\\ \bar\psi_a(x)\end{pmatrix} O_a(0) &= \sum_{n\in Z}\left[\begin{pmatrix}u_n(x)\\ \bar u_n(x)\end{pmatrix} a_{-n} + \begin{pmatrix}v_n(x)\\ \bar v_n(x)\end{pmatrix}\bar a_{-n}\right] O_a(0)\,,\\ \begin{pmatrix}\psi_b(x)\\ \bar\psi_b(x)\end{pmatrix} O_b(0) &= \sum_{n\in Z}\left[\begin{pmatrix}u_n(x)\\ \bar u_n(x)\end{pmatrix} b_{-n} + \begin{pmatrix}v_n(x)\\ \bar v_n(x)\end{pmatrix}\bar b_{-n}\right] O_b(0)\,,\end{aligned} \tag{5.29}$$

which is the definition of two sets of operators $a_n, \bar a_n$ and $b_n, \bar b_n$ associated with the two copies. These operators obey the same canonical anti-commutation relations, for example

$$\{a_n, a_m\} = \delta_{n+m,0}\,, \qquad \{b_n, b_m\} = \delta_{n+m,0}\,, \qquad \{a_n, b_m\} = 0\,, \tag{5.30}$$

and a's commute with b's, and identically for $\bar a$'s and $\bar b$'s. Then one can express the action of the generators on the products of the type O_aO_b through the actions of these mode operators.

I am not going to repeat these calculation here, because it is not much different from the derivation of the action of $(\mathbf{P}, \bar{\mathbf{P}})$ in a single IFT which we did the last time. For instance, for the generator $\mathbf{Z}_0$ straightforward calculation leads to

$$\big[\mathbf{Z}_0, O_aO_b\big] = -i\left(a_0b_0 + \bar a_0\bar b_0 + \sum_{n=1}^{\infty}\big(a_{-n}b_n - b_{-n}a_n + \bar a_{-n}\bar b_n - \bar b_{-n}\bar a_n\big)\right) O_aO_b\,. \tag{5.31}$$

Similar expressions can be derived for the actions of the rest of the generating elements.

Exercise: Check that

$$\mathbf{Y}_1 = -i \sum_{n=0}^{\infty} \left[(n+1/2)\left(a_{-n-1}b_n + b_{-n-1}a_n\right) + \left(\frac{m}{2}\right)^2 \frac{1}{n+\frac{1}{2}} \left(\bar{a}_{-n}\bar{b}_{n+1} + \bar{b}_{-n}\bar{a}_{n+1}\right)\right] \tag{5.32}$$

and derive the action of $\mathbf{Z}_2$ in terms of the a and b operators.

The relations simplify when the basic fields (5.27), i.e. $\sigma_a\sigma_b$, etc, are taken, since the positive modes a_n, b_n, etc, with $n > 0$, annihilate these fields. For instance

$$\begin{aligned}\left[\mathbf{Z}_0, \sigma_a\sigma_b\right] &= -i\left(a_0 b_0 + \bar{a}_0\bar{b}_0\right)\sigma_a\sigma_b = -i\left(a_0\sigma_a\ b_0\sigma_b + \bar{a}_0\sigma_a\ \bar{b}_0\sigma_b\right) = \\ &= -i\left(\frac{\omega}{\sqrt{2}}\mu_a\ \frac{\omega}{\sqrt{2}}\mu_b + \frac{\bar{\omega}}{\sqrt{2}}\mu_a\ \frac{\bar{\omega}}{\sqrt{2}}\mu_b\right) = 0\,.\end{aligned} \tag{5.33}$$

But

$$\begin{aligned}\left[\mathbf{Z}_0, \sigma_a\mu_b\right] &= -i\left(a_0\sigma_a\ b_0\mu_b + \bar{a}_0\sigma_a\ \bar{b}_0\mu_b\right) = \\ = -i\left(\frac{\omega}{\sqrt{2}}\ \mu_a\ \frac{\bar{\omega}}{\sqrt{2}}\ \sigma_b + \frac{\bar{\omega}}{\sqrt{2}}\ \mu_a\ \frac{\omega}{\sqrt{2}}\ \sigma_b\right) &= -i\,\mu_a\,\sigma_b\,.\end{aligned} \tag{5.34}$$

Similar way one finds

$$\left[\mathbf{Z}_0, \mu_a\mu_b\right] = 0\,, \qquad \left[\mathbf{Z}_0, \mu_a\sigma_b\right] = i\,\sigma_a\,\mu_b\,. \tag{5.35}$$

Similar analysis allows one to derive the all basic commutation relations. They have the form

$$[\mathbf{Y}_1, \sigma_a\sigma_b] = -\partial\mu_a\,\mu_b + \mu_a\partial\mu_b\,, \qquad [\mathbf{Y}_1, \mu_a\mu_b] = -\partial\sigma_a\,\sigma_b + \sigma_a\partial\sigma_b\,, \tag{5.36a}$$

$$[\mathbf{Y}_1, \sigma_a\mu_b] = i\,\partial\mu_a\,\sigma_b - i\,\mu_a\partial\sigma_b\,, \qquad [\mathbf{Y}_1, \mu_a\sigma_b] = -i\,\partial\sigma_a\,\mu_b + i\,\sigma_a\partial\mu_b\,. \tag{5.36b}$$

The generators $\mathbf{Y}_{-1}$ obey the same commutation relations with obvious changes:

$$\mathbf{Y}_{-1}: \quad (\partial \to \bar{\partial}\ \text{ and } i \to -i)\,. \tag{5.37}$$

Let me also display some commutation relations involving the operator $\mathbf{Z}_2 = -(i/2)\,[\mathbf{X}_1, \mathbf{Y}_1]$,

$$[\mathbf{Z}_2, \sigma_a\sigma_b] = 2\,\partial\mu_a\,\partial\mu_b - \partial^2\mu_a\,\mu_b - \mu_a\,\partial^2\mu_b\,,$$

$$[\mathbf{Z}_2, \mu_a \mu_b] = 2\, \partial\sigma_a\, \partial\sigma_b - \partial^2\sigma_a\, \sigma_b - \sigma_a\, \partial^2\sigma_b\,. \tag{5.38}$$

Now we are prepared to deriving closed differential equations for the correlation functions. I will use the following notations

$$G(x) = \langle\, \sigma(x)\sigma(0)\, \rangle\,, \qquad \tilde{G}(x) = \langle\, \mu(x)\mu(0)\, \rangle\,. \tag{3.39}$$

Consider first the following identity

$$\langle\, \big[\mathbf{Z}_2, \sigma_a(x)\sigma_b(x)\, \mu_a(0)\mu_b(0)\big]\, \rangle = 0\,. \tag{3.40}$$

For the correlation functions, the commutator here is understood as the contour integral

$$\big[\mathbf{Z}_2, O_1(x_1)\cdots O_n(x_n)\big] = \frac{1}{2\pi} \oint_C \left[\partial\psi_a\partial\psi_b\, dz + \left(\frac{m}{2}\right)^2 \psi_a\psi_b\, d\bar{z}\right] O_1(x_1)\cdots O_n(x_n)\,, \tag{3.41}$$

where C encircles *all* the insertion points $x_1, \cdots, x_n$, as in **Fig.1**

The expression (3.40) equals to zero because the contour can be moved away to infinity. In the operator formalism the correlation function appears as the vacuum expectation value,

$$\langle \ldots \rangle = \langle\, 0 \mid \ldots \mid 0\, \rangle\,,$$

and the above statement follows from

$$\mathbf{Z}_2 \mid 0\, \rangle = 0\,. \tag{3.42}$$

Now, the contour C can be split into the sum of small contours each surrounding one of the points x_i. Or, equivalently, the commutator in (5.31) is written as the sum of commutators each involving one of the insertions $O_i(x_i)$. Either way, the identity (3.40) leads to

$$0 = \langle\, \big[\mathbf{Z}_2, \sigma_a(x)\sigma_b(x)\big]\, \mu_a(0)\mu_b(0)\, \rangle + \langle\, \sigma_a(x)\sigma_b(x)\, \big[\mathbf{Z}_2, \mu_a(0)\mu_b(0)\big]\, \rangle\,. \tag{5.43}$$

The elementary commutators in (5.43) are already known, and we find

$$0 = \langle\, \big(2\partial\mu_a(x)\partial\mu_b(x) - \partial^2\mu_a(x)\mu_b(x) - \mu_a(x)\partial^2\mu_b(x)\big)\, \mu_a(0)\mu_b(0)\,\rangle +$$

$$+\langle\, \sigma_a(x)\sigma_b(x)\, \big(2\partial\sigma_a(0)\partial\sigma_b(0) - \partial^2\sigma_a(0)\sigma_b(0) - \sigma_a(0)\partial^2\sigma_b(0)\big)\,\rangle\,. \qquad (5.44)$$

Each term here can be factorized in terms of the correlation functions of single-copy IFT. In doing so one has to be careful to keep track of the Klein factors which make the operators μ from different copies anti-commute (remember, we have made the change $\mu_a \to \eta_a\mu_a$, etc); the result is the equation

$$\partial G\,\partial G - G\,\partial^2 G - \partial\tilde{G}\,\partial\tilde{G} + \tilde{G}\,\partial^2\tilde{G} = 0\,. \qquad (5.45)$$

Of course, similar equation with ∂ replaced by $\bar{\partial}$ are derived if one puts $\mathbf{Z}_{-2}$ instead of $\mathbf{Z}_2$ in (5.40).

Eq.(5.45) is just one of many equations which follow from the symmetry of the doubled IFT. Let me demonstrate simple way to derive such relations. Assuming

$$\mathbf{X}_{\pm 1}\mid 0\,\rangle = \mathbf{Y}_{\pm 1}\mid 0\,\rangle = 0\,, \qquad (5.46)$$

we have

$$\langle\, 0\mid\, [\mathbf{X}_{-1}, A][\mathbf{Y}_1, B] - [\mathbf{Y}_1, A][\mathbf{X}_{-1}, B]\,\mid 0\,\rangle = \langle\, 0\mid\, A\,[\mathbf{Y}_1, \mathbf{X}_{-1}]\,B\,\mid 0\,\rangle \qquad (5.47)$$

for any A, B; this is identity. Setting here $A = \sigma_a(x)\sigma_b(x)$ and $B = \mu_a(0)\mu_b(0)$ and using the commutators we have

$$\langle\, \big(-i\bar{\partial}\sigma_a(x)\,\sigma_b(x) + i\sigma_a(x)\,\bar{\partial}\sigma_b(x)\big)\big(-\partial\sigma_a(0)\,\sigma_b(0) + \sigma_a(0)\partial\sigma_b(0)\big)\,\rangle -$$

$$-\langle\, \big(-\partial\mu_a(x)\,\mu_b(x) + \mu_a(x)\,\partial\mu_b(x)\big)\big(-i\bar{\partial}\mu_a(0)\,\mu_b(0) + i\mu_a(0)\,\bar{\partial}\mu_b(0)\big)\,\rangle =$$

$$= -i\frac{m^2}{2}\,\langle\, \sigma_a(x)\sigma_b(x)\,\mathbf{Z}_0\,\mu_a(0)\mu_b(0)\,\rangle\,, \qquad (5.48)$$

where in the r.h.s. the commutation relation

$$[\mathbf{Y}_1, \mathbf{X}_{-1}] = -i\frac{m^2}{2}\,\mathbf{Z}_0 \qquad (5.49)$$

was used.

Since we have found already that $\mathbf{Z}_0$ commutes with $\sigma_a\sigma_b$, (see (5.33) above) the r.h.s. of (5.48) is zero. Factorizing the l.h.s into pieces associated with each copy, one derives from (5.48)

$$G\,\partial\bar{\partial}G - \partial G\,\bar{\partial}G + \tilde{G}\,\partial\bar{\partial}\tilde{G} - \partial\tilde{G}\,\bar{\partial}\tilde{G} = 0\,. \qquad (5.49)$$

Yet another equation can be derived if one takes $A = \sigma_a(x)\mu_b(x)$ and $B = \mu_a(0)\sigma_b(0)$. The l.h.s. of (5.47) is evaluated similarly, with the use of the commutators (5.36) of $\mathbf{X}$ and $\mathbf{Y}$ found above. But the r.h.s is not zero now, because $\mathbf{Z}_0$ does not commute with the mixed product $A = \sigma_a\mu_b$. Instead, as we have found in (5.34),

$$[\mathbf{Z}_0, \sigma_a\mu_b] = -i\,\mu_a\sigma_b\,.$$

With this, the r.h.s. of (5.47) can be evaluated by moving $\mathbf{Z}_0$ to the left, yielding

$$i\,\frac{m^2}{2}\,\langle\,[\mathbf{Z}_0, \sigma_a(x)\mu_b(x)]\,\mu_a(0)\sigma_b(0)\,\rangle = \frac{m^2}{2}\,\langle\,\mu_a(x)\sigma_b(x)\mu_a(0)\sigma_b(0)\,\rangle = \frac{m^2}{2}\,G\,\tilde{G}\,. \tag{5.50}$$

Thus one derives

$$G\,\partial\bar{\partial}\tilde{G} - \partial G\,\bar{\partial}\tilde{G} + \tilde{G}\,\partial\bar{\partial}G - \bar{\partial}G\,\partial\tilde{G} = \left(\frac{m}{2}\right)^2 G\,\tilde{G}\,. \tag{5.51a}$$

I list again the other two equations we have obtained above:

$$G\,\partial\bar{\partial}G - \partial G\,\bar{\partial}G + \tilde{G}\,\partial\bar{\partial}\tilde{G} - \partial\tilde{G}\,\bar{\partial}\tilde{G} = 0\,. \tag{5.51b}$$

$$\partial G\,\partial G - G\,\partial^2 G - \partial\tilde{G}\,\partial\tilde{G} + \tilde{G}\,\partial^2\tilde{G} = 0\,. \tag{5.51c}$$

The above equations are known as the quadratic form of of the famous differential equations first derived by Wu, McCoy, Tracy and Barouch in 1976. They can be brought to somewhat nicer form by making substitution

$$G = m^{\frac{1}{4}}\,e^{\chi/2}\,\cosh\big(\varphi/2\big)\,, \qquad \tilde{G} = m^{\frac{1}{4}}\,e^{\chi/2}\,\sinh\big(\varphi/2\big)\,, \tag{5.52}$$

in terms of two auxiliary functions $\chi(x)$ and $\varphi(x)$. The equations (5.51) reduce to

$$\partial\bar{\partial}\chi = \frac{m^2}{8}\Big[1 - \cosh(2\varphi)\Big]\,, \qquad \partial^2\chi + \big(\partial\varphi\big)^2 = 0 \quad \text{and} \quad \partial \to \bar{\partial}\,, \tag{5.53}$$

and

$$\partial\bar{\partial}\varphi = \frac{m^2}{8}\,\sinh\big(2\varphi\big)\,. \tag{5.54}$$

I will postpone detailed discussion of what kind of solutions of these equations we are interested in in order to find the correlation functions, and how to analyze these solutions. At the moment I just want to note that the last equation involves only one of the functions $\varphi(x)$. It is the Euclidean version of the well-known *sinh-Gordon* equation, which is known to be integrable field equation. I want to stress that here I am talking about the *classical* sinh-Gordon equation. Although its quantum field theory version exists, it needs not to concern us now, we are interested in solutions of classical field equations (5.53),(5.54). The equation (5.54) admits a "Lax pair", or "zero curvature" representation. Namely, consider a system of linear equations for a pair of functions $\Psi_+(x),\ \Psi_-(x)$,

$$\partial\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\,\partial\varphi & \lambda\, e^{\varphi}\\ -\lambda e^{-\varphi} & \frac{1}{2}\,\partial\varphi\end{pmatrix}\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}, \quad \bar\partial\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\,\bar\partial\varphi & -\,\bar\lambda\, e^{-\varphi}\\ \bar\lambda e^{\varphi} & -\,\frac{1}{2}\,\partial\varphi\end{pmatrix}\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}. \tag{5.55}$$

Here λ and $\bar\lambda$ are complex parameters such that

$$\lambda\bar\lambda = \frac{m^2}{16} \quad \text{i.e.} \quad \bar\lambda = m^2/16\lambda\,; \tag{5.56}$$

λ is usually referred to as the "spectral parameter". The two equations in (5.55) are compatible at any value of λ (i.e. the $SL(2)$ connection defined by these equations is flat) if $\varphi(x)$ satisfies (5.54), and vice versa.

Just like I said, I'll come back to these equations a bit later, but now I want to discuss the particle theory associated with the free fermion theory we are looking at. Of course, it is a theory of free particles, but we will be interested in various matrix elements of the spin fields between these free particle states.

First of all, I want to make a trivial remark. When one wants to quantize a field theory using operator formalism, one has to define a space of states the field operators act in. Usually there is more then one choice. One has to choose a space-like hyper-surface of co-dimension one (in our case of 2D space time it is a curve), sometimes referred to as the "equal time slice". In classical theory it is the surface where the initial data are specified. In quantum theory the space of states is associated to such a surface. Before, I have used the space $\mathcal{F}$ of local fields, and have defined various operators actin there. Obviously, the "equal-time slice" associated with this space is a small circle surrounding a given point x, the point where the fields from $\mathcal{F}$ are inserted, see **Fig.2**. In this picture the "Euclidean time" is the radial coordinate, for this reason this choice of the space of states is often called the "radial quantization".

But if our field theory is defined on an infinite plane $\mathbf{R}^2$, there is more conventional choice, which is to take just infinite line for the equal-time slice. One chooses arbitrary direction on the plane, say the y-axis, for the Euclidean time, and then the equal-time slice is any line $y =$const. This choice leads to a conventional quantization which is done in the beginning of virtually any textbook on QFT.

In the case of Conformal Field Theory the "radial quantization" and the "textbook quantization" lead to identical spaces of states, there is an isomorphism between the two. This is because when one considers CFT on a plane it is usually assumed that the plane is compactified to a sphere (this means that certain rigid asymptotic conditions at infinity are imposed on the fields), so that conventional equal-time line is a circle, and on a sphere any circle can be transformed to another given circle by suitable conformal transformation. When the theory is *not* conformally invariant, the spaces of states appearing from the "radial" and from the "textbook" quantizations are certainly different.

Remark: Let me say yet another few words about CFT. From many points of view, and especially in string theory applications, it is important to study CFT defined on the manifolds different from sphere, in particular on compact surfaces of higher genius. This is developed business, and much insight was gained from these analysis (I refer again to the fat yellow book by DiFrancesco, Mathieu and Senechal). But it also might be interesting to look at the CFT on a non-compact surface, and understand the structure of the space of states associated with non-compact equal-time slices. This question is closely related to the question which could have been risen when discussing possible analytic properties of the holomorphic energy-momentum tensor $T(z)$ in CFT. We have found that the correlation function $\langle T(z) \cdots \rangle$ was a single-valued function with isolated singularities which I said were poles. But from mathematical point of view nothing prevents from having more complicated single-valued essential singularities. I think it is interesting open question if such possibilities have any physical significance. If so, it would probably lead to new structures associated with CFT; for instance, we would now have a possibility to choose for quantization a contour which starts and ends at such essential singularity; structure of associated space of states might be sort of interesting. **End of Remark**.

OK, let me now do the textbook quantization of my free fermion theory. I'll take the x-axis for the equal-time slice. Appropriate solutions of the Dirac equations which are bounded at $|x| \to \infty$ are plane waves, each characterized by the wave vector p which is to become the particle momentum. In fact, in 2D it is very convenient to use the so-called rapidity parametrization of the momenta. Thus, for the energy-momentum $p^\mu = (p^0, p^1) = (\omega, p)$ of a particle of mass m we write the components in terms of the rapidity θ,

$$p^0 = m\,\cosh\theta\,, \qquad p^1 = m\,\sinh\theta\,, \tag{5.57}$$

so that the mass-shell condition

$$(p^0)^2 - (p^1)^2 = m^2$$

is automatically satisfied.

I denote $\mathcal{H}$ the space of states of the free fermion theory associated with the textbook quantization. Since I still want to take advantage of the "doubling" of the IFT, our actual space is $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$. We write

$$\begin{pmatrix} \psi_a(x) \\ \bar\psi_a(x) \end{pmatrix} = \sqrt{\frac{m}{2}} \int_{-\infty}^{\infty} \frac{d\theta}{\sqrt{2\pi}} \Big[A^\dagger(\theta) \begin{pmatrix} e^{\theta/2} \\ e^{-\theta/2} \end{pmatrix} e^{\mathrm{y}\, m\cosh\theta - i\mathrm{x}\, m\sinh\theta} +$$

$$+ A(\theta) \begin{pmatrix} -i\, e^{\theta/2} \\ i\, e^{-\theta/2} \end{pmatrix} e^{-\mathrm{y}\, m\cosh\theta + i\mathrm{x}\, m\sinh\theta} \Big] \qquad (5.58)\,,$$

where (x, y) are Cartesian coordinates of the point x, and $A^\dagger(\theta),\ A(\theta)$ are creation and annihilation operators of the particles of the copy a. All the details of the plane waves are obtained by solving the Dirac equation in a usual manner. Similar expression for the fields $\psi_b(x), \bar\psi_b(x)$ can be written down, with associated creation and annihilation operators $B^\dagger(\theta),\ B(\theta)$ replacing $A^\dagger(\theta),\ A(\theta)$. The operators obey canonical anti-commutation relations

$$\{A^\dagger(\theta), A(\theta')\} = 2\pi\,\delta(\theta - \theta')\,, \quad \{A^\dagger(\theta), A^\dagger(\theta')\} = \{A(\theta), A(\theta')\} = 0\,. \tag{5.59}$$

The operators $B^\dagger(\theta),\ B(\theta)$ satisfy identical relations, and A and B anti-commute.

Note that these relations are similar but different from the relations for the mode operators $a_n,\ \bar a_n$ in the radial quantization; in particular, this time the operators are labelled by continuous rapidity variable θ.

The space $\mathcal{H}_a$ of a single copy IFT is the Fock space built on the vacuum $\mid 0\,\rangle_a$ which satisfies

$$A(\theta) \mid 0\,\rangle = 0 \quad \text{for all} \quad \theta\,. \tag{5.60}$$

The states in $\mathcal{H}_a$ are N-particle states of the type

$$\mid A(\theta_1)\dots A(\theta_N)\,\rangle = A^\dagger(\theta_1)\dots A^\dagger(\theta_N)\ \mid 0\,\rangle_a\,. \tag{5.61}$$

The space $\mathcal{H}_b$ is the identical Fock space for the algebra of the operators $B^\dagger,\ B$. Generic states in $\mathcal{H}$ are $N+M$-particle states with N particles A and M particles B.

The conserved currents of the doubled IFT give rise to operators $\mathbf{Z}_0$, $\mathbf{X}_{\pm 1}$ and $\mathbf{Y}_{\pm 1}$ acting in $\mathcal{H}$, which are integrals of motion. These operators are defined by the contour integrals of the currents, but now we take the contour to be the "equal-time slice" associated with the textbook quantization, that is the x-axis. For instance since for this slice

$$dz = d\bar{z} = d\mathrm{x} \tag{5.62}$$

we have

$$\mathbf{Z}_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\psi_a\psi_b - \bar{\psi}_a\bar{\psi}_b\right] d\mathrm{x}\,, \tag{5.63}$$

and similarly for other generators. The action of these generators in $\mathcal{H}$ can be expressed through the basic fermion creation and annihilation operators; one just uses the plane-wave decompositions of ψ_a and ψ_b in the expressions like (5.63). Thus, for instance, from (5.63)

$$\mathbf{Z}_0 = -i\int_{-\infty}^{\infty} \Big[A^\dagger(\theta)B(\theta) - B^\dagger(\theta)A(\theta)\Big]\,\frac{d\theta}{2\pi}\,. \tag{5.64}$$

Likewise

$$\mathbf{Y}_1 = \frac{m}{2}\int_{-\infty}^{\infty} e^{\theta}\,\Big[A^\dagger(\theta)B(\theta) + B^\dagger(\theta)A(\theta)\Big]\,\frac{d\theta}{2\pi}\,. \tag{5.65}$$

The expression for $\mathbf{Y}_{-1}$ is the same as this one with e^{θ} replaced by $e^{-\theta}$,

$$\mathbf{Y}_{-1}\ :\qquad e^{\theta} \to e^{-\theta}\,. \tag{5.66}$$

In this picture of quantization the local fields $O(x)$ are the Heisenberg field operators and we are naturally interested in the matrix elements of these operators between the particle states. Let me show how the general structure of such matrix elements come out from the symmetry of the doubled IFT.

First of all, to simplify arguments, let me make some physical input. Let us assume that m is positive and (since $m \sim K - K_c$) our IFT corresponds to the scaling limit of the Ising model with $T < T_c$. In this regime we expect that the model exhibits spontaneous magnetization, i.e. that

$$\langle\, \sigma(x)\,\rangle = \langle\, 0 \mid \sigma(x) \mid 0\,\rangle \neq 0\,.$$

More precisely, it means that the theory in the infinite volume has two exactly degenerate and orthogonal ground states $\mid 0_\pm\,\rangle$ such that

$$\langle\, 0_- \mid 0_+\,\rangle = 0\,, \quad \langle\, 0_\pm \mid \sigma(x) \mid 0_\pm\,\rangle = \pm\bar{\sigma} \neq 0\,, \tag{5.67}$$

where $\bar{\sigma}$ is some constant (we are going to evaluate it shortly). We can choose the space based on $\mid 0_+\,\rangle$ so that

$$\langle\, \sigma(x)\,\rangle = \bar{\sigma}\,. \tag{5.68}$$

At the same time the expectation value of $\mu(x)$ must be zero. The field $\mu(x)$ is dual to $\sigma(x)$, so its behavior at $T < T_c$ is identical to the behavior of $\sigma(x)$ at $T > T_c$ which is not expected to develop non-zero expectation value. This can be confirmed on the formal level, by taking the identity

$$0 = \langle\, [\mathbf{Z}_0, \sigma_a(x)\mu_b(x)]\,\rangle = -i\,\langle\, \mu_a(x)\,\rangle\langle\, \sigma_b(x)\,\rangle\,, \tag{5.69}$$

so that either $\langle\sigma\rangle$ or $\langle\sigma\rangle$ must vanish in both phases, i.e. at $m > 0$

$$\langle\, \mu(x)\,\rangle = 0\,. \tag{5.70}$$

Now, what about the matrix elements of σ and mu between the states involving particles? Our qualitative discussion of the Ising model has made us to conclude that in the low-T phase, i.e. at $m > 0$, the particles are the "domain walls" separating domains of two vacua with positive and negative magnetization. The local field $\sigma(x)$ cannot emit such particle, hence we expect that for $m > 0$ the matrix elements

$$\langle\, 0 \mid \sigma(x) \mid A(\theta)\,\rangle = 0 \tag{5.71}$$

vanish. On the other hand, since at $m > 0$ the disorder field $\mu(x)$ behaves as $\sigma(x)$ does at $m < 0$ (i.e. at high-T), and we have seen that in the high-T regime σ does emit particle, we expect that the matrix element

$$\langle\, 0 \mid \mu(x) \mid A(\theta)\,\rangle = \bar{\mu} \neq 0 \tag{5.72}$$

does not vanish. By Lorentz invariance it does not depend on θ, so it is some nonzero constant (again, we will compute it soon).

These data are sufficient to find all matrix elements of the type

$$\langle\, A(\theta_1') \cdots A(\theta_M')\mid O(x) \mid A(\theta_1)\cdots A(\theta_N)\,\rangle\,. \tag{5.73}$$

where O is either σ or μ, using the symmetry generators of the doubled IFT. Let me demonstrate how it works for more simple class of matrix elements

$$F_N^O(\theta_1,\cdots,\theta_N) = \langle\, 0 \mid O(x) \mid A(\theta_1)\cdots A(\theta_N)\,\rangle\,. \tag{5.74}$$

For some historical reason in 2D QFT such matrix elements are generally referred to as the "form-factors".

Consider the doubled IFT, and take the identity

$$0 = \langle\, 0 \mid \mathbf{Y}_1\, O_{ab}(0) \mid A(\theta_1)\cdots A(\theta_N)\,\rangle\,, \tag{5.75}$$

where O_{ab} will be a product of spin fields like $\mu_a\mu_b$ or $\sigma_a\mu_b$. This can be transformed by moving the operator $\mathbf{Y}_1$ to the right of O_{ab}, and then computing its action on the state by using (5.65),

$$\frac{m}{2}\int_{-\infty}^{\infty}\frac{e^\theta\, d\theta}{2\pi}\Big[A^\dagger(\theta)B(\theta)+B^\dagger(\theta)A(\theta)\Big] \mid A(\theta_1)\cdots A(\theta_N)\,\rangle =$$

$$= \frac{m}{2}\sum_{j=1}^{N} e^{\theta_j}\,\mid A(\theta_1)\cdots A(\theta_{j-1})B(\theta_j)A(\theta_{j+1})\cdots A(\theta_N)\,\rangle\,, \tag{5.76}$$

i.e. it acts by replacing one of the A-particles by the B particle of the same rapidity, and multiplying by corresponding e^θ.

Let us take $O_{ab} = \mu_a\mu_b$. The term involving the state (5.76) in this case factorizes as

$$\frac{m}{2}\,\bar\mu\sum_{j=1}^{N}(-)^{j-1}\, e^{\theta_j}\, F_{N-1}^{\mu}(\theta_1,\cdots,\theta_{j-1},\theta_{j+1},\cdots,\theta_N)\,, \tag{5.77}$$

Also, since

$$[\mathbf{Y}_1, \mu_a\mu_b] = -\partial\sigma_a\,\sigma_b + \sigma_a\,\partial\sigma_b \tag{5.77}$$

the commutator term yields

$$-\langle\, 0 \mid \partial\sigma_a \mid A(\theta_1)\cdots A(\theta_N)\rangle \;\; \langle\, 0 \;\mid \sigma_b \mid 0\,\rangle + 0\,, \tag{5.78}$$

where the second term vanishes since it involves

$$\langle\, \partial\sigma_b\,\rangle = 0\,. \tag{5.79}$$

The derivative of the matrix element in (5.78) can be replaced by the factor

$$\sum_{j=1}^{N} i\,\frac{m}{2}\,e^{\theta_j}$$

since ∂ is generated by the **P** component of the total momentum [2]. Thus, the identity (5.75) leads to the relation

$$\bar{\sigma}\left(\sum_{j=1}^{N} e^{\theta_j}\right) F_N^{\sigma}(\theta_1,\cdots,\theta_N) =$$

$$= i\,\bar{\mu}\sum_{j=1}^{N}(-)^{j-1}\,e^{\theta_j}\,F_{N-1}^{\mu}(\theta_1,\cdots,\theta_{j-1},\theta_{j+1},\cdots,\theta_N)\,, \tag{5.80}$$

Putting in (5.75) $O_{ab} = \sigma_a\mu_b$ and repeating these steps one finds similar relation

$$\bar{\sigma}\left(\sum_{j=1}^{N} e^{\theta_j}\right) F_N^{\mu}(\theta_1,\cdots,\theta_N) =$$

$$= \bar{\mu}\sum_{j=1}^{N}(-)^{j-1}\,e^{\theta_j}\,F_{N-1}^{\sigma}(\theta_1,\cdots,\theta_{j-1},\theta_{j+1},\cdots,\theta_N)\,, \tag{5.81}$$

The equations (5.80) and (5.81) are the recurrent relations which explicitly express the N-particle form-factors in terms of the $N-1$-particle ones. Ones the

[2] We have $\partial O = -i[\mathbf{P}, O]$ so that $\langle\, 0 \mid \partial O \mid A(\theta_1)\cdots A(\theta_N)\,\rangle = i\,\langle\, 0 \mid O\,\mathbf{P} \mid A(\theta_1)\cdots A(\theta_N)\,\rangle$. We have also $\mathbf{P} \mid A(\theta)\,\rangle = \frac{m}{2}\,e^{\theta} \mid A(\theta)\,\rangle$.

constants $\bar\sigma$ and $\bar\mu$ are given, they determine all the form-factors uniquely. In particular, one finds that in our case of $m > 0$

$$\begin{aligned} F^{\sigma}_N &= 0 \quad \text{for any odd} \quad N\,, \\ F^{\mu}_N &= 0 \quad \text{for any even} \quad N\,, \end{aligned} \tag{5.82}$$

as one expects from qualitative considerations. Moreover, it is possible to show that the nonzero form-factors both for σ and μ are given by the products

$$F_N(\theta_1, \cdots, \theta_N) = i^{[N/2]}\,\bar\sigma\,g^N\,\prod_{i<j}\tanh\left(\frac{\theta_i - \theta_j}{2}\right), \tag{5.83}$$

where $[N/2]$ is the integer part of $N/2$, and $g = \bar\mu/\bar\sigma$ (we will see that in the infinite system $\bar\mu = \bar\sigma$ so that $g = 1$).