L4

The last time we have considered basic properties of the Conformal field theory (CFT) associated with the Ising field theory with zero magnetic field and zero mass parameter $m = 0$ (which corresponds to $K = K_c$). At $m = 0$ the EM tensor is traceless and its independent components

$$T(z) = -\frac{1}{2} \, :\psi\partial\psi:(z) \, , \qquad \bar{T}(z) = -\frac{1}{2} \, :\bar{\psi}\bar{\partial}\bar{\psi}:(z) \tag{4.1}$$

generate two copies of the Virasoro algebra

$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{c}{12}\,(n^3 - n)\,\delta_{n+m,0} \tag{4.2}$$

with

$$c = \frac{1}{2} \, . \tag{4.3}$$

We have found that the action of L_n's in the space $\mathcal{R}$ of spin fields can be expressed through the fermion mode operators a_n. For any $O \in \mathcal{R}$

$$L_{n+m}\, O(0) = \frac{1}{4}\,(1/4 - m^2)\,\delta_{n+m,0}\, O(0) +$$

$$+\frac{1}{2} \sum_{k=0}^{\infty} k \left[a_{m-k}a_{n+k} + a_{n-k}a_{m+k}\right] O(0) \, . \tag{4.4}$$

Setting here, for instance, $n = m = 0$ we have

$$L_0\, O = \left\{ \frac{1}{16} + \sum_{k=0}^{\infty} k \ a_{-k}a_k \right\} O \, . \tag{4.5}$$

Recall now that at the bottom of the space $\mathcal{R}$ there is Fock vacuum, in fact two-dimensional space of vacua spanned by the order field σ and its dual disorder field μ, which satisfy

$$a_n\, \sigma = 0 \, , \qquad a_n\, \mu = 0 \quad \text{for all} \quad n > 0 \, , \tag{4.6}$$

and the same relations with $\bar{a}_n,\ n>0$, and also

$$a_0\sigma = \frac{\omega}{\sqrt{2}}\,\mu\,, \qquad a_0\mu = \frac{\bar{\omega}}{\sqrt{2}}\,\sigma\,,$$

$$\bar{a}_0\sigma = \frac{\bar{\omega}}{\sqrt{2}}\,\mu\,, \qquad \bar{a}_0\mu = \frac{\omega}{\sqrt{2}}\,\sigma\,, \tag{4.7}$$

where $\omega = e^{\frac{i\pi}{4}}$, $\bar{\omega} = e^{-\frac{i\pi}{4}}$. We find from (4.5)

$$L_0\,\sigma = \bar{L}_0\,\sigma = \frac{1}{16}\,\sigma\,, \qquad L_0\,\mu = \bar{L}_0\,\mu = \frac{1}{16}\,\mu\,. \tag{4.8}$$

It is also elementary exercise to verify from (4.4) that

$$L_n\,\sigma = L_n\,\mu = 0 \qquad \text{for} \quad n>0 \tag{4.9}$$

(and the same with $\bar{L}$'s), i.e. σ and μ are conformal primary fields with the conformal dimensions

$$\big(\Delta_\sigma, \bar{\Delta}_\sigma\big) = \big(\Delta_\mu, \bar{\Delta}_\mu\big) = \big(1/16, 1/16\big)\,, \tag{4.10}$$

In other words, both have spin zero and scale dimension 1/8. The dimensions determine the two-point correlation functions up to overall constant. To simplify notations, I will assume that the fields $\sigma(x)$ and $\mu(x)$ are normalized in such a way that in the CIFT (i.e. at $m=0$)

$$\langle\,\sigma(x)\,\sigma(x')\,\rangle = \frac{1}{|x-x'|^{\frac{1}{4}}}\,, \qquad \langle\,\mu(x)\,\mu(x')\,\rangle = \frac{1}{|x-x'|^{\frac{1}{4}}}\,, \tag{4.11}$$

with the coefficient 1.

Remark: On relation between lattice $\sigma_{\mathbf{x}}$ and IFT $\sigma(x)$. **End of Remark**.

Let us now take the relation (4.4) with $n=-1,\ m=0$, and the same relation with $n=-1,\ m=-1$

$$L_{-1}\,O = \sum_{k=0}^{\infty}\big(k{+}1/2\big)\,a_{-k-1}a_k\,O = \Big\{\frac{1}{2}\,a_{-1}a_0 + \sum_{k=1}^{\infty}\big(k{+}1/2\big)\,a_{-k-1}a_k\Big\}\,O\,. \tag{4.12}$$

$$L_{-2}\,O = \sum_{k=0}^{\infty}\big(k+1\big)\,a_{-k-2}a_k\,O = \Big\{a_{-2}a_0 + \sum_{k=1}^{\infty}\big(k+1\big)\,a_{-k-2}a_k\Big\}\,O\,. \tag{4.13}$$

Plugging here, say, σ for O one finds

$$L_{-1}\sigma = \frac{\omega}{2\sqrt{2}}\, a_{-1}\mu\,, \qquad L_{-2}\,\sigma = \frac{\omega}{\sqrt{2}}\, a_{-2}\mu\,. \tag{4.14}$$

Similar relation (with ω replaced by $\bar{\omega}$) is obtained for the action of these operators on μ. Next, let's consider

$$L_{-1}^2\,\sigma = L_{-1}L_{-1}\sigma = \frac{\omega}{2\sqrt{2}}\, L_{-1}a_{-1}\,\mu\,.$$

Here in the expression (4.6) for L_{-1} the term $\sim a_{-1}a_0$ gives vanishing result since $a_{-1}^2 = 0$. But the term with $k = 1$, i.e. $(3/2)\, a_{-2}a_1$ leads to nonzero contribution. As the result

$$L_{-1}^2\,\sigma = \frac{\omega}{\sqrt{2}}\,\frac{3}{4}\, a_{-2}\,\mu\,. \tag{4.15}$$

Again, similar equation, with $\bar{\omega}$ replacing ω, is valid for $L_{-1}^2\,\mu$.

There two lessons to draw from the equations (4.14), (4.15). One is that the field σ (as well as μ) satisfy the so called null-vector equation

$$\left[L_{-2} - \frac{4}{3}\, L_{-1}^2\right]\sigma = 0\,. \tag{4.16}$$

One of the consequences of such equation is that any correlation function involving $\sigma(x)$ (or $\mu(x)$) satisfies linear differential equation. Let me briefly remind how it works. Consider, say, the correlation function

$$\langle\, \sigma(z)\,\sigma(z_1)\sigma(z_2)\cdots\sigma(z_n)\,\rangle\,, \tag{4.17}$$

where for simplicity I ignore the arguments $\bar{z},\ \bar{z}_1,\ \cdots$ in the fields. The null-vector equation (4.16) means that

$$\langle\, \big(L_{-2}\sigma\big)(z)\,\sigma(z_1)\ \cdots\sigma(z_n)\,\rangle = \frac{4}{3}\,\partial_z^2\,\langle\, \sigma(z)\,\sigma(z_1)\cdots\sigma(z_n)\,\rangle\,, \tag{4.18}$$

where I have made use of the fact that L_{-1} is just the derivative

$$L_{-1}O(z) = \partial_z O(z)\,. \tag{4.19}$$

Now, recall that by definition

$$L_{-2}O(z) = \oint_{C_z} \frac{dw}{2\pi i}\, T(w)\,(w-z)^{-1}\, O(z)\,, \tag{4.20}$$

where C_z encircles the point z.

$$= -\sum_{i=1}^{n}$$

Let us insert this into the l.h.s. of the equation (4.18). As we discussed before, the correlation function with the insertion $T(w)$ has singularities (poles) at all points where other field insertions are present. In our case we have, besides the point $w = z$ itself, all the points $w = z_i \;\; i = 1, \cdots, n$. We can deform the contour C_z to the sum of contours C_i encircling the points z_i. Now, each such integral has the form

$$\oint_{C_i} \frac{dw}{2\pi i}\, T(w)\, (w-z)^{-1}\, \sigma(z_i)\,. \tag{4.21}$$

When w becomes close to z_i we have

$$(w-z)^{-1} = \frac{1}{z_i - z} - \frac{(w-z_i)}{(z_i-z)^2} + O\big((w-z_i)^2\big)\,. \tag{4.22}$$

We recall that

$$L_n O(z_i) = \oint \frac{dw}{2\pi i}\, T(w)\, (w-z_i)^{n+1}\, \sigma(z_i)\,,$$

i.e. the first two terms of the expansion in (4.22) represent the action of the operators L_{-1} and L_0 on $\sigma(z_i)$, while the higher terms in (4.22) correspond to L_n's with $n > 0$ and they can be ignored since σ is a primary field. Thus, the expression (4.21) is

$$(4.21) \;=\; -\frac{1}{z-z_i}\, \partial_{z_i}\sigma(z_i) - \frac{\Delta_\sigma}{(z-z_i)^2}\, \sigma(z_i)\,, \tag{4.23}$$

where $\Delta_\sigma = 1/16$ is the eigenvalue of L_0 on σ. Combining (4.23) with (4.18) we have

$$\left[-\frac{4}{3}\, \partial_z^2 + \sum_{i=1}^{n} \left(\frac{1}{z-z_i}\, \partial_{z_i} + \frac{1/16}{(z-z_i)^2} \right) \right] \langle\, \sigma(z)\, \sigma(z_1) \cdots \sigma(z_n)\,\rangle = 0 \tag{4.24}$$

Note that changing $z \leftrightarrow z_i$ we can derive n differential equations for the same correlation function (4.17).

The differential equations (4.24) allow one to find all correlation functions. I will not stop to discuss neither exactly how it is done nor the structure of resulting functions. This is a standard routine of conformal field theory. I only mention here that the null-vector equations similar to (4.16), and differential equations which follow from them, are characteristic features of the so-called *Minimal Models* of CFT. The minimal models constitute special class of "solvable" CFT in which the space of fields $\mathcal{F}$ is built from finite number of irreducible representations of the Virasoro algebra. The CIFT is the simplest of the unitary minimal CFT. Its space $\mathcal{F}$ contains three irreducible representations with

$$\Delta = 0\,, \quad 1/16\,, \quad 1/2\,. \tag{4.25}$$

The first is associated with the identity operator whose conformal dimensions are $(0, 0)$, the second clearly involves the spin fields σ or μ with dimensions $(1/16, 1/16)$, and the third includes the scalar field

$$\varepsilon(x) = i\,\bar{\psi}\psi(x)\,. \tag{4.26}$$

with dimensions $(1/2, 1/2)$. Of course the fermion fields ψ, $\bar{\psi}$ themselves also are the primary fields (with $(\Delta, \bar{\Delta}) = (1/2, 0)$ and $(0, 1/2)$, respectively. However, these fields are not local with respect to the spin field σ. For complete space of *mutually local* fields one can take, for instance

$$\mathcal{F}_{\text{local}} = [I] \oplus [\sigma] \oplus [\varepsilon] \tag{4.27}$$

where $[O]$ stands for the irreducible representation of the algebra of L_n, $\bar{L}_n$ with the primary field O. There are other possibilities to choose mutually local fields, but (4.27) is the only one which includes σ; so it will be suitable choice when we are ready to take nonzero H.

Another way to look at the relations (4.14) and (4.15),

$$L_{-1}\mu = \frac{\bar{\omega}}{2\sqrt{2}}\, a_{-1}\sigma\,, \qquad L_{-1}^2\mu = \frac{\bar{\omega}}{\sqrt{2}}\,\frac{3}{4}\, a_{-2}\sigma \tag{4.28}$$

is to read them "from left to right",

$$a_{-1}\sigma = \frac{\omega}{\sqrt{2}}\, 4\,\partial\mu\,, \qquad a_{-2}\sigma = \frac{\omega}{\sqrt{2}}\,\frac{8}{3}\,\partial^2\mu\,, \tag{4.29}$$

where $\partial O(z, \bar{z}) = \partial_z O(z, \bar{z})$ is the derivative, and I have used $L_{-1}O = \partial O$. Again, same relations with $\sigma \leftrightarrow \mu$, $\omega \leftrightarrow \bar{\omega}$ hold as well. These relations tell us that if we take the OPE of $\psi(z)\sigma(w, \bar{w})$,

$$\psi(z)\,\sigma(w, \bar{w}) = \frac{\omega}{\sqrt{2}}\Big[(z - w)^{-1/2}\,\mu(w, \bar{w}) +$$

$$+4\,(z-w)^{1/2}\,\partial_w\mu(w,\bar w)+\frac{8}{3}\,(z-w)^{3/2}\,\partial_w^2\mu(w,\bar w)+\cdots\Big] \tag{4.30}$$

the first three most singular terms are expressed in terms of μ and its derivatives. This property can be used for alternative derivation of the differential equation for the correlation functions of the spin operators. But more importantly, similar property remains valid if one shifts away from the critical point, i.e. takes $m \neq 0$. In this more general case the property similar to (4.30) helps to derive closed equations determining the correlation functions. This is what I am going to turn to now.

We consider now the IFT with $m \neq 0$ (but still with $H = 0$). It is still the theory of free fermions, this time having nonzero mass. I'll not write down the action again, but the equations of motion for the fields ψ and $\bar\psi$ get modify as

$$\bar\partial\psi = \frac{im}{2}\,\bar\psi\,, \qquad \partial\bar\psi = -\frac{im}{2}\,\psi\,. \tag{4.31}$$

In regard to the fields σ, μ, we still can consider the space $\mathcal{R}$ of spin fields, with the defining property that for any $O_R \in \mathcal{R}$ the product

$$\psi(x)O_R(x') \to -\psi(x)O_R(x') \quad \text{when} \quad x \quad \text{is brought around} \quad x'\,. \tag{4.32}$$

Although $\psi, \bar\psi$ are no longer holomorphic fields, they are still free fields satisfying linear equations of motion. Hence we still can try to walk the same walk as in the conformal case. Introduce a complete set $(\psi_i(x), \bar\psi_i(x))$ of c-number solutions of (4.31) having the property (4.32) with respect to given point x'. In the following equations I set

$$x' = 0$$

to simplify notations. Then one could write

$$\psi(x)O_R(0) = \sum_i \psi_i(x)\,a_i O_R(0)\,,$$

thus defining a set of operators a_i acting in $\mathcal{R}$.

Let us work this out in some details. In the case $m = 0$ we had two sets of solutions labelled by integer $n \in Z$,

$$(*) : \quad \begin{pmatrix}\psi\\ \bar\psi\end{pmatrix} = \begin{pmatrix} z^{n-1/2}\\ 0\end{pmatrix}, \qquad (**) : \quad \begin{pmatrix}\psi\\ \bar\psi\end{pmatrix} = \begin{pmatrix} 0\\ \bar z^{n-1/2}\end{pmatrix} \tag{4.33}$$

which for $n \leq 0$ are singular at $x = (z, \bar{z}) \to 0$. At $m \neq 0$ two similar sets are naturally defined. In the following I assume $m >$, i.e. the $K > K_c$ low-T phase of the Ising theory. Modifications needed for the case $m < 0$ will be straightforward.

Introduce the polar coordinates centered at 0,

$$z = r\, e^{i\theta}\,, \qquad \bar{z} = r\, e^{-i\theta}\,. \tag{4.34}$$

Now, appropriate sets of solutions can be defined as

$$(*) \to \begin{pmatrix} u_n(x) \\ \bar{u}_n(x) \end{pmatrix} = \left(\frac{m}{2}\right)^{\frac{1}{2}-n} \Gamma\big(n+1/2\big) \begin{pmatrix} e^{i(n-1/2)\theta}\, I_{n-1/2}(mr) \\ -i\, e^{i(n+1/2)\theta}\, I_{n+1/2}(mr) \end{pmatrix} \tag{4.35}$$

and

$$(*) \to \begin{pmatrix} v_n(x) \\ \bar{v}_n(x) \end{pmatrix} = \left(\frac{m}{2}\right)^{\frac{1}{2}-n} \Gamma\big(n+1/2\big) \begin{pmatrix} i\, e^{-i(n+1/2)\theta}\, I_{n+1/2}(mr) \\ e^{-i(n-1/2)\theta}\, I_{n-1/2}(mr) \end{pmatrix} \tag{4.36}$$

Here $I_k(x)$ are modified Bessel functions

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{\big(x/2\big)^{\nu+2k}}{k!\,\Gamma(k+\nu+1)}\,. \tag{4.37}$$

The modified Bessel functions satisfy relations

$$I'_\nu(x) + \frac{\nu}{x}\, I_\nu(x) = I_{\nu-1}(x) \quad \text{and} \quad I'_\nu(x) - \frac{\nu}{x}\, I_\nu(x) = I_{\nu+1}(x) \tag{4.38}$$

which help to check that (4.35), (4.36) indeed satisfy the Dirac equations (4.31). In fact, the Bessel functions with half-integer ν are expressed through elementary functions [1], but this expression is not too useful. It is easy to check that

$$\begin{pmatrix} u_n(x) \\ \bar{u}_n(x) \end{pmatrix} = \begin{pmatrix} z^{n-1/2} + \frac{m^2\, r^2}{(2n+1)(2n+3)}\, z^{n-1/2} + O(m^4) \\ 0 \quad -\, i\,\frac{m}{2n+1}\, z^{n+1/2} \;+\; O(m^3) \end{pmatrix} \tag{4.39}$$

$$\begin{pmatrix} v_n(x) \\ \bar{v}_n(x) \end{pmatrix} = \begin{pmatrix} 0 \quad i\,\frac{m}{2n+1}\, \bar{z}^{n+1/2} \;+\; O(m^3) \\ \bar{z}^{n-1/2} + \frac{m^2\, r^2}{(2n+1)(2n+3)}\, \bar{z}^{n-1/2} + O(m^4) \end{pmatrix}, \tag{4.40}$$

so that (4.39),(4.40) are the massive deformations of (4.33).

---

[1] For instance $I_{-1/2}(x) = \sqrt{\frac{2}{\pi\, x}}\, \cosh x$ and $I_{1/2}(x) = \sqrt{\frac{2}{\pi\, x}}\, \sinh x$

We now can define operators a_n, $\bar{a}_n$ with $n \in Z$, acting in $\mathcal{R}$, according to the terms in the OPE

$$\psi(x) O_R(0) = \sum_{n\in Z} \big[u_{-n}(x)\, a_n + v_{-n}(x)\, \bar{a}_n\big] O_R(0)\,,$$

$$\bar{\psi}(x) O_R(0) = \sum_{n\in Z} \big[\bar{u}_{-n}(x)\, a_n + \bar{v}_{-n}(x)\, \bar{a}_n\big] O_R(0)\,. \tag{4.41}$$

Like in the conformal case, the operators a_n, $\bar{a}_n$ can be written in terms of the contour integrals. The following simple relation is useful. Let $\big(U_1(x), \bar{U}_1(x)\big)$ and $\big(U_2(x), \bar{U}_2(x)\big)$ be two solutions of the Dirac equation (4.31), i.e.

$$\bar{\partial} U_a = \frac{im}{2}\, \bar{U}_a\,, \qquad \partial \bar{U}_a = -\frac{im}{2}\, U_a\,. \tag{4.42}$$

Then, by elementary calculation, one verifies that

$$\bar{\partial}\big(U_1 U_2\big) = \frac{im}{2}\, \bar{U}_1 U_2 + \frac{im}{2}\, U_1 \bar{U}_2 = -\partial\big(\bar{U}_1 \bar{U}_2\big)\,. \tag{4.43}$$

This is a continuity equation

$$\bar{\partial} A = \partial \bar{A}\,,$$

from which by the Stokes theorem it follows that the contour integral

$$\int_C \big[A\, dz + \bar{A}\, d\bar{z}\big]$$

does not change under trivial deformations of the contour C. Thus, the integral

$$\oint_C \big[U_1 U_2\, dz - \bar{U}_1 \bar{U}_2\, d\bar{z}\big] \tag{4.44}$$

does not depend on the contour.

For the solutions defined in (4.35), (4.36) let me denote, say

$$< u_n, u_m >_{(0)} \equiv \frac{1}{2\pi i} \oint_{C_0} \big[u_n u_m\, dz - \bar{u}_n \bar{u}_m\, d\bar{z}\big]\,, \tag{4.45}$$

where C_0 surrounds the point 0. It is easy to verify that

$$< u_n, u_m >_{(0)} = \delta_{n+m,0}\,. \tag{4.46}$$

Indeed, one can take C_0 in (4.45) to be a circle, so that

$$dz = ir\, e^{i\theta}\, d\theta\,, \qquad d\bar{z} = -ir\, e^{-i\theta}\, d\theta\,; \tag{4.47}$$

Since

$$u_n \sim e^{i(n-1/2)\theta}\,, \qquad \bar{u}_n \sim e^{i(n+1/2)\theta}$$

the angular integration in (4.45) yields zero unless $n + m = 0$. If $n + m = 0$ the integral over θ is trivial,

$$< u_n, u_{-n} >_{(0)} = \frac{mr}{2}\, \Gamma\big(1/2+n\big)\Gamma\big(1/2-n\big) \times \tag{4.48}$$

$$\big[I_{n-1/2}(mr)I_{-n-1/2}(mr) - I_{n+1/2}(mr)I_{-n+1/2}(mr)\big]$$

Since the quantity cannot depend on the contour, this combination of the Bessel functions must be a constant, independent of r; this is indeed the case, since this combination can be related to the Wronskian of the Bessel differential equation;

$$\text{combination} = \frac{2\, \sin\big(\pi(n+1/2)\big)}{\pi m r}\,,$$

and (4.46) follows. Alternative and perhaps simpler way to derive this relation is to take the contour C_0 in (4.45) very small and to replace the amplitudes in (4.45) by their leading $(z, \bar{z}) \to 0$ asymptotics

$$u_n \to z^{n-1/2}\,, \qquad \bar{u}_n \to -\frac{im}{2n+1}\, z^{n+1/2} \qquad \text{as} \quad (z,\bar{z}) \to 0\,, \tag{4.49}$$

then in this limit only the first term in (4.45) contributes, yielding (4.46). Similarly, using

$$v_n \to \frac{im}{2n+1}\, \bar{z}^{n+1/2}\,, \qquad \bar{v}_n \to \bar{z}^{n-1/2} \qquad \text{as} \quad (z,\bar{z}) \to 0 \tag{4.50}$$

one finds

$$< u_n, u_m >_{(0)} = \delta_{n+m,0}\,, \qquad < u_n, v_m >_{(0)} = 0\,, \qquad < v_n, v_m >_{(0)} = \delta_{n+m,0}\,. \tag{4.51}$$

The operators a_n, $\bar{a}_n$ in (4.41) now can be defined as the contour integrals

$$a_n O_R(0) = \frac{1}{2\pi i} \oint_{C_0} \big[u_n(z,\bar{z})\, \psi(z,\bar{z})\, dz - \bar{u}_n(z,\bar{z})\, \bar{\psi}(z,\bar{z})\, d\bar{z}\big]\, O_R(0)\,,$$

$$\bar{a}_n O_R(0) = \frac{1}{2\pi i} \oint_{C_0} \left[v_n(z,\bar{z})\, \psi(z,\bar{z})\, dz - \bar{v}_n(z,\bar{z})\, \bar{\psi}(z,\bar{z})\, d\bar{z} \right] O_R(0)\,, \tag{4.52}$$

And it is not difficult to check that these operators satisfy the same canonical anti-commutators as in the massless case,

$$\{a_n, a_m\} = \delta_{n+m,0} = \{\bar{a}_n, \bar{a}_m\}\,, \qquad \{a_n, \bar{a}_m\} = 0\,. \tag{4.53}$$

These are derived more or less the same way we did it in the case $m = 0$. For instance, the anti-commutator

$$\big(a_n a_m + a_m a_n\big)\, O_R(0)\,,$$

being written in terms of the contour integrals (4.52), reduces to the integral

$$\left(\frac{1}{2\pi i}\right)^2 \oint_{x'\in C_0} \oint_{x\in C_{x'}} \left[u_n(x)\psi(x)\, dz(x) - \bar{u}_n(x)\bar{\psi}(x)\, d\bar{z}(x) \right] \times$$
$$\left[u_m(x')\psi(x')\, dz(x') - \bar{u}_m(x')\bar{\psi}(x')\, d\bar{z}(x') \right] O_R(0)\,, \tag{4.54}$$

where $z(x)$ and $z(x')$ stand for the z-coordinates of the points x and x', and the integration is performed in the following order: first one integrates over x along small contour $C_{x'}$ surrounding the point x', and then the integral over x' is taken along the contour C_0 around the point 0 where the field O_R is inserted, see **Fig.1**

The x-integral over $C_{x'}$ is controlled by the singular part of the OPEs of the components ψ, $\bar{\psi}$,

$$\psi(x)\psi(x') = \langle\, \psi(x)\psi(x')\,\rangle + :\psi(x)\psi(x'):\,,$$
$$\psi(x)\bar{\psi}(x') = \langle\, \psi(x)\bar{\psi}(x')\,\rangle + :\psi(x)\bar{\psi}(x'):\,, \tag{4.55}$$

and similarly for $\bar{\psi}(x)\psi(x')$ and $\bar{\psi}(x)\bar{\psi}(x')$. The terms with the Wick ordered products are non-singular at $x = x'$ and do not contribute to the x-integral. The contributions come from the singularities of the two-point functions. The latter can be found explicitly in terms of the Bessel functions [2], but we do not need these expressions. Instead we recall that the two-point function of the free field $(\psi, \bar{\psi})$ satisfies the inhomogeneous equations

$$\langle\, \bar\partial\psi(x)\psi(x')\,\rangle - \frac{im}{2}\,\langle\, \bar\psi(x)\psi(x')\,\rangle = \pi\,\delta(x-x')\,,$$

$$\langle\, \bar\partial\psi(x)\bar\psi(x')\,\rangle - \frac{im}{2}\,\langle\, \bar\psi(x)\bar\psi(x')\,\rangle = 0\,, \tag{4.56}$$

which are obtained by standard textbook manipulations with the functional integral. With this,

$$\oint_{x\in C_{x'}} \big[u_n(x)\psi(x)\,dz(x) - \bar u_n(x)\bar\psi(x)\,d\bar z(x)\big] =$$

$$2i\int_{D_{x'}} \big[\bar\partial\big(u_n(x)\psi(x)\big) + \partial\big(\bar u_n(x)\bar\psi(x)\big)\big]\,d^2x \tag{4.57}$$

Here $D_{x'}$ is a small domain incorporating the point x' whose boundary is $C_{x'}$. The integrand in (4.57) is zero everywhere except the point $x = x'$, where it is controlled by the delta-function in (4.56). With this we find

$$(4.54) = \delta_{n+m,0}\,. \tag{4.58}$$

Other relations (4.53) are obtained the same way.

We see that the structure of the space $\mathcal{R}$ at $m \neq 0$ remains essentially the same as it was in the massless case. At this point I simply assume that the product $\psi(x)\sigma(x')$ (as well as similar products with $\bar\psi$ and μ) at $m \neq 0$ cannot be more singular at $x \to x'$ then it was at $m = 0$ (Later I will discuss general argument for this this property of the OPE in more general context). Then we still have two Fock vacua σ and μ which are annihilated by all a_n, $\bar a_n$ with $n > 0$, and are turned one to another by the actions of a_0 and $\bar a_0$, via the same equations which we had in the case $m = 0$. Some other properties, in particular that a_{-1} and a_{-2} applied to σ are expressed through the derivatives of μ also remain valid. Let us check this now.

We start with the general identities valid for any local field O,

$$\partial_w O(w,\bar w) = \frac{1}{2\pi i}\oint_{C_w} \big[T(z,\bar z)\,dz + \Theta(z,\bar z)\,d\bar z\big]\,O(w,\bar w)\,, \tag{4.59a}$$

[2] Explicitly $\langle\, \psi(x)\psi(0)\,\rangle = m\,\frac{|z|}{z}\,K_1(mr)$, and $\langle\, \psi(x)\bar\psi(x)\,\rangle = -im\,K_0(mr)$

$$\partial_{\bar{w}} O(w,\bar{w}) = -\frac{1}{2\pi i} \oint_{C_w} \left[\bar{T}(z,\bar{z})\, d\bar{z} + \Theta(z,\bar{z})\, dz\right] O(w,\bar{w})\,, \tag{4.59b}$$

where the integration is over a contour C_w which goes around w in the counter-clockwise direction, and

$$T = -\frac{1}{2} \,:\psi\partial\psi:\,, \qquad \Theta = -\frac{im}{4} \,:\bar{\psi}\psi:\,, \qquad \bar{T} = -\frac{1}{2} \,:\bar{\psi}\bar{\partial}\bar{\psi}: \tag{4.60}$$

are the components of the EM tensor. The integrals are invariant under the contour deformations thanks to the continuity equations for the EM tensor.

Now, again, let $(U_1, \bar{U}_1)$ and $(U_2, \bar{U}_2)$ be two solutions of the Dirac equation,

$$\bar{\partial} U = \frac{im}{2}\, \bar{U}\,, \qquad \partial \bar{U} = -\frac{im}{2}\, U\,. \tag{4.61}$$

One checks that

$$\bar{\partial}\big(U_1 \partial U_2\big) = \frac{im}{2}\, \partial\big(\bar{U}_1 U_2\big) \quad \text{and} \quad \partial\big(\bar{U}_1 \bar{\partial}\bar{U}_2\big) = -\frac{im}{2}\, \bar{\partial}\big(U_1 \bar{U}_2\big)\,. \tag{4.62}$$

These are continuity equations showing that the forms

$$< U_1, U_2 >_{(1)} = -\frac{1}{2\pi i} \oint_C \left[U_1 \partial U_2\, dz + \frac{im}{2}\, \bar{U}_1 U_2\, d\bar{z}\right], \tag{4.62}$$

$$< U_1, U_2 >_{(-1)} = \frac{1}{2\pi i} \oint_C \left[\bar{U}_1 \bar{\partial} \bar{U}_2\, d\bar{z} - \frac{im}{2}\, U_1 \bar{U}_2\, dz\right], \tag{4.63}$$

are invariant under the deformations of the contour C.

Exercise: Check that for the solutions $\big(u_n, \bar{u}_n\big)$ and $\big(v_n, \bar{v}_n\big)$

$$< u_n, u_k >_{(1)} = \big(n - 1/2\big)\, \delta_{n+k-1}\,, \qquad < u_n, v_k >_{(1)} = 0\,, \tag{4.64a}$$

and

$$< v_n, v_k >_{(1)} = \frac{m^2}{2(2n+1)}\, \delta_{n+k+1}\,, \tag{4.64b}$$

and derive similar expressions for $< ... >_{(-1)}$.

Using these relations it is easy now to compute the integrals in (4.59) when $O \in \mathcal{R}$. In this case we take

$$\psi = \sum_{n\in Z} \left[u_n\, a_{-n} + v_n\, \bar{a}_{-n}\right],$$

$$\bar{\psi} = \sum_{n \in Z} \left[\bar{u}_n \, a_{-n} + \bar{v}_n \, \bar{a}_{-n} \right] ,$$

and assume that C surrounds the point where $O \in \mathcal{R}$ is inserted. One finds

$$\partial O_R = \frac{1}{2} \sum_{n=0}^{\infty} \left[(2n+1) \, a_{-n-1} a_n + \frac{m^2}{2n+1} \, \bar{a}_{-n} \bar{a}_{n+1} \right] O_R \, , \tag{4.65}$$

$$\bar{\partial} O_R = \frac{1}{2} \sum_{n=0}^{\infty} \left[(2n+1) \, \bar{a}_{-n-1} \bar{a}_n + \frac{m^2}{2n+1} \, a_{-n} a_{n+1} \right] O_R \, , \tag{4.66}$$

In particular, taking $O_R = \sigma$ one finds

$$\partial \sigma = \frac{1}{2} \, a_{-1} a_0 \sigma = \frac{\omega}{2\sqrt{2}} \, a_{-1} \mu \, , \tag{4.67}$$

$$\partial^2 \sigma = \frac{\omega}{2\sqrt{2}} \left[\frac{1}{2} a_{-1} a_0 + \frac{3}{2} \, a_{-2} a_1 + \ldots \right] a_{-1} \mu = \frac{3}{4} \, \frac{\omega}{\sqrt{2}} \, a_{-2} \mu \, . \tag{4.68}$$

Of course, similar relations are derived for $\bar{\partial}$ replacing ∂, and for μ instead of σ.

I'd like to mention in passing that the Integrals of Motion

$$P = \frac{1}{2\pi} \int_C \left[T \, dz + \Theta \, d\bar{z} \right] \tag{4.69a}$$

and

$$\bar{P} = -\frac{1}{2\pi} \int_C \left[\bar{T} \, d\bar{z} + \Theta \, dz \right] , \tag{4.69b}$$

which of course are just the light-cone components of the 2-momentum, are the first representatives of infinite set of commuting local IM of the free-fermion theory. While the conservation (i.e. independence of the contour C) of (4.69) follows from the continuity equations for the EM tensor, there are infinitely many currents, all quadratic in the fermion field, which satisfy similar continuity equations,

$$\bar{\partial} T_{2k} = \partial \Theta_{2k-2} \, , \qquad k = 2, 3, \cdots \tag{4.70}$$

where

$$T_{2k} = -\frac{1}{2} \, \partial^{k-1} \psi \partial^k \psi \, , \qquad \Theta_{2k-2} = -\frac{1}{2} \left(\frac{m}{2} \right)^2 \partial^{k-2} \psi \partial^{k-1} \psi \, , \tag{4.71}$$

(and of course similar set of equations exist with $\psi \to \bar{\psi}$ and $\partial \to \bar{\partial}$). The infinite set of conserved currents of the type (4.70) is the signature of *Integrable Field Theory.* I'll come back to this subject later in relation to the IFT with nonzero H. The IFT with $H = 0$ which we are discussing now is not only integrable but free. In this case it is seen that in (4.70)

$$\Theta_{2k-2} \sim T_{2k-2}\,.$$

Changing normalization to

$$T_{2k} = -\frac{1}{2}\left(\frac{m}{2}\right)^{2-2k} \partial^{k-1}\psi\partial^{k}\psi \tag{4.72}$$

one brings the system of the continuity equations (4.70) to the "chain" form

$$\bar{\partial}T_{2k} = \partial T_{2k-2}\,. \tag{4.73}$$

It can be shown that any integrable FT with this form of local IM is a free field theory.

Although the IFT at $H = 0$ is a free field theory, the correlation functions of the spin fields σ and/or μ are rather complicated objects. At the same time we need such correlation functions if we want to study the theory at $H \neq 0$ by perturbation theory in H. It turns out that these correlation functions can be expressed through solutions of certain *nonlinear* integrable differential equations, the so-called Painleve equations. This result, which constitutes major achievement of mathematical physics, was first obtained by Wu, McCoy, Tracy and Barouch. Let me turn to this part of the theory. I will derive this result by slightly different method.

Let me first try to make a straightforward approach to the problem, more or less following the strategy used in conformal field theories. Suppose I am interested in two-point correlation function, say

$$G(x_1 - x_2) = \langle\, \sigma(x_1)\sigma(x_2)\,\rangle\,. \tag{4.74}$$

I can start with the three-point correlation functions which involve ψ or $\bar{\psi}$,

$$\Psi(x|x_1, x_2) = \langle\, \psi(x)\sigma(x_1)\mu(x_2)\,\rangle\,, \tag{4.75a}$$

$$\bar{\Psi}(x|x, x') = \langle\, \bar{\psi}(x)\sigma(x_1)\mu(x_2)\,\rangle\,, \tag{4.75b}$$

(these correlation functions are nonzero since $\psi \sim \sigma\mu$). As the functions of x, (4.75) satisfy the Dirac equation,

$$\bar{\partial}\Psi(x) = \frac{im}{2}\,\bar{\Psi}(x)\,, \qquad \partial\bar{\Psi}(x) = -\frac{im}{2}\,\Psi(x)\,, \tag{4.76}$$

and also the monodromy condition that $\Psi(x)$ and $\bar{\Psi}(x)$ change sign when x is brought around either of the points x_1 or x_2, as in **Fig.2**

Of course, when x is close to x_1 (or x_2) these functions admit expansions in terms of the local solutions $u_n(x-x_1)$ and $v_n(x-x_1)$ for Ψ, and in $\bar{u}$'s and $\bar{v}$'s for $\bar{\Psi}$, for instance

$$\Psi(x|x_1,x_2) = \sum_{n=0}^{\infty}\big[u_n(x-x_1)\,\langle\, a_{-n}\sigma(x_1)\,\mu(x_2)\,\rangle + v_n(x-x_1)\,\langle\, \bar{a}_{-n}\sigma(x_1)\,\mu(x_2)\,\rangle\,. \tag{4.77}$$

That is, when $x \to x_1$, $\Psi(x)$ is as singular as $u_0(x-x_1)$, and similarly for x_2.

Now, suppose I knew explicitly a solution

$$\big(U(x|x_1,x_2), \bar{U}(x|x_1,x_2)\big) \tag{4.78}$$

of the Dirac equation (4.76), which obeys the same monodromy property as (4.75), namely changes sign when x is brought around x_1 or x_2, and also decays as $|x| \to \infty$. I also assume that (4.78) is as singular as (4.75) near x_2, but is one step more singular near x_1, say

$$\begin{aligned} U(x|x_1,x_2) &\to u_{-1}(x-x_1)\,A(x_1,x_2)\,, \\ \bar{U}(x|x_1,x_2) &\to \bar{u}_{-1}(x-x_1)\,A(x_1,x_2)\,. \end{aligned} \tag{4.79}$$

Of course, these conditions do not define the solution uniquely. First of all, one can always add arbitrary x-independent factor. Let's assume I use this freedom to make

$$A(x_1,x_2) = 1 \tag{4.80}$$

in (4.79). Some ambiguity still remains (it can be shown that there is 2-dimensional space of solutions satisfying all the above requirements), but it suffices to know only one solution. Ones we have such solution we can write down an identity

$$0 = \frac{1}{2\pi i}\oint_{C_\infty}\big[U(x)\Psi(x)\,dz(x) - \bar{U}(x)\bar{\Psi}(x)\,d\bar{z}(x)\big]\,, \tag{4.81}$$

where $z(x), \bar{z}(x)$ are the complex coordinates associated with the point x, and I have suppressed the arguments $x-1, x_2$. The contour C_∞ encircles both x_1 and x_2; the integral is equal to zero because the contour can be moved away to infinity.

On the other hand we could evaluate the integral (4.81) closing the contour around the points x_1 and x_2. Suppose

$$U(x) = u_{-1}(x-x_1) + u_0(x-x_1)\, B(x_1,x_2) + v_0(x-x_1)\, B'(x_1,x_2) + \cdots \quad \text{near} \quad x_1\,, \tag{4.82}$$

and

$$U(x) = u_0(x-x_2)\, C(x_1,x_2) + v_0(x-x_2)\, C'(x_1,x_2) + \cdots \quad \text{near} \quad x_2\,, \tag{4.83}$$

with corresponding expansions for $\bar{U}(x)$, where the coefficients B, B' and C, C' are derived from the solution $U(x), \bar{U}(x)$ which we assumed was known. Using the orthogonality conditions for the local solutions u_n, v_n, one finds

$$0 = \langle\, \big[a_{-1}\sigma(x_1) + B\, a_0\sigma(x_1) + B'\, \bar{a}_0\sigma(x_1)\big]\, \mu(x_2)\,\rangle + \langle\, \sigma(x_1)\, \big[C\, a_0\mu(x_2) + C'\, \bar{a}_0\mu(x_2)\big]\,\rangle$$

or, using our expressions for $a_{-1}\sigma$ and the actions of a_0, $\bar{a}_0$ on σ and μ,

$$\begin{aligned} 4\,\partial_{z_1}\langle\, \mu(x_1)\mu(x_2)\,\rangle + \big[B(x_1,x_2) - iB'(x_1,x_2)\big]\,\langle\, \mu(x_1)\mu(x_2)\,\rangle + \\ + \big[C'(x_1,x_2) - iC(x_1,x_2)\big]\,\langle\, \sigma(x_1)\sigma(x_2)\,\rangle = 0 \end{aligned} \tag{4.84}$$

This is a differential equation for the correlation functions. Three more differential equations appear when one interchange σ and mu, or replace z by $\bar{z}$; this in principle yields complete system which would determine the correlation functions.

The problem with this straightforward approach is that the solutions like $\big(U(x), \bar{U}(x)\big)$ are not available in closed form, and hence the coefficients B, B', C, C' in (4.84) are not explicitly known. One can still pursue this approach without explicit knowledge of the solutions, using instead the theory of isomonodromic deformations of differential equations. This was done with success by Sato, Miwa and Jimbo. I am not going to explain details of that approach here, partly because I did not get through all mathematical details myself.

Instead, I will use another the following idea. Although we do not know solutions like the above $\big(U(x), \bar{U}(x)\big)$ explicitly, after all appropriate correlation functions of the IFT themselves provide such solutions. For instance, the correlation functions

$$\begin{pmatrix} U(x) \\ \bar{U}(x) \end{pmatrix} \sim \begin{pmatrix} \langle\, \psi(x)\, a_{-1}\mu(x_1)\mu(x_2)\,\rangle \\ \langle \bar{\psi}(x)\, a_{-1}\mu(x_1)\mu(x_2)\,\rangle \end{pmatrix} \tag{4.85}$$

satisfy all the conditions we have assumed about $\big(U(x), \bar{U}(x)\big)$ above. Then the coefficients B, B', C, C' are expressed through the two-point functions of the spin operators, and (4.84) and its siblings become closed, albeit nonlinear differential equations for these correlation functions.

One can think of the solutions like (4.85) used in (4.81) as coming from additional "spectator" copy of the IFT. Therefore we are effectively dealing with the system which involves two non-interacting copies of the original Ising field theory. This step of adding extra copy is often referred to as the "doubling". We will see that the "doubled" IFT has exhibits rich symmetry, and the differential equations of Wu, McCoy, Tracy and Barouch appear directly as the Ward identities associated with this symmetry.

It must be mentioned that the idea of "doubling" in the Ising model is not new, but has long history. Two copies of the Majorana fermion constitute a single charged Dirac fermion, and Dirac fermion in two dimensions is more convenient object in many respects. Most technical advantages of the "doubling" are apparent in the critical case $m = 0$. The theory of massless Dirac fermions is equivalent to free massless bosons - I mean the famous Bosonization technique. This in particular allows one to write down, in the massless case, arbitrary n-point correlation functions of the Ising spins in simple closed form. This is an interesting and important subject, but I will skip it, referring you to, say, textbook on CFT by DiFrancesco, Mathieu and Senechal.