L3

The last time we have discussed the Ising field theory (i.e. the scaling limit of the Ising model) in the case $H = 0$. We have found that it can be described as the theory of free Majorana fermions. It has the action which is quadratic in fermion fields, and it describes free particles of the mass $|m|$, where $m \sim K - K_c$. Important particular case is the critical theory which appears at $K = K_c$, hence has $m = 0$. I will call it the Critical Ising Field Theory (CIFT). Of course, it is still a theory of free fermions, as described by the action

$$\mathcal{A}_{\rm FF} = \frac{1}{2\pi} \int \left[\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}\right] d^2x\,. \tag{3.1}$$

As there is no dimensional parameter in (3.1), the physics is essentially the same at all length scales -we say that the CIFT is scale invariant. Indeed the action is invariant with respect to the scale transformations

$$(z,\bar{z}) \to (\lambda\, z, \lambda\, \bar{z})\,, \qquad \big(\psi, \bar{\psi}\big) \to \big(\lambda^{-1/2}\,\psi, \lambda^{-1/2}\,\bar{\psi}\big)\,; \tag{3.2}$$

the last transformation law of ψ is often expressed by saying that the fields $(\psi, \bar{\psi})$ has scale dimensions

$$\psi,\ \bar{\psi} \sim [\text{length}]^{-1/2}\,. \tag{3.3}$$

As the result, the correlation functions of any fields of the CIFT are homogeneous functions of the coordinates, as exemplified by already familiar two-point function

$$\langle\, \psi(z)\psi(z')\,\rangle = \frac{1}{z - z'}\,.$$

Also, the last time we have discussed the "spin fields" of the IFT, the fields O_R whose characteristic property is that the products

$$\psi(z)O_R(x) \to -\psi(z)O_R(x)$$

change sign when z is brought around x (with the same property holding for $\bar{\psi}$). We have found that there is an infinite-dimensional space $\mathcal{R}$ of such spin fields, and there is an infinite set of operators a_n, $\bar{a}_n$ with $n = 0, \pm 1, \pm 2, \ldots$, which act in $\mathcal{R}$, and satisfy the canonical anti-commutation relations

$$\{a_n, a_m\} = \delta_{n+m,0}\,, \qquad \{\bar{a}_n, \bar{a}_m\} = \delta_{n+m,0}\,, \tag{3.4}$$

and a_n's anti-commute with all $\bar{a}_m$'s. In other words, the space $\mathcal{R}$ supports representation of the algebra (3.4).

Now, it is clear from the definition

$$a_n O(z_1, \bar{z}_1) = \oint_{z_1} \big(z - z_1\big)^{n-1/2}\, \psi(z)\, O(z_1, \bar{z}_1)\, \frac{dz}{2\pi i}\,, \tag{3.5}$$

(and from similar formula defining $\bar{a}_n$) that the operator a_n has the scale dimension

$$a_n \sim [\text{length}]^n\,, \qquad \bar{a}_n \sim [\text{length}]^n\,, \tag{3.6}$$

that is, if $O_D \in \mathcal{R}$ has the length dimension $-D$, or the mass dimension D, then $a_n O_D$ has the mass dimension $D - n$:

$$O_D \sim [\text{length}]^{-D} \quad \to \quad a_n O_D \sim [\text{length}]^{-D+n}\,. \tag{3.7}$$

In the scale invariant theory the correlation functions are homogeneous functions of the coordinates, with the degree determined by the scale dimensions of the fields involved. It follows from this simple dimensional analysis that

$$\langle\, a_n O_D(x_1)\, a_n O_D(x_2)\,\rangle \sim (x_1 - x_2)^{-2D+2n}\,. \tag{3.8}$$

Since n here is arbitrary we are facing the problem of having some fields whose correlation functions grow with the distance. Such growing behavior of the correlations in the Ising model seems highly implausible. In fact, it is possible to prove rigorously that in the Ising model (and indeed in the whole class of statistical models with positive Boltzman weights and with the nearest-neighbor interaction; I'll say on this point more later in the course) the correlation functions cannot grow. But I think intuitive reasonings are sufficient to rule the growing correlations out. With this additional input, the equation (3.8) implies that the field $a_n O_D$ vanishes for sufficiently large n,

$$a_n O_D = 0 \qquad \text{for} \quad n > D\,.$$

This property implies that the space $\mathcal{R}$ contains at least one "Fock vacuum", i.e. the field (which I denote O_v) such that

$$O_v \in \mathcal{R}: \qquad a_n O_v = 0\,, \quad \bar{a}_n O_v = 0 \qquad \text{for all} \quad n > 0\,. \tag{3.9}$$

Now, the "Fock vacuum" field O_v cannot be unique. Indeed, by our dimensional analysis the fields

$$a_0 O_v\,, \qquad \bar{a}_0 O_v \tag{3.10}$$

have the same scale dimensions as O_v, and they also satisfy the vacuum conditions (3.9). At the same time the fields (3.10) cannot be just proportional to O_v since the algebra

$$\{a_0, a_0\} = 2\,a_0^2 = 1\,, \qquad \{\bar{a}_0, \bar{a}_0\} = 2\,\bar{a}_0^2 = 1\,, \qquad \{a_0, \bar{a}_0\} = 0 \tag{3.11}$$

does not have one-dimensional representations. The minimal degeneracy of O_v sufficient to support the relations is two. But that is exactly what we expect. Remember, the lattice Ising model exhibits order-disorder duality: it admits two equivalent descriptions in terms of the original spins $\sigma_{\mathbf{x}}$ and the "dual spins" $\mu_{\tilde{\mathbf{x}}}$, the parameter K being replaced by $\tilde{K}$: $\exp\big(-2K\big) = \tanh\tilde{K}$. The duality transformation maps the high-T domain for the σ's to the low-T domain of the μ's, and vice versa. In particular, at the critical point $K = K_c$ the model is self-dual, $K_c = \tilde{K}_c$, i.e. the correlations of the dual spins are exactly the same as the correlations of the original spins. The critical point corresponds to the massless theory (3.1).

For that reasons one expects the vacuum O_v of the space $\mathcal{R}$ to be two-dimensional, spanned by the fields $\sigma(x)$ and $\mu(x)$, the continuous limits of the order and disorder variables $\sigma_{\mathbf{x}}$ and $\mu_{\tilde{\mathbf{x}}}$ of the lattice theory,

$$O_v = \big(\sigma, \mu\big)\,.$$

Moreover, we must have

$$a_0\sigma \sim \mu\,, \qquad a_0\mu \sim \sigma\,, \tag{3.12}$$

and similarly with $\bar{a}_0$. Indeed, the action of a_0 on some field $O \in \mathcal{R}$ is essentially the result of the fusion of ψ and O,

$$a_0 O(z_1) = \lim_{z\to z_1} \sqrt{z - z_1}\;\psi(z)\,O(z_1)\,.$$

If one also recalls that microscopically the fermions $\psi, \bar{\psi}$ are products of the type $\sigma\mu$, the relations (3.12) appear the only consistent possibility. This is also consistent with the property that the product $\sigma(x)\mu(x')$ changes sign when x is brought around x' (one has to remember that $\psi(x)\sigma(x')$ and $\psi(x)\mu(x')$ change signs under such move).

Nicely symmetric representation of the anti-commutation relations (3.11) is given by the equations

$$a_0\sigma(x) = \frac{\omega}{\sqrt{2}}\,\mu(x)\,, \qquad a_0\mu(x) = \frac{\bar{\omega}}{\sqrt{2}}\,\sigma(x)\,,$$

$$\bar{a}_0\sigma(x) = \frac{\bar{\omega}}{\sqrt{2}}\,\mu(x)\,, \qquad \bar{a}_0\mu(x) = \frac{\omega}{\sqrt{2}}\,\sigma(x)\,, \tag{3.13}$$

where

$$\omega = e^{\frac{i\pi}{4}}\,, \qquad \bar{\omega} = e^{-\frac{i\pi}{4}}\,. \tag{3.14}$$

The equations (3.14) together with

$$a_n\sigma(x) = \bar{a}_n\sigma(x) = a_n\mu(x) = \bar{a}_n\mu(x) = 0 \quad \text{for} \quad n > 0 \tag{3.15}$$

can be taken as the definition of the spin fields in the Ising field theory, and if fact I could have started my discussion of the IFT with writing down the free fermion theory and defining the spin field by (3.14),(3.15). The main point of the discussion of the lattice model in the previous lectures was to show why these definitions are natural.

My next goal will be to determine the scale dimensions of the spin fields σ and μ and their correlation functions. Conformal invariance of the CIFT (3.1) turns out to be very useful ingredient in this analysis.

As any local field theory, the CIFT (3.1) has special field, the energy momentum (EM) tensor $T^{\mu\nu}(x)$, which is symmetric

$$T^{\mu\nu}(x) = T^{\nu\mu}(x) \tag{3.16}$$

and satisfies the continuity equation

$$\partial_\mu T^{\mu\nu}(x) = 0\,. \tag{3.17}$$

As usual, the continuity equation (3.17) is the manifestation of translational invariance of the field theory, while the symmetry (3.16) comes from rotational symmetry (in both cases I am talking about the symmetries of the local action; the global settings of the theory, like boundaries and such, can break these symmetries on the global level, but this has nothing to do with the local conditions (3.16),(3.17) which remain valid regardless). In 2D space it is convenient to use components of the tensor $T^{\mu\nu}(x)$ corresponding to the complex coordinates $z = \mathrm{x} + i\mathrm{y}$, $\bar{z} = \mathrm{x} - i\mathrm{y}$. Namely, it is conventional to use the notations

$$T = -(2\pi)\, T_{zz} = \frac{\pi}{2}\left(T_{\mathrm{yy}} - T_{\mathrm{xx}} + 2i\, T_{\mathrm{xy}}\right),$$

$$\bar{T} = -(2\pi)\, T_{\bar{z}\bar{z}} = \frac{\pi}{2}\left(T_{\mathrm{yy}} - T_{\mathrm{xx}} - 2i\, T_{\mathrm{xy}}\right),$$

$$\Theta = (2\pi)\, T_{z\bar{z}} = (2\pi)\, T_{\bar{z}z} = \frac{\pi}{2}\left(T_{\mathrm{yy}} + T_{\mathrm{xx}}\right). \tag{3.18}$$

Note that Θ is essentially the trace of the EM tensor,

$$\Theta = \frac{\pi}{2}\, T^{\mu}_{\mu}\,.$$

Note also that the continuity equations (3.16) take the form

$$\bar{\partial} T = \partial \Theta\,, \qquad \partial \bar{T} = \bar{\partial} \Theta\,. \tag{3.19}$$

For the free fermion theory

$$\mathcal{A} = \frac{1}{2\pi} \int \left[\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} + im\, \bar{\psi} \psi\right] d^2 z \tag{3.20}$$

the EM tensor is derived by the standard textbook manipulations,

$$T = -\frac{1}{2}\, \psi \partial \psi\,, \qquad \bar{T} = -\frac{1}{2}\, \bar{\psi} \bar{\partial} \bar{\psi}\,, \qquad \Theta = -\frac{im}{4}\, \bar{\psi} \psi\,. \tag{3.21}$$

Here I have restored the mass parameter m for future references. It is straightforward to check that (3.21) satisfy (3.19) in virtue of the equations of motion of the theory (3.21).

At this point, I would like to remind some general properties of the EM tensor in local QFT. By definition, the energy-momentum tensor describes response of the system to the infinitesimal diffeomorphisms

$$x^{\mu} \to x^{\mu} + \varepsilon^{\mu}(x)\,. \tag{3.22}$$

Namely, the variation of the action under (3.22) has general form

$$\delta_{\varepsilon} \mathcal{A} = \int d^2 x\, \partial_{\mu} \varepsilon_{\nu}(x)\, T^{\mu\nu}(x)\,, \tag{3.23}$$

which defines $T^{\mu\nu}(x)$. The transformation (3.22) is understood as the change of variables in the functional integral

$$\langle\, O_1(x_1)\cdots O_n(x_n)\,\rangle = Z^{-1}\int\, [D\Phi]\, O_1(x_1)\cdots O_n(x_n)\, e^{-\mathcal{A}[\Phi]} \tag{3.24}$$

defining the theory, where the integration is over some "fundamental field" $\Phi(x)$ (say, $(\psi(x),\bar\psi(x))$ in (3.1)), and $O_i(x)$ are some local insertions built from $\Phi(x)$ and its derivatives. Consider an infinitesimal change of the functional variable $\Phi(x)$,

$$\Phi(x)\to\Phi'(x)=\Phi(x)+\delta_\varepsilon\Phi(x) \tag{3.25}$$

where $\delta_\varepsilon\Phi(x)$ has the form

$$\delta_\varepsilon\Phi(x)=\varepsilon^\mu(x)\partial_\mu\Phi(x)+\partial_\mu\varepsilon_\nu(x)\,\Delta^{\mu,\nu}(\Phi(x))+\cdots\,, \tag{3.26}$$

where $\Delta^{\mu\nu}(\Phi)$ are some functions of Φ at the point x, and $\cdots$ may contain higher-order derivatives of $\varepsilon(x)$. Obvious ambiguity in (3.26) leads to corresponding ambiguity in the energy-momentum tensor defined through (3.23), which cannot be completely fixed on general grounds (unless we know something about interaction with gravity). The ambiguity is usually used to make $T^{\mu\nu}$ symmetric in rotationally invariant theories, and further to achieve most convenient form of the energy-momentum tensor.

The transformation (3.25) induces the transformations

$$O_i(x)\to O'_i(x)=O_i(x)+\delta_\varepsilon O_i(x)\,. \tag{3.27}$$

Since change of variables does not alter the value of the integral in the r.h.s of (3.24), the linear in ε terms coming from the variations (3.27) of the insertions must cancel the term (3.23) coming from the variation of the action. This leads to the identity

$$\sum_{i=1}^{n}\langle\, O_1(x_1)\cdots\delta_\varepsilon O_i(x_i)\cdots O_n(x_n)\,\rangle=\int\, d^2x\,\langle\,\partial_\mu\varepsilon_\nu(x)\,T^{\mu\nu}(x)\,O_1(x_1)\cdots O_n(x_n)\,\rangle\,. \tag{3.28}$$

Now, let D_i be small regions of the Euclidean plane such that $x_i\in D_i$, and different D_i have no intersections (see **Fig.1**).

The integral in the r.h.s. of (3.28) can be split into n integrals over D_i, and the integral over the remaining part $\bar D$ of the Euclidean plane. Since $\partial_\mu T^{\mu\nu}(x)=0$

everywhere in $\bar{D}$, the integrand in (3.28) is total divergence

$$\partial_\mu \varepsilon_\nu(x)\, T^{\mu\nu}(x) = \partial_\mu \big(\varepsilon_\nu(x)\, T^{\mu\nu}(x)\big)$$

everywhere inside $\bar{D}$, and it can be reduced (using the Gauss's theorem) to the integrals over the boundary of this domain, which coincides with the combination of the boundaries $C_i = \partial D_i$ of the domains D_i (**Fig.1**). The r.h.s. of (3.28) then splits into the sum of the terms

$$\text{r.h.s. of} \quad (3.28) = \sum_{i=1}^{n} \int_{D_i} d^2x \,\langle\, \partial_\mu \varepsilon_\nu(x)\, T^{\mu\nu}(x)\, O_1(x_1) \cdots O_n(x_n)\,\rangle +$$

$$+ \sum_{i=1}^{n} \int_{C_i} \epsilon_{\lambda\mu} d\xi^\lambda(x)\, \varepsilon_\nu(x)\, \langle\, T^{\mu\nu}(x)\, O_1(x_1) \cdots O_n(x_n)\,\rangle \,. \tag{3.29}$$

where $d\xi^\lambda(x)$ is the counterclockwise directed differential tangent to the boundary contour C_i, and $\epsilon_{\lambda\mu}$ is unit antisymmetric tensor ($\epsilon_{12} = 1$). Since the domains D_i can be made arbitrarily small, it is clear that the equality between the l.h.s. of (3.28) and (3.29) must hold term by term in i. Therefore we can write

$$\delta_\varepsilon O(x') = \int_{D'} d^2x\, \partial_\mu \varepsilon_\nu(x)\, T^{\mu\nu}(x)\, O(x') + \int_{C'} \epsilon_{\lambda\mu} d\xi^\lambda(x)\, \varepsilon_\nu(x)\, T^{\mu\nu}(x)\, O(x') \,, \tag{3.30}$$

where the r.h.s. should be understood in terms of operator product expansions.

In the CIFT (3.1) we have $m = 0$ and from explicit expressions (3.21)

$$\text{CIFT}: \qquad \Theta = 0 \,. \tag{3.31}$$

This equation is of course related to the scale invariance (3.2) of the CIFT (3.1). For the dilations

$$x^\mu \to x^\mu + \varepsilon\, x^\mu \tag{3.32}$$

we have

$$\delta \mathcal{A} = \varepsilon \int T^\mu_\mu(x)\, d^2x = 0 \,. \tag{3.33}$$

Hence the scale symmetry of the theory suggests $\Theta = 0$.

Remark: Of course, in general (3.31) does not follow from vanishing of (3.33). It suffices to have

$$T^\mu_\mu(x) = \partial_\mu A^\mu(x) \tag{3.34}$$

with some local field $A^\mu(x)$ to make (3.33) vanish. If $A^\mu(x)$ in tern is a gradient,

$$A^\mu(x) = \partial^\mu \Phi(x) \tag{3.35}$$

then it is possible to redefine the EM tensor

$$\hat{T}^{\mu\nu}(x) \to T^{\mu\nu}(x) + \partial^\mu \partial^\nu \Phi - \delta^{\mu\nu}\, \partial^2 \Phi \tag{3.36}$$

to make it traceless. Moreover, it is possible to prove that in any unitary 2D field theory $A^\mu(x)$ in (3.34) is necessarily a gradient (3.35) [1]. It is not known (to me) if interesting non-unitary field theories exist in which (3.33) vanishes but (3.31) does not hold. **End of remark**.

In fact, vanishing trace

$$T_\mu^\mu = 0 \tag{3.37}$$

signals much larger *conformal* symmetry of the theory, which includes the scale transformations (3.32) as small (but important) part. By definition, the conformal transformations are diffeomorphisms

$$x^\mu \to y^\mu = f^\mu(x)$$

which preserve the "conformally flat" form of the metric

$$ds^2 = dx^\mu dx^\mu = \Lambda(y)\, dy^\mu dy^\mu\,. \tag{3.38}$$

For infinitesimal conformal transformations this means

$$\partial_\mu \varepsilon_\nu(x) + \partial_\nu \varepsilon_\mu(x) = \delta_{\mu\nu}\, \partial_\lambda \varepsilon^\lambda(x)\,, \tag{3.39}$$

hence

$$\delta_\varepsilon \mathcal{A} = \int d^2x\, \partial_\lambda \varepsilon^\lambda(x)\, T_\mu^\mu(x) = 0$$

as the consequence of the zero trace (3.37).

In 2D all such transformations can be described as analytic maps of the complex variables $(z, \bar{z})$,

$$z \to w = f(z)\,, \qquad \bar{z} \to \bar{w} = \bar{f}(\bar{z})\,, \tag{3.40}$$

[1] In rotational and scale invariant theory we have $\langle\, T(z)\, T(0)\,\rangle = C/2\, z^4$, $\langle\, T(z)\, \Theta(0)\,\rangle = B/2\, z^3\, \bar{z}$ $\langle\, \Theta(z)\, T(0)\,\rangle = B/2\, z^3\, \bar{z}$, and $\langle\, \Theta(z)\, \Theta(0)\,\rangle = A/z^2\, \bar{z}^2$. From (3.19) we have $0 = B$, $-B = -2\,A$, hence $\langle\, \Theta(z)\, \Theta(0)\,\rangle = 0$. In unitary theory this implies $\Theta = 0$.

with holomorphic functions f and $\bar{f}$. Infinitesimal form of (3.40) is

$$z \to z + \varepsilon(z)\,, \qquad \bar{z} \to \bar{z} + \bar{\varepsilon}(\bar{z})\,, \tag{3.41}$$

where ε and $\bar{\varepsilon}$ are infinitesimal holomorphic functions of z and $\bar{z}$ respectively.

We observe that 2D scale invariant field theory generally has infinite-dimensional conformal symmetry.

Conformal invariance of the CIFT is evident from (3.31), but also it is not difficult to check directly that the CIFT action (3.1) does not change its form under the transformations (3.40), provided one also transforms to new fields $(\psi, \bar{\psi}) \to (\psi_f, \bar{\psi}_f)$ defined as

$$\psi_f(z,\bar{z}) = \big(f'(z)\big)^{1/2}\, \psi(f(z), \bar{f}(\bar{z}))\,, \tag{3.42a}$$

$$\bar{\psi}_f(z,\bar{z}) = \big(\bar{f}'(\bar{z})\big)^{1/2}\, \psi(f(z), \bar{f}(\bar{z}))\,, \tag{3.42b}$$

where $f'(z)$ denotes the derivative of the function f. Generally, the transformation (3.32) should be understood as the change of variables in the functional integral over $(\psi, \bar{\psi})$, in the functional integral, that is why I am careful to write $\psi(z,\bar{z})$, not $\psi(z)$ in (3.42). Of course, the functional integration variable ψ is not a holomorphic function, but the correlation function $\langle \cdots \psi(z) \cdots \rangle$ is.

With vanishing trace component Θ, the equations (3.19) take the form

$$\bar{\partial} T = 0\,, \qquad \partial \bar{T} = 0\,. \tag{3.42}$$

These equations are common for any field theory with vanishing trace of the EM tensor. Such theories are known as Conformal Field Theories (CFT). At this point let me abandon specific discussion of the Ising field theory and describe some generalities of CFT.

The equations (3.42) express the holomorphic nature of the field T (anti-holomorphic for $\bar{T}$),

$$T = T(z)\,, \qquad \bar{T} = \bar{T}(\bar{z})\,. \tag{3.43}$$

Just like in the case of the fields ψ, $\bar{\psi}$, these equations mean that, say, any correlation function of the form

$$\langle\, T(z)\, O_1(x_1)\, \cdots O_n(x_n)\,\rangle \tag{3.44}$$

is a holomorphic function of z. It is also a single-valued function since the energy-momentum tensor is local with respect to all fields of the theory. Therefore the correlation function (3.44) has single-valued singularities (poles) at the points $z_1, \cdots, z_n$.

It admits Lorant expansions near these singular points, which can be expressed through the operator product expansions

$$T(z)\,O(w,\bar{w}) = \sum_{n\in\mathbf{Z}} (z-w)^{-n-2}\, L_n O(w,\bar{w})\,, \tag{3.45}$$

where $L_n O$ are some local fields of the theory. Just like we had with a_n, this expansion (and similar expansion with $\bar{T}(\bar{z})$) defines an infinite set of operators L_n, $\bar{L}_n$ acting in the space $\mathcal{F}$ of local fields of the theory,

$$L_n\,,\;\bar{L}_n\;: \qquad \mathcal{F}\;\to\;\mathcal{F} \tag{3.46}$$

Of course, these operators can be represented in terms of the contour integrals

$$L_n O(w,\bar{w}) = \oint_{C_w} \frac{dz}{2\pi i}\,(z-w)^{n+1}\,T(z)\,O(w,\bar{w})\,, \tag{3.47a}$$

$$\bar{L}_n O(w,\bar{w}) = \oint_{C_{\bar{w}}} \frac{d\bar{z}}{2\pi i}\,(\bar{z}-\bar{w})^{n+1}\,\bar{T}(\bar{z})\,O(w,\bar{w})\,, \tag{3.47b}$$

The last form (3.47) makes it explicit that the action of the operators L and $\bar{L}$ on the field O describes the transformation law of this field under infinitesimal conformal transformations. Recall that for general coordinate transformation

$$x^\mu \to x^\mu + \varepsilon^\mu(x)$$

the variation of the field O can be written in terms of the energy-momentum tensor as the sum of two terms

$$\delta_\varepsilon O(w,\bar{w}) = \int_{D_1} (\cdots) + \int_{C_1} (\cdots)\,, \tag{3.48}$$

see Eq.(3.30), where D_w is an arbitrary small domain containing the point $(w,\bar{w})$, and C_w is its boundary. The integrand in the first (volume) term in (3.48) involves the expression

$$\partial_\mu \varepsilon_\nu(x)\,T^{\mu\nu}(x) = 0 \qquad \text{for conformal} \quad \varepsilon^\mu(x)\,, \tag{3.49}$$

which vanishes for any conformal transformation. Therefore, for conformal transformations

$$z \to z + \varepsilon(z)\,, \qquad \bar{z} \to \bar{z} + \bar{\varepsilon}(\bar{z})\,. \tag{3.50}$$

only the contour integral remains,

$$\int_{C_w} \epsilon_{\lambda\mu} d\xi^\lambda(x)\, \varepsilon_\nu(x)\, T^{\mu\nu}(x)\, O(w)\,. \tag{3.51}$$

In terms of the complex coordinates and the components T, $\bar{T}$ it can be written as

$$\delta_\varepsilon O(w,\bar{w}) = \oint_{C_w} \frac{dz}{2\pi i}\, \varepsilon(z)\, T(z)\, O(w,\bar{w}) + \oint_{\bar{C}_{\bar{w}}} \frac{d\bar{z}}{2\pi i}\, \bar{\varepsilon}(\bar{z})\, \bar{T}(\bar{z})\, O(w,\bar{w})\,, \tag{3.52}$$

where $\varepsilon(z)$ and $\bar{\varepsilon}(\bar{z})$ are the functions in (3.50), and the contour C_w in the z-plane goes around the point w in the counterclockwise direction, while the contour $\bar{C}$ does the same in the $\bar{z}$ plane.

Consider particular case

$$\varepsilon(z) = \varepsilon\,, \quad \bar{\varepsilon}(\bar{z}) = \bar{\varepsilon} \qquad \text{constants}\,. \tag{3.52}$$

This case corresponds to homogeneous translations, and therefore

$$\delta_\varepsilon O(w,\bar{w} = \varepsilon\, \partial_w O(w,\bar{w}) + \bar{\varepsilon}\, \partial_{\bar{w}} O(w,\bar{w})\,, \tag{3.53}$$

that is, according to (3.52)

$$\oint_{C_w} \frac{dz}{2\pi i}\, \varepsilon(z)\, T(z)\, O(w,\bar{w}) = \partial O(w,\bar{w})\,, \quad \oint_{\bar{C}_{\bar{w}}} \frac{d\bar{z}}{2\pi i}\, \bar{\varepsilon}(\bar{z})\, \bar{T}(\bar{z})\, O(w,\bar{w}) = \bar{\partial} O(w,\bar{w})\,. \tag{3.54}$$

Comparing this with the definitions (3.47) of the operators L_n, $\bar{L}_n$ we find

$$L_{-1} O(w,\bar{w}) = \partial O(w,\bar{w})\,, \qquad \bar{L}_{-1} O(w,\bar{w}) = \bar{\partial} O(w,\bar{w})\,. \tag{3.55}$$

Next, consider dilations and rotations centered at the point $(w,\bar{w})$,

$$(z-w) \to \lambda\,(z-w)\,, \qquad (\bar{z}-\bar{w}) \to \bar{\lambda}\,(\bar{z}-\bar{w})\,, \tag{3.56}$$

where $|\lambda|$ represents the dilation factor, and $\arg\lambda$ is the rotation angle. Infinitesimal transformation of this form is

$$z \to z + a\,(z-w)\,, \qquad \bar{z} \to \bar{z} + \bar{a}\,(\bar{z}-\bar{w})\,, \tag{3.57}$$

where the real and imaginary parts of $a = \alpha + i\beta$ are infinitesimal dilation factor and infinitesimal rotation angle, respectively. Using this form of ε in (3.52) and comparing with (3.47) we find that the variation of $O(w, \bar{w})$ under the transformation (3.57) generated by the operators L_0 and $\bar{L}_0$,

$$\delta O(w, \bar{w}) = \big(a\, L_0 + \bar{a}\, \bar{L}_0\big)\, O(w, \bar{w})\,. \tag{3.58}$$

In other words, $L_0 + \bar{L}_0$ and $L_0 - \bar{L}_0$ describe response to dilations and rotations, respectively,

$$\text{Dilation} = L_0 + \bar{L}_0\,, \qquad \text{Rotation} = L_0 - \bar{L}_0\,. \tag{3.59}$$

The field O having the mass dimension D and spin S must be eigenvector of these operators,

$$(L_0 + \bar{L}_0) O(w, \bar{w}) = D\, O(w, \bar{w})\,, \qquad (L_0 - \bar{L}_0) O(w, \bar{w}) = S\, O(w, \bar{w})\,. \tag{3.60}$$

The separate eigenvalues Δ and $\bar{\Delta}$ of L_0 and $\bar{L}_0$

$$L_0 O(w, \bar{w}) = \Delta\, O(w, \bar{w})\,, \qquad \bar{L}_0 O(w, \bar{w}) = \bar{\Delta}\, O(w, \bar{w}) \tag{3.61}$$

are usually called the left and right conformal dimensions of O; obviously $D = \Delta + \bar{\Delta}$, $S = \Delta - \bar{\Delta}$.

For general n, one can consider infinitesimal conformal transformations of the form

$$z \to z + \varepsilon_n(z)\,, \qquad \bar{z} \to \bar{z} + \bar{\varepsilon}_n(\bar{z}) \tag{3.62}$$

with

$$\varepsilon_n(z) = \alpha\, (z - w)^{n+1}\,, \qquad \bar{\varepsilon}_n(\bar{z}) = \bar{\alpha}\, (\bar{z} - \bar{w})^{n+1}\,. \tag{3.63}$$

Variations of $O(w, \bar{w})$ under this transformation is described by the operators L_n, $\bar{L}_n$,

$$\delta_{\varepsilon_n, \bar{\varepsilon}_n} O(w, \bar{w}) = \big(\alpha\, L_n + \bar{\alpha}\, \bar{L}_n\big) O(w, \bar{w})\,. \tag{3.64}$$

I would like to note that the operators L_{-n} and $\bar{L}_{-n}$ with $n \geq 2$ describe the response of the field $O(w, \bar{w})$ to infinitesimal transformation which is singular at the insertion point $(w, \bar{w})$.

The operators L_n, form the Virasoro algebra,

$$[L_n, L_m] = (n - m)\, L_{n+m} + \frac{c}{12}\, (n^3 - n)\, \delta_{n+m,0}\,, \tag{3.65}$$

and $\bar{L}_n$'s obey the same commutation relations, and

$$[L_n, \bar{L}_m] = 0\,. \tag{3.66}$$

Here c is numerical parameter, conventionally called the central charge, which is important characteristic of the conformal field theory.

The commutation relations (3.65) can be established on rather general grounds. First, one notices that the fields T and $\bar{T}$ both have mass dimension 2; this follows directly from the definition (3.23) of the EM tensor. Also, one can check that the components T and $\bar{T}$ have spins $+2$ and -2, respectively. In other words, the conformal dimensions of these fields are

$$T\ : \quad (\Delta, \bar{\Delta}) = (2, 0)\,, \qquad \bar{T}\ : \quad (\Delta, \bar{\Delta}) = (0, 2)\,. \tag{3.67}$$

In the following argument I will also assume that the theory has no fields with negative mass dimensions (no growing correlations), and that the only field of zero dimension is the identity operator I.

Consider the product

$$T(z)\, T(z')\,. \tag{3.68}$$

It is holomorphic function of z with a single-valued singularity at $z = z'$. One can write the general form of the operator product expansion

$$T(z)\, T(z') = \sum_{n\in Z} \frac{R_n(z')}{(z-z')^n}\,, \tag{3.69}$$

where $R_n(z)$ are some local (holomorphic) fields. It is easy to see that R_n has $D = S = 4 - n$,

$$R_n\ : \quad (\Delta, \bar{\Delta}) = (4 - n, 0)\,. \tag{3.70}$$

the assumption of the absence of negative mass dimensions requires that $R_n = 0$ for $n > 4$, and that R_4 must be a multiple of identity; we will denote it $R_4 = c/2$ where c is a constant. Then the OPE (3.69) takes the form

$$T(z)\, T(z') = \frac{c}{2\,(z-z')^4} + \frac{R_3(z')}{(z-z')^3} + \frac{R_2(z')}{(z-z')^2} + \frac{R_1(z')}{z-z'} + \text{reg}\,. \tag{3.71}$$

The same arguments can be applied to the product (3.68) viewed as the function of z', leading to

$$T(z)\, T(z') = \frac{c}{2\,(z-z')^4} + \frac{R_3(z)}{(z'-z)^3} + \frac{R_2(z)}{(z'-z)^2} + \frac{R_1(z)}{z'-z} + \text{reg}\,. \tag{3.72}$$

The Eq.(3.72) is consistent with (3.71) only if

$$R_3(z) = 0\,, \qquad \text{and} \qquad 2\,R_1(z) = \partial R_2(z)\,. \tag{3.73}$$

Next, we have by definition

$$\partial T(z') = L_{-1} T(z') = \oint_{C_{z'}} \frac{dz}{2\pi i}\, T(z)\, T(z')\,, \tag{3.74}$$

which shows that

$$R_1(z) = \partial T(z)\,, \qquad \text{and hence} \quad R_2(z) = 2\,T(z)\,. \tag{3.75}$$

therefore, under our assumptions, the OPE (3.69) has the general form

$$T(z)\,T(z') = \frac{c}{2\,(z-z')^4} + \frac{2\,T(z')}{(z-z')^2} + \frac{\partial T(z')}{z-z'} + \text{reg}\,. \tag{3.76}$$

Exercise: Using the definition of L_n's as the contour integrals, and using contour deformations, show that (3.65) follow from (3.76)

The operators L_n, $\bar{L}_n$ act in the space $\mathcal{F}$ of local fields of the CFT. The commutators

$$[L_n, L_0] = n\, L_n \tag{3.77}$$

show that the operator L_n with positive n lowers the mass dimension by n units. Absence of negative mass dimensions requires that for any $O \in \mathcal{F}$ $L_n O$ must vanish for $n > D(O)$. It follows that $\mathcal{F}$ contains the *primary fields* Φ_a which satisfy

$$L_n \Phi_a = \bar{L}_n \Phi_a = 0 \quad \text{for} \quad n > 0\,,$$

$$L_0 \Phi_a = \Delta_a \Phi_a\,, \qquad \bar{L}_0 \Phi_a = \bar{\Delta}_a \Phi_a\,.$$

Thus, typical structure of $\mathcal{F}$ is the sum of irreducible representations of the Virasoro algebra,

$$\mathcal{F} = \oplus_a \mathcal{F}_a\,,$$

where $\mathcal{F}_a$ contains Φ_a as well as all its independent "descendants", i.e. local fields generated from Φ_a by applying L_n and $\bar{L}_n$ with negative n.

After this overview of general structure of CFT, let us get back to the CIFT. Characteristic feature of the CIFT is the presence of the holomorphic fermions $\psi(z), \bar{\psi}(\bar{z})$, and the expressions

$$T = -\frac{1}{2}\,\psi\partial\psi\,, \qquad \bar{T} = \bar{\psi}\bar{\partial}\bar{\psi} \tag{3.81}$$

These expressions are understood as follows. Consider again the OPE

$$\psi(z)\psi(z') = \frac{1}{z-z'} + \text{reg}\,. \tag{3.82}$$

In fact, in this case it is easy to understand the structure of the regular terms as well. We have by definition

$$\psi(z)\psi(z') = \frac{1}{z-z'} + :\psi(z)\psi(z'): \tag{3.83}$$

where the first term represents the wick contraction of the two fields, and the symbol : ... : stands for the Wick normal ordering (which means that all contractions inside : ... : are excluded). It is easy to see looking at arbitrary correlation function that the Wick ordered product is regular at $z = z'$. It can be expanded in Taylor series in the powers of the difference $z - z'$. Since $:\psi(z)\psi(z): = 0$, we have

$$\psi(z)\psi(z') = \frac{1}{z-z'} - (z-z')\ :\psi(z')\partial\psi(z'): + O((z-z')^2) \tag{3.84}$$

The regular term explicitly written in (3.84) is proportional to

$$T(z') = -\frac{1}{2}\ :\psi(z')\partial\psi(z'): \tag{3.85}$$

Thus, (3.84) reads

$$\psi(z)\psi(z') = \frac{1}{z-z'} + 2\,(z-z')\,T(z') + O((z-z')^2) \tag{3.86}$$

Equivalent statement is

$$T(w) = \frac{1}{2}\oint_{C_w}\frac{dz}{2\pi i}\,(z-w)^{-2}\,\psi(z)\psi(w)\,. \tag{3.87}$$

It is now possible to express the action of L_n's on the spin states $O \in \mathcal{R}$ in terms of the operators a_n. Consider the integral

$$I_{n,m}O(0) = \oint_{C_0}\frac{dz}{2\pi i}\,z^{n+1/2}\oint_{C_z}\frac{dw}{2\pi i}\,w^{m+1/2}\,(w-z)^{-2}\,\psi(w)\psi(z)\,O(0) \tag{3.88}$$

where C_z encircles the point z, see **Fig.2**.

Using (3.86) we can evaluate the integral over w

$$\oint_{C_z} \frac{dw}{2\pi i}\, w^{m+1/2}\,(w-z)^{-2}\,\psi(w)\psi(z) =$$

$$= \oint_{C_z} \frac{dw}{2\pi i}\, w^{m+1/2}\,(w-z)^{-2}\left[\frac{1}{w-z} + 2\,(w-z)\,T(z) + ...\right] =$$

$$= \frac{1}{2}\,(m^2 - 1/4)\, z^{m-3/2} + 2\, z^{m+1/2}\, T(z)\,. \tag{3.89}$$

Then the z integral evaluates to

$$I_{n,m} = 2\, L_{n+m} + \frac{1}{2}\,(m^2 - 1/4)\,\delta_{n+m,0}\,. \tag{3.90}$$

On the other hand, the contour C_z can be represented as the difference

$$\oint_{C_z} = \oint_{C_+} - \oint_{C_-}\,, \tag{3.91}$$

where C_+ is placed outside C_0 and C_- is inside C_0. (**Fig.3**).

Then

$$I_{n,m}O(0) = \oint_{C_+} \frac{dw}{2\pi i}\, w^{m+1/2} \oint_{C_0} \frac{dz}{2\pi i}\, z^{n+1/2}\,(w-z)^{-2}\,\psi(w)\psi(z)\,O(0) -$$

$$
-\oint_{C_0} \frac{dz}{2\pi i}\, z^{n+1/2} \oint_{C_-} \frac{dw}{2\pi i}\, w^{m+1/2}\, (w-z)^{-2}\, \psi(w)\psi(z)\, O(0)\,. \tag{3.92}
$$

In the first term we have $|w| > |z|$ and write

$$
(z-w)^{-2} = \sum_{k=0}^{\infty} k\, \frac{z^{k-1}}{w^{k+1}}\,.
$$

Then this integral evaluates to

$$
\sum_{k=0}^{\infty} k\, a_{m-k} a_{n+k}\, O(0)\,. \tag{3.94}
$$

In the second term, we have instead $|z| > |w|$, so we can expand

$$
(z-w)^{-2} = \sum_{k=0}^{\infty} k\, \frac{w^{k-1}}{z^{k+1}}\,,
$$

hence the second term evaluates to

$$
\sum_{k=0}^{\infty} k\, a_{n-k} a_{m+k}\, O(0)\,. \tag{3.95}
$$

Finally

$$
\begin{aligned}
L_{n+m}\, O(0) &= \frac{1}{4}\,(1/4 - m^2)\, \delta_{n+m,0}\, O(0) + \\
&+ \frac{1}{2} \sum_{k=0}^{\infty} k \left[a_{m-k} a_{n+k} + a_{n-k} a_{m+k}\right] O(0)\,.
\end{aligned} \tag{3.96}
$$