L2

The last time we have introduced the disorder parameter for the square-lattice Ising model with $H = 0$. The disorder variables $\mu_{\tilde{\mathbf{x}}}$ are attached to the cites $\tilde{\mathbf{x}}$ of the dual lattice. Insertion of the pair $\mu_{\tilde{\mathbf{x}}_1}\,\mu_{\tilde{\mathbf{x}}_2}$ into any correlation function is equivalent to introducing some sort of "dislocation line" $\Gamma_{\tilde{\mathbf{x}}_1\,\tilde{\mathbf{x}}_2}$ on the dual lattice, with the end-points at $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$. Namely, in the distribution function of the spins $\sigma_{\mathbf{x}}$

$$P\{\sigma\} \sim e^{K\sum_{nn}\,\sigma_{\mathbf{x}}\sigma_{\mathbf{y}}} \tag{2.1}$$

one changes the signs of K for all links of the original lattice which cross the contour $\Gamma_{\tilde{\mathbf{x}}_1\,\tilde{\mathbf{x}}_2}$. This of course is equivalent to inserting factor

$$e^{-2K\,\sigma_{\mathbf{x}}\sigma_{\mathbf{y}}} \tag{2.2}$$

for every link $(\mathbf{xy})$ which crosses Γ.

In what follows it will be convenient to split the pair

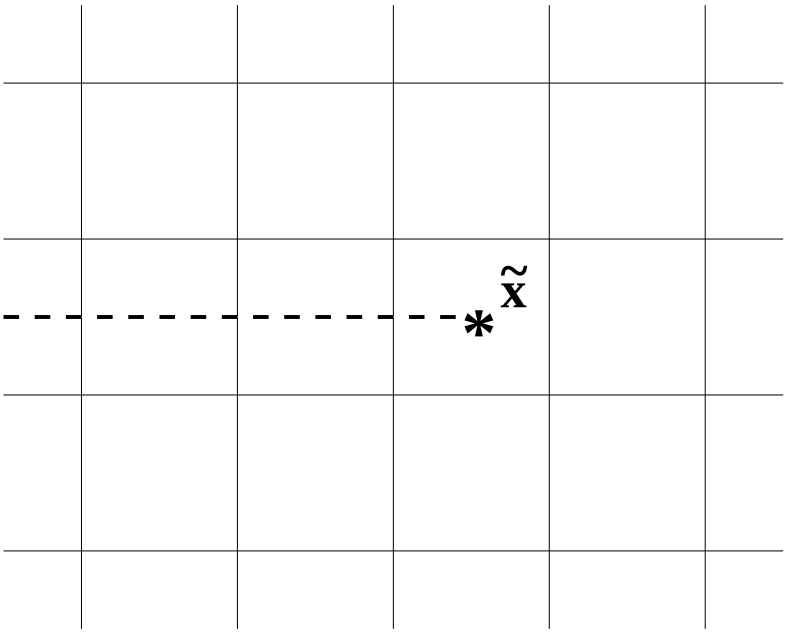
$$\mu_{\tilde{\mathbf{x}}_1}\,\mu_{\tilde{\mathbf{x}}_2} = T\{\Gamma_{\tilde{\mathbf{x}}_1\,\tilde{\mathbf{x}}_2}\} \tag{2.3}$$

into separate factors associated with individual insertions of μ. Since correlation functions involving insertions (2.3) essentially depend only on the end-points $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ but not on the exact form of the contour Γ (with the subtlety about the overall sign, as we discussed the last time), one can split

$$\Gamma_{\tilde{\mathbf{x}}_1\,\tilde{\mathbf{x}}_2} = \Gamma_{\tilde{\mathbf{x}}_1} + \Gamma_{\tilde{\mathbf{x}}_2}\,, \tag{2.4}$$

where the dual-lattice contour $\Gamma_{\tilde{\mathbf{x}}}$ extends from some reference point $\tilde{\mathbf{x}}_0$ to $\tilde{\mathbf{x}}$. In fact, when an infinite lattice is considered, the most convenient thing to do is to throw the point $\tilde{\mathbf{x}}_0$ away to infinity. In what follows I will always assume that the contour $\Gamma_{\tilde{\mathbf{x}}}$ starts at the left horizontal infinity and ends at $\tilde{\mathbf{x}}$ (see **Fig.1**), and identify

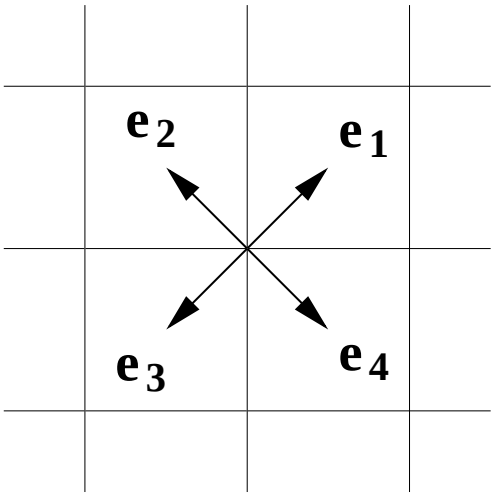
$$\mu_{\tilde{\mathbf{x}}} = T\{\Gamma_{\tilde{\mathbf{x}}}\}\,. \tag{2.5}$$



Let us now show that the Ising model with $H = 0$ is the theory of free fermions, which makes it completely solvable. The fermions $\psi(\mathbf{x})$ appear as the products of the spin variable $\sigma_{\mathbf{x}}$ and the disorder variable $\mu_{\tilde{\mathbf{x}}}$ sitting at the nearby cite $\tilde{\mathbf{x}}$ of the dual lattice. There are four closest dual cites $\tilde{\mathbf{x}}$ to every cite $\mathbf{x}$, and to label them I introduce four vectors

$$\mathbf{e}_a\,, \quad a = 1, 2, 3, 4\,,$$

each of the length $1/\sqrt{2}$, and each pointing at 45^o to the original lattice axes, in four possible directions NE, NW, SW, and SE



$$\mathbf{e}_1 \to \mathrm{NE}\,, \quad \mathbf{e}_2 \to \mathrm{NW}\,, \quad \mathbf{e}_3 \to \mathrm{SW}\,, \quad \mathbf{e}_4 \to \mathrm{SE}\,. \tag{2.6}$$

(of course this set has redundancy, $\mathbf{e}_3 = -\mathbf{e}_1$, $\mathbf{e}_4 = -\mathbf{e}_2$, but it is convenient to keep separate notations for all four, as in (2.6)). The four dual cites closest to $\mathbf{x}$ are

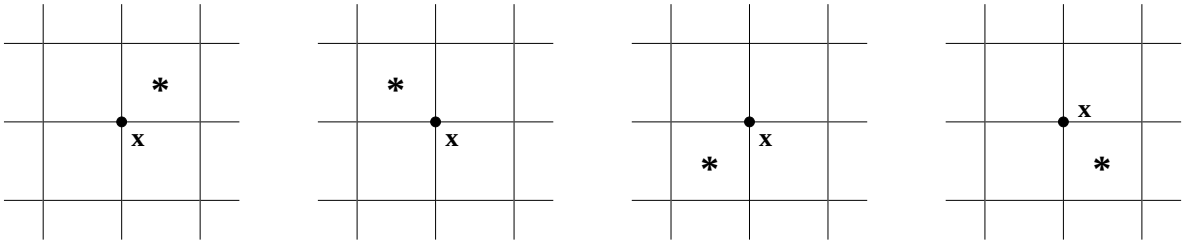
$$\mathbf{x} + \mathbf{e}_a\,, \quad a = 1, 2, 3, 4\,. \tag{2.7}$$

as is shown in **Fig.2**.

The fermion variables $\psi_{a\,,\mathbf{x}}$ are defined as

$$\psi_{a\,,\mathbf{x}} = \sigma_{\mathbf{x}}\, \mu_{\mathbf{x}+\mathbf{e}_a}\,. \tag{2.8}$$

As these objects involve both σ and μ, there is the sign ambiguity which I have mentioned already. Precise way how this ambiguity is fixed is not important, but to make things as simple as possible I will always assume that the contour $\Gamma_{\tilde{\mathbf{x}}}$ associated with $\mu_{\tilde{\mathbf{x}}}$ is a horizontal straight line from minus infinity to $\tilde{\mathbf{x}}$, or it is deformable to such straight line. It is convenient to use pictorial representations of the objects (2.8) shown in **Fig.3**, where bold dots and crosses represent insertions of σ and μ, respectively.



Let us show that the variables (2.8), being inserted into any correlation function

$$\langle \cdots \psi_{a\,,\mathbf{x}} \cdots \rangle \tag{2.9}$$

satisfy closed linear difference equation, which is the lattice version of the Dirac equation.

Consider for instance

$$\psi_{1\,,\mathbf{x}} = \sigma_{\mathbf{x}}\, \mu_{\mathbf{x}+\mathbf{e}_1}\,. \tag{2.10}$$

The disorder part $\mu_{\mathbf{x}+\mathbf{e}_1}$ is by definition a product of the factors (2.2) (or (2.13) below) along associated (horizontal) contour. One can split this product into the product representing insertion μ at the next dual cite to the left, times the factor associated with the last link $(\mathbf{x}+\mathbf{e}_2, \mathbf{x}+\mathbf{e}_1)$ (see **Fig.4**)

$$\mu_{\mathbf{x}+\mathbf{e}_1} = \mu_{\mathbf{x}+\mathbf{e}_2}\, e^{-2K\,\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{\Delta}_2}}\,, \tag{2.11}$$

where $\mathbf{\Delta}_2$ is one of the basic vectors of the lattice

$$\mathbf{\Delta}_1 = \mathbf{e}_1 + \mathbf{e}_4 = (1,0)\,, \qquad \mathbf{\Delta}_2 = \mathbf{e}_1 + \mathbf{e}_2 = (0,1)\,. \tag{2.12}$$

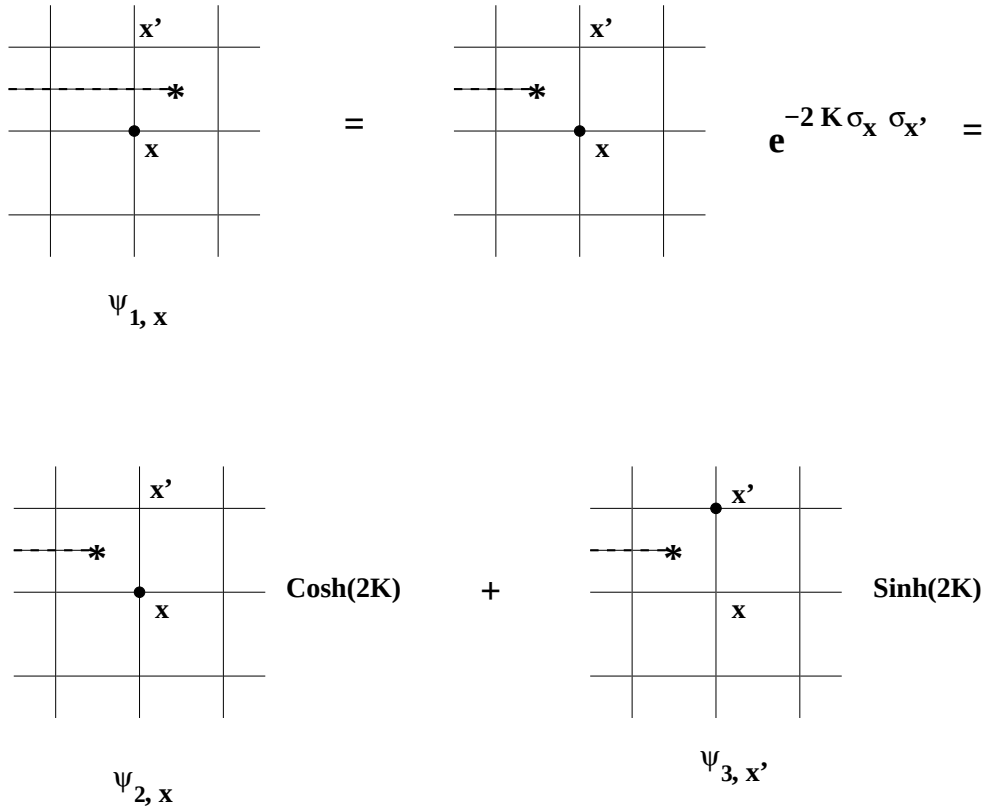
Now, as usual

$$e^{-2K\,\sigma_{\mathbf{x}}\sigma_{\mathbf{x}'}} = \cosh 2K - \sigma_{\mathbf{x}}\sigma_{\mathbf{x}'}\ \sinh 2K\,. \tag{2.13}$$

Substituting (2.11) into (2.10), and using (2.13) as well as the fact that $\sigma_{\mathbf{x}}^2 = 1$, one finds

$$\psi_{1\,,\mathbf{x}} = \big(\cosh 2K\big)\ \psi_{2\,,\mathbf{x}} - \big(\sinh 2K\big)\ \psi_{3\,,\mathbf{x}+\mathbf{\Delta}_2}\,. \tag{2.14}$$

This calculation is illustrated in **Fig.5**.



Similar equations can be derived for the other components ψ_a. This can be done exactly as above provided one first makes appropriate deformation of the contour associated with the disorder variable. To illustrate this last point, consider

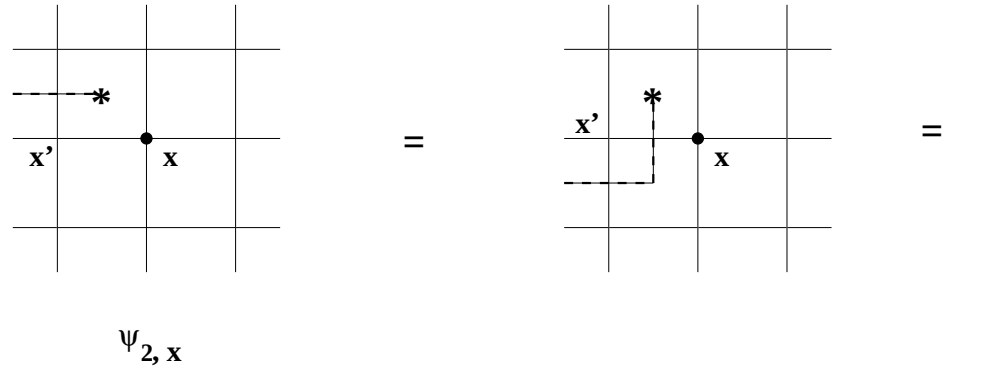
$$\psi_{2\,,\mathbf{x}} = \sigma_{\mathbf{x}}\,\mu_{\mathbf{x}+\mathbf{e}_2}\,. \tag{2.15}$$

Let us first deform the contour associated with the μ insertion here as is shown in the **Fig.6**, and then apply the same transformation as in (2.11) to the last link of this contour. We obtain

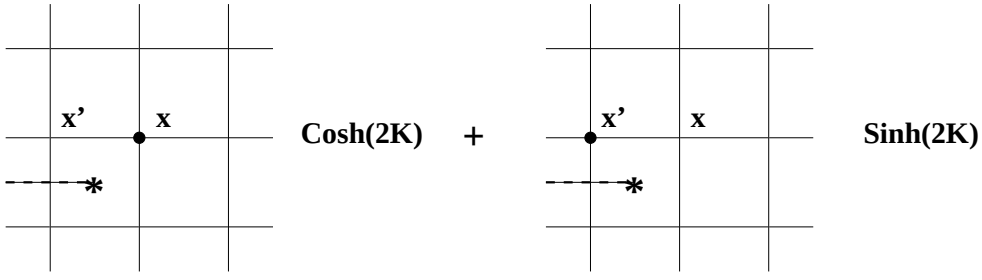
$$\mu_{\mathbf{x}+\mathbf{e}_2} = \mu_{\mathbf{x}+\mathbf{e}_3}\,e^{-2K\,\sigma_{\mathbf{x}-\boldsymbol{\Delta}_1}\sigma_{\mathbf{x}}} = \mu_{\mathbf{x}+\mathbf{e}_3}\left[\big(\cosh 2K\big)\,-\big(\sinh 2K\big)\,\sigma_{\mathbf{x}-\boldsymbol{\Delta}_1}\sigma_{\mathbf{x}}\right],$$

and then

$$\psi_{2\,,\mathbf{x}} = \big(\cosh 2K\big)\,\psi_{3\,,\mathbf{x}} - \big(\sinh 2K\big)\,\psi_{4\,,\mathbf{x}-\boldsymbol{\Delta}_1}\,. \tag{2.16}$$



$\Psi_{2,\,x}$



$\Psi_{3,\,x}$ $\qquad$ $\Psi_{4,\,x'}$

The equations (2.14), (2.16), and similar equations for the other two components of ψ can be written in the following symmetric form

$$\psi_{a\,,\mathbf{x}} = \big(\cosh 2K\big)\,\psi_{a+1\,,\mathbf{x}} - \big(\sinh 2K\big)\,\psi_{a+2\,,\mathbf{x}+\boldsymbol{\Delta}_{a+1}}\,. \tag{2.17}$$

where ψ_a with $a \neq 1, 2, 3, 4$ are defined by the equations

$$\psi_{a+4\,,\mathbf{x}} = -\psi_{a\,,\mathbf{x}}\,. \tag{2.18}$$

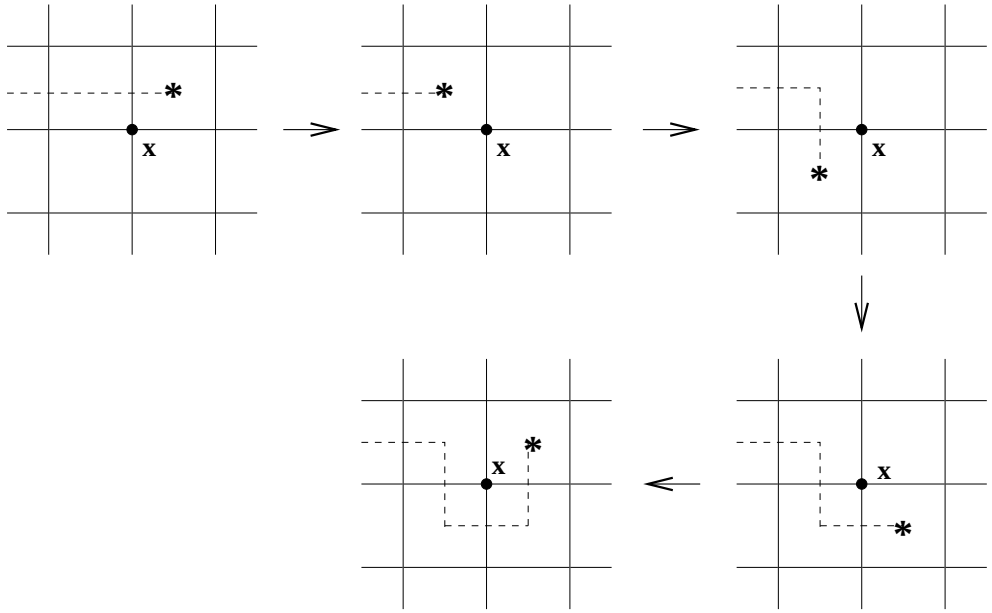
Also, by definition,

$$\mathbf{e}_{a+4} = \mathbf{e}_a\,, \qquad \mathbf{\Delta}_{s+2} = -\mathbf{\Delta}_a\,.$$

The equation (2.18) is natural if one thinks of ψ_{a+4} as the result of successive 90^o rotations of the object

$$\psi_{a\,,\mathbf{x}} = \sigma_{\mathbf{x}}\,\mu_{\mathbf{x}+\mathbf{e}_a}\,.$$

Although the full 360^o rotation returns σ and μ to the original positions, it is important to remember that after such rotation the contour associated with μ winds once around the point $\mathbf{x}$ (**Fig.7**); the minus sign in (2.18) appears as the result of "unwinding" of this contour.



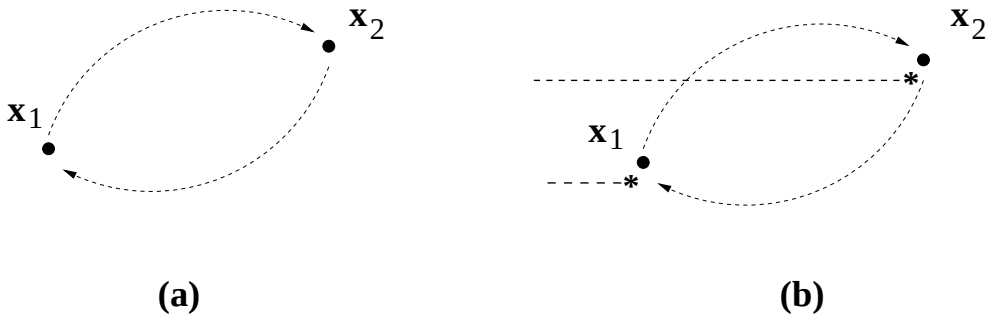
We observe that the composite objects $\psi_{a\,,\mathbf{x}}$ obey linear "equations of motion" (2.17). This means that ψ_a is a free field. It is easy to see that ψ_a is fermi field. In the language of the lattice correlation functions the signature of a fermi field is the following property. Consider arbitrary correlation function of the form

$$\langle \cdots \psi_{a\,,\mathbf{x}_1}\, \psi_{a\,,\mathbf{x}_2} \cdots \rangle\,. \tag{2.19}$$

Let us move the points $\mathbf{x}_1$ and $\mathbf{x}_2$ in such a way that as the result of the move they interchange their positions, i.e.$\mathbf{x}_1 \to \mathbf{x}_1$ and $\mathbf{x}_2 \to \mathbf{x}_1$. For a fermi field such move results in the change of the sign of the correlation function (2.19),

$$(2.19) \to -\left\langle \cdots \psi_{a\,,\mathbf{x}_2}\, \psi_{a\,,\mathbf{x}_1} \cdots \right\rangle . \tag{2.20}$$

In simple words, (2.19) is antisymmetric in $\mathbf{x}_1$ and $\mathbf{x}_2$. It is easy to see that the construction (2.15) guarantees this fermion exchange property - when one moves the points to interchange their positions, either the contour associated with $\psi_{\mathbf{x}_1}$ crosses $\mathbf{x}_2$, or the other way round, the contour attached to $\psi_{\mathbf{x}_2}$ crosses $\mathbf{x}_1$ (see **Fig.8**), leading to the minus sign. And it is easy to check that if in (2.19) one just brings $\mathbf{x}_1$ around $\mathbf{x}_2$ all extra signs cancel and (2.19) returns to its original value - the correlation function (2.19) is single valued, which means that $\psi_{a\,,\mathbf{x}}$ is *local* fermi field.



On the other hand, consider correlation function involving, besides ψ_a, any number of σ, or any number of μ, or both, for instance

$$\left\langle \psi_{a\,,\mathbf{x}}\, \sigma_{\mathbf{x}_1}\, \mu_{\tilde{\mathbf{x}}_2}\, \cdots \right\rangle \tag{2.21}$$

It follows from the properties of σ and μ, and from our construction of ψ that such correlation function changes sign every time $\mathbf{x}$ is brought around either $\mathbf{x}_1$ or $\tilde{\mathbf{x}}_2$. One says that the fermi field $\psi_{a\,,\mathbf{x}}$ is not local with respect to σ and μ.

Let me mention here few simple identities involving these fermi fields. As follows directly from the definition (2.8), we have for instance

$$\sigma_{\mathbf{x}}\, \sigma_{\mathbf{x}+\boldsymbol{\Delta}_1} = \psi_{1\,,\mathbf{x}}\, \psi_{2\,,\mathbf{x}+\boldsymbol{\Delta}_1} = \psi_{4\,,\mathbf{x}}\, \psi_{3\,,\mathbf{x}+\boldsymbol{\Delta}_1} \tag{2.22}$$

and similarly for $\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\boldsymbol{\Delta}_2}$, i.e. the energy density of the model

$$\varepsilon_{\mathbf{x}} = -\frac{J}{2} \sum_{a=1}^{4} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\boldsymbol{\Delta}_a} \tag{2.23}$$

is expressed as the fermion bilinear. Similar expressions exist for the nearest-neighbor products

$$\mu_{\tilde{\mathbf{x}}}\, \mu_{\tilde{\mathbf{x}}+\mathbf{\Delta}_a}\,.$$

In principle, the linear equations (2.17) (with suitable boundary conditions) can be used to find exact solution of the Ising model directly on the lattice. The solution shows critical point at

$$K = K_c\,, \qquad K_c = \frac{1}{2}\,\log\left(\sqrt{2}+1\right), \tag{2.24}$$

exactly as duality predicts. When $K \to K_c$ the correlation length becomes large as compared to the lattice spacing, and one can obtain continuous field theory by taking the scaling limit, i.e. the limit $K \to K_c$ accompanied by an appropriate change of the length scale in order to keep the correlation length finite. This procedure is straightforward but somewhat cumbersome. We can get to the same result by taking the continuous limit directly in the linear equation (2.17).

Note that at the critical point $K = K_c$ we have

$$\cosh 2K_c = \sqrt{2}\,, \qquad \sinh 2K_c = 1\,. \tag{2.25}$$

One can check that with this coefficients the linear equations admit constant (i.e. $\mathbf{x}$-independent) solutions of the form

$$\psi_a = \omega^a\; C + \bar{\omega}^a\; \bar{C}\,, \tag{2.26}$$

where

$$\omega = e^{\frac{i\pi}{4}}\,, \qquad \bar{\omega} = e^{-\frac{i\pi}{4}}\,, \tag{2.27}$$

and C and $\bar{C}$ are arbitrary constants. This signals appearance of gapless modes with infinite correlation radius. Indeed, if one writes

$$\psi_a = \frac{\omega^a}{\sqrt{\pi}}\,\psi(\mathbf{x}) + \frac{\bar{\omega}^a}{\sqrt{\pi}}\,\bar{\psi}(\mathbf{x}) \tag{2.28}$$

and assumes that $\psi(\mathbf{x})$ and $\bar{\psi}(\mathbf{x})$ have very slow rate of change at the lattice scales, so that the lattice shift in the r.h.s. of (2.17) can be replaced by the derivative,

$$\psi(\mathbf{x}+\mathbf{\Delta}_a) \to \psi(\mathbf{x}) + \mathbf{\Delta}_a\,\partial_a\psi(\mathbf{x}) \tag{2.29}$$

the equation (2.17) leads to

$$\left(\partial_1 + i\partial_2\right)\psi(\mathbf{x}) = 0\,, \qquad \left(\partial_1 - i\partial_2\right)\bar{\psi}(\mathbf{x}) = 0\,. \tag{2.30}$$

This has the form of massless Dirac equation in 2D (more precisely, (2.30) involves self-conjugated spinor fields, i.e. it is the 2D Majorana equation).

If we slightly shift away from the critical point, i.e. set

$$K = K_c + k\,. \tag{2.31}$$

with $k << 1$, the above gapless modes remain soft. Using

$$\cosh 2K = \sqrt{2} + 2k + O(k^2)\,, \qquad \sinh 2K = 1 + 2\sqrt{2}\,k + O(k^2)\,, \tag{2.32}$$

and neglecting all terms beyond the linear one in k, one finds instead of (2.30)

$$\begin{aligned} \big(\partial_1 + i\partial_2\big)\,\psi(\mathbf{x}) &= im\,\bar\psi(\mathbf{x})\,, \\ \big(\partial_1 - i\partial_2\big)\,\bar\psi(\mathbf{x}) &= -im\,\psi(\mathbf{x})\,, \end{aligned} \tag{2.33}$$

which is massive Majorana equation, with the mass related to k

$$\varepsilon\, m = 4\,k\,, \tag{2.34}$$

where I have restored the notation ε for the lattice spacing to make dimensional counting straightforward. Taking the scaling limit amounts to sending k to zero, while looking at the theory at the length scales of the order of the correlation length

$$R \sim R_c = m^{-1}\,, \qquad R >> \varepsilon\,. \tag{2.35}$$

We see that in this limit the Ising model reduces to the free M Majorana theory described by the equations of motion (2.33).

Strictly speaking, establishing the equations of motion (2.33) is not sufficient to prove equivalence. One has to find the boundary conditions which would determine the correlation functions, most important of which concerns with the singularities of the correlation functions at the coincident points. In deriving the equations (2.17) we have ignored possibility of other insertions in the correlation function, assuming that such extra insertions are located at finite lattice distance from the point $\mathbf{x}$. More careful analysis shows that if other fermion insertions are present at some points $\mathbf{x}_1,\ \cdots\ \mathbf{x}_n$, the equations (2.17) are violated by some constant (i.e. field independent) terms when $\mathbf{x}$ hits one of the points $\mathbf{x}_k$. In the scaling limit these terms modify (rather complete) the equations (2.33) as follows

$$\big(\partial_{x_1} + i\partial_{x_2}\big)\langle\,\psi(x)\,\psi(y)\;X\,\rangle = im\,\langle\,\bar\psi(x)\,\psi(y)\;X\,\rangle + i\pi\,\delta(x-y)\,\langle\,X\,\rangle + \cdots\,, \tag{2.36}$$

where X stands for any combination of the fermion insertions, and the r.h.s. can have other delta-function terms if X contains ψ at other points. The second of the equations (2.33) is completed by similar delta-function terms.

In short, the scaling limit of the Ising model at $H = 0$ is the free fermion field theory whose properties can be encoded in the action

$$\mathcal{A}_{\rm FF} = \frac{1}{2\pi} \int \left[\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi} + im\,\bar{\psi}\psi\right] d^2x \tag{2.37}$$

In what follows I will use the complex coordinates

$$z = x_1 + ix_2\,, \qquad \bar{z} = x_1 - ix_2 \tag{2.38}$$

on the Euclidean plane. The derivatives in (2.37) stand for the complex derivatives

$$\partial = \partial_z = \frac{1}{2}\left(\partial_1 - i\partial_2\right), \qquad \bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}\left(\partial_1 + i\partial_2\right). \tag{2.39}$$

The correlation functions in this free theory can be understood in terms of the gaussian functional integral

$$\langle\,\cdots\rangle = Z^{-1} \int \left[D\psi, D\bar{\psi}\right] \left(\,\cdots\right) e^{-\mathcal{A}_{\rm FF}[\psi,\bar{\psi}]} \tag{2.40}$$

over the Grassmanian (anticommuting) field variables $\psi(x),\ \bar{\psi}(x)$.

As the field theory, (2.40) is not terribly interesting. It contains a single sort of neutral particles with fermion statistics, which otherwise do not interact. The particle's mass is $|m|$ (remember, the parameter $m \sim K - K_c$, it can be positive or negative depending on whether we are in the low or in the high T phase). From the point of view of the functional integral (2.40) itself the sign of m is irrelevant. It can be changed by a simple change of variables in (2.40)

$$\psi \to \psi\,, \quad \bar{\psi} \to -\bar{\psi}\,, \qquad m \to -m\,. \tag{2.41}$$

The symmetry (2.41) is what the duality transformation of the Ising model does to the fermion field of (2.40). According to my convention $m\,\varepsilon = 4(K - K_c)$ the parameter m is positive in the low-T phase and it is negative in the high-T phase

$$\begin{aligned} {\rm high-T}: &\qquad m < 0\,, \\ {\rm low-T}: &\qquad m > 0\,. \end{aligned} \tag{2.42}$$

$m = 0$ corresponds to the critical point. In the high-T domain the free particles are the "spin-particles" - we will see that the spin insertion $\sigma(x)$ can emit a single particle. In the low-T phase the particles are rather interpreted as the "kinks" separating domains with opposite orientations of the spins.

Basic thermodynamic properties of the Ising theory near criticality are readily derived from the gaussian functional integral (2.40). I'll skip explicit calculation. The specific free energy

$$F = -\frac{\log Z}{V} \tag{2.43}$$

(V is the 2-volume of the space) develops the famous Onsager's singularity

$$F_{\text{sing}} = \frac{m^2}{8\pi} \log m^2 \tag{2.44}$$

which leads to logarithmic divergence of the heat capacity near the critical point. Magnetization and other related thermodynamic quantities will be discussed later on.

As the particle theory (2.37) is rather boring. We would like to add more fun by turning on the external field H (which, I remind, was set to zero in the previous discussion). We would like to modify the action by adding corresponding term, something proportional to the "spin density"

$$\mathcal{A} = \mathcal{A}_{\text{FF}} + H \int \sigma(x)\, d^2x\,. \tag{2.45}$$

But at this point it is not quite clear how to do that, how to understand the last term in (2.45) in terms of the free-fermion theory. We first need to give useful definition of the "spin" field $\sigma(x)$ in the free fermion theory (2.37) and find out all we can about its properties. That is what we will be doing for the next few lectures.

The main property to start with was already observed in the lattice theory (when looking at the correlation functions of the type (2.21)). Namely, if we consider a correlation function which involves $\psi(x)$ (or $\bar\psi(x)$) as well as several σ-insertions, i.e.

$$\langle\, \psi(x)\, \sigma(x_1) \cdots \sigma(x_n) \cdots \rangle \tag{2.46}$$

it is a double-valued function of the Euclidean point x, which changes the sign every time the point x is brought around any one of the points x_1, x_2, $\cdots$ x_n, as is shown in **Fig.9**. The same property holds true if one replaces ψ by $\bar\psi$.

However, this property alone does not define the field $\sigma(x)$ uniquely. In fact, there are infinitely many local fields which satisfy this property. Indeed, within the lattice theory we could have taken a product of three spins in some neighboring points, say

$$\sigma_{\mathbf{x}}\,\sigma_{\mathbf{x}+\mathbf{\Delta}_1}\,\sigma_{\mathbf{x}-\mathbf{\Delta}_2}\,. \tag{2.47}$$

(see **Fig.10**). By taking the scaling limit $R_c >> \varepsilon$ we still shrink all such configurations to a point, thus producing a local field. Obviously, this field (let's for the moment call it $\sigma_3(x)$) has the same property that the product

$$\psi(x)\,\sigma_3(x_1) \to -\psi(x)\,\sigma_3(x_1) \tag{2.48}$$

changes sign when x goes around x_1. One can throw in any odd number of the lattice σ insertions at different finite lattice separations, and all will become local fields with the same property (2.47). On top of that, there is the dual field $\mu(x)$, which also brings along an infinite number of new fields.

Thus we have an infinite-dimensional space of "spin fields", the fields whose product with $\psi(x)$ has the property (2.48). I will denote this space $\mathcal{R}$. By the definition, for any $O \in \mathcal{R}$ the products

$$\psi(x)\,O(x_1) \to -\psi(x)\,O(x_1)\,, \qquad \bar\psi(x)\,O(x_1) \to -\bar\psi(x)\,O(x_1) \tag{2.49}$$

change sign when x is brought around x_1. $\mathcal{R}$ is the vector space since the sum of any two fields satisfying the property (2.49) satisfies this property as well. On the contrast, I will use the notation $\mathcal{NS}$ for the space of fields which are local with respect to our free fermions $(\psi, \bar\psi)$. That means that for any field $O \in \mathcal{NS}$ the product $\psi(x)\,O(x_1) \to \psi(x)\,O(x_1)$, i.e. does not change when x is brought around x_1. The structure of the space $\mathcal{NS}$ is more or less clear - it consists of all local composite fields built from the fermions ψ and $\bar\psi$ and their derivatives, like

$$\psi\partial\psi\,, \quad \bar\psi\partial^n\psi\,, \quad \text{etc}\,. \tag{2.50}$$

We need some tools to sort out the content of the space $\mathcal{R}$ of the "spin fields". The notations $\mathcal{R}$ and $\mathcal{NS}$ are motivated by the terminology of the string theory - these spaces are analogous to the Ramond and the Neveu-Schwartz sectors of the string theory.

To make the analysis as simple as possible, let me consider first the case $m = 0$, i.e. the critical point itself. In this case the theory (2.37) has no dimensional parameters and hence enjoys the scaling symmetry. In fact it has larger conformal

symmetry, and can be analyzed by methods of conformal field theory. I am planning to discuss this point of view a bit later.

For now, it is important to note that at $m = 0$ the equations of motion (2.33) simplify to

$$\bar{\partial}\psi = 0\,, \qquad \partial\bar{\psi} = 0\,. \tag{2.51}$$

These equations state that ψ is holomorphic function of $z = x_1 + ix_2$, and likewise $\bar{\psi}$ is a holomorphic function of $\bar{z} = x_1 - ix_2$. For that reason I will write

$$\psi = \psi(z)\,, \qquad \bar{\psi} = \bar{\psi}(\bar{z})\,. \tag{2.52}$$

To be more precise, consider correlation function of the form

$$\langle\, \psi(z)\, O_1(x_1) \cdots O_n(x_n)\, \rangle\,. \tag{2.53}$$

It is holomorphic function of z with possible singularities at the insertion points $x_1, x_2, \cdots, x_n$ where the equations of motion (2.52) are not valid. For instance, if $O_1 \in \mathcal{NS}$, the singularity at $z = x_1$ is single-valued, i.e. it is generally a pole. Simple example is the two-point correlation function

$$\langle\, \psi(z)\, \psi(z')\, \rangle = \frac{1}{z - z'}\,, \tag{2.54}$$

which is easily derived from the free-fermion theory (2.40) with $m = 0$ (in fact, the special normalization of ψ responsible for the factor $1/\pi$ in the action (2.37) was chosen to make (2.54) simple). If the product $\psi(z)\psi(z')$ appears in more complicated correlation function, say

$$\langle\, \psi(z)\, \psi(z')\; O_2(x_2) \cdots O_n(x_n)\, \rangle\,, \tag{2.55}$$

the first-order pole at $z = z'$ is still there - after all, in the free-field theory it appears from the Wick contraction of the pair of ψ's - but the remaining part does not have singularity at $z = z'$. This fact can be expressed in terms of the *operator product expansions*

$$\psi(z)\, \psi(z') = \frac{1}{z - z'} + \mathrm{reg}\,, \qquad \bar{\psi}(\bar{z})\, \bar{\psi}(\bar{z}') = \frac{1}{\bar{z} - \bar{z}'} + \mathrm{reg}\,, \tag{2.56}$$

where "reg" stands for terms which are regular at $z = z'$ (or $\bar{z} = \bar{z}'$). The operator product expansions like (2.56) in Euclidean theory play role analogous to canonical (anti)commutation relations of canonical approach.

If, say, O_1 in (2.53) is more complicated field from $\mathcal{NS}$, the singularity at $z = z_1$ can be second or higher order pole.

Exercise 2: Consider composite fields (related to the components of the energy-momentum tensor)

$$T(z) = -\frac{1}{2} : \psi\partial\psi : (z)\,, \qquad \bar{T}(\bar{z}) = -\frac{1}{2} : \bar{\psi}\bar{\partial}\bar{\psi} : (\bar{z})\,, \tag{2.57}$$

where $: \cdots :$ denotes the Wick ordering in the usual sense: all Wick contractions inside $: \cdots :$ are excluded. Using Wick contractions (2.54) (and similar one for $\bar{\psi}$ derive the operator product expansions

$$T(z)T(z') = \frac{1}{4\,(z-z')^4} + \frac{2}{(z-z')^2}\,T(z') + \frac{1}{z-z'}\,\partial T(z') + \mathrm{reg}\,, \tag{2.58}$$

and similar one for the $\bar{T}$'s.

For the fields from $\mathcal{NS}$ the operator product expansions are just fancy way to describe the usual Wick rules. The situation is a bit different if we consider the "spin fields". Consider again correlation function

$$\langle\, \psi(z)\, O_1(z_1, \bar{z}_1)\, \cdots \rangle\,, \tag{2.59}$$

but this time assume that $O_1 \in \mathcal{R}$. By definition of the spin fields, this correlation function has square-root brunching point at $z = z_1$, i.e. the analytic structure of (2.59) at z sufficiently close to z_1 can be described by the expansion

$$\psi(z)\, O_1(z_1, \bar{z}_1) = \sum_{n\in\mathbf{Z}} (z - z_1)^{-n-1/2}\, O_1^{(n)}(z_1, \bar{z}_1)\,. \tag{2.60}$$

The defining monodromy property of the product in the l.h.s. (the product changes sign when z is brought around z_1) is reflected in the half-integer powers in the r.h.s. This equation can be understood as the operator product expansion, with the coefficients $O_1^{(n)}$ in the r.h.s. being some fields belonging (as one easily checks) to the space $\mathcal{R}$,

$$O_1^{(n)} \in \mathcal{R}\,. \tag{2.61}$$

Given a field $O_1 \in \mathcal{R}$, the expansion (2.60) provides definition of the fields $O_1^{(n)}$. Perhaps this role of the expansion (2.60) is more clear if one rewrites it in equivalent form

$$O_1^{(n)}(z_1, \bar{z}_1) = \oint_{z_1} \big(z - z_1\big)^{n-1/2}\, \psi(z)\, O_1(z_1, \bar{z}_1)\, \frac{dz}{2\pi i}\,, \tag{2.62}$$

where the integration is performed over closed contour which encircles the point z_1. Note that the integrand in (2.62) is single-valued in the vicinity of z_1, and the contour is indeed a closed one. Note also that the contour can be made arbitrary small, close to the point z_1; this makes it clear that the r.h.s of (2.62) defines a local field (provided O_1 is local) at the point $(z_1, \bar{z}_1)$.

The equation (2.62) defines a set of linear operators, labelled by the integer n, acting in the space $\mathcal{R}$. I will denote these operators a_n

$$a_n : \qquad \mathcal{R} \to \mathcal{R} \,.$$

The operator a_n applied to any spin field O_1 returns the l.h.s of (2.62). By this definition,

$$a_n \, O_1(z_1, \bar{z}_1) = \text{the r.h.s. of the Eq.(2.62)}$$

Note that this explicit construction, the eq.(2.62), is given relative to the 'reference point" z_1.

Of course, the antiholomorphc component $\bar{\psi}$ gives rise to similar set of operators $\bar{a}_n$,

$$\bar{a}_n \, O(z_1, \bar{z}_1) = \oint_{\bar{z}_1} \left(\bar{z} - \bar{z}_1\right)^{n-1/2} \bar{\psi}(\bar{z}) \, O(z_1, \bar{z}_1) \, \frac{d\bar{z}}{2\pi i} \,, \tag{2.63}$$

where the integration is over the counterclockwise contour in the $\bar{z}$ plane.

Our nearest goal is to derive the commutation relations among the operators a_n and $\bar{a}_n$; we will show that

$$\{a_n, a_m\} = \delta_{n+m,0} \,, \qquad \{\bar{a}_n, \bar{a}_m\} = \delta_{n+m,0} \,, \tag{2.64}$$

and

$$\{a_n, \bar{a}_m\} = 0 \,, \tag{2.65}$$

and that the space $\mathcal{R}$ has the structure of the fermion Fock space generated by these operators.

let us consider the field $a_n a_m O(z_1)$, the result of successive application of the operators a_m and a_n,

$$a_n a_m O(z_1) = \oint_C \frac{dz}{2\pi i} \, \psi(z) \, (z - z_1)^{n-1/2} \oint_{C'} \frac{dz'}{2\pi i} \, \psi(z') \, (z' - z_1)^{m-1/2} \, O(z_1) \,. \tag{2.66}$$

The integration contours C and C' both encircle the point z_1, but by definition, according to the order of the operators $a_n a_m$ in (2.66), the integration over z' generating the action of a_m is performed first. In fact, the actual order of integration is not important if we assume that the contour C' lays inside the contour C (**Fig 11**)- this is the arrangement which corresponds to order of operators written in (2.66). On the other hand

$$a_m a_n O(z_1) \oint_{C'} \frac{dz'}{2\pi i}\, \psi(z')\, (z'-z_1)^{m-1/2} \oint_{C} \frac{dz}{2\pi i}\, \psi(z)\, (z-z_1)^{n-1/2}\, O(z_1)\,. \quad (2.67)$$

where this time we assume that C lays inside C' (**Fig 12**), since the operator a_n acts first.

If not for the order of integrations, the expressions (2.66) and (2.67) are almost identical - (2.66) can be rewritten as (2.67) if one replaces the contours in **Fig.11** by those in **Fig 12** and also interchanges the positions of $\psi(z)$ and $\psi(z')$. The interchange of the ψ's results only in the change of sign,

$$\psi(z)\psi(z') = -\psi(z')\psi(z)\,.$$

Now, the contours shown in **Fig.11** can be transformed to those in **Fig.12** by moving C inside C'. Since the operator product in the integrand in (2.66) is singular at $z = z'$,

$$\psi(z)\psi(z') = \frac{1}{z-z'} + \mathrm{reg}\,, \quad (2.68)$$

one has to take into account the contribution of the residue of the pole in (2.67) when moving C inside C'. This contribution is

$$\oint_{C'} \frac{dz'}{2\pi i} \oint_{C_{z'}} \frac{dz}{2\pi i}\, \psi(z)\, \psi(z')\, (z-z_1)^{n-1/2}\, (z'-z_1)^{m-1/2}\, O(z_1)\,, \quad (2.69)$$

where $C_{z'}$ is a small contour surrounding the point z (**Fig.13**). This integral is done by the residue calculation using (2.68), with the result

$$(2.69) = \oint_{C'} \frac{dz'}{2\pi i}\, (z'-z_1)^{n+m-1}\, O(z_1) = \delta_{n+m,0}\, O(z_1)\,. \quad (2.70)$$

As the result,

$$(2.66) = -(2.67) + (2.70)\,,$$

which is the first of the anti-commutation relations (2.64). The rest of the anti-commutators (2.64),(2.65) are derived in a similar way.