L10

The last time I have started to describe the properties of the IFT in the high-T domain

$$High - T: \quad \eta \equiv \frac{m}{|h|^{8/15}} < 0\,. \tag{10.1}$$

I have mentioned that at $\eta = 0$, where the theory is integrable, there are eight stable particles, but only three of them are below the two-meson threshold [1],

$$\eta = 0: \quad M_1,\ M_2,\ M_3 < 2\,M_1 < M_4,\ ...,\ M_8\,. \tag{10.2}$$

Ones η is shifted away from zero five particles above the threshold loose their stability becoming resonance states. The fact that the decay channel $M_4,\ ..., M_8 \to M_1 + M_1$ opens at any small eta can be established using perturbation theory around the integrable theory $\eta = 0$. So, for $\eta < 0$ we have three or less stable particles.

To describe the most important properties of the IFT at $\eta < 0$, it is useful to discuss in terms of the elastic $2 \to 2$ scattering amplitude. This we define as follows. Let us denote $A_1(\theta)$ the lightest particle, the one which has the mass M_1, having the rapidity θ, i.e. the energy-momentum $P^\mu = (E, P) = (M_1 \cosh\theta,\ M_1 \sinh\theta)$. Now, consider $2 \to 2$ scattering

$$A_1(\theta_1) + A_1(\theta_2) \to A_1(\theta_1') + A_1(\theta_2')\,. \tag{10.3}$$

It follows from the energy-momentum conservation that the rapidities of the outgoing particles must coincide with the rapidities of the incoming ones, $\theta_1' = \theta_1$, $\theta_2' = \theta_2$, up to the interchange $1 \leftrightarrow 2$. Therefore the two-particle in-state can be expanded into the out-states as follows

$$|\ A_1(\theta_1)A_1(\theta_2)\,\rangle_{in} = S_{11}(\theta_1, \theta_2)\ |\ A_1(\theta_1)A_1(\theta_2)\,\rangle_{out} + \text{inelastic terms}\,, \tag{10.4}$$

---

[1] The mass ratios at this point are known exactly, i.e.

$$M_2 = 2M_1\ \cos\frac{\pi}{5}\,, \quad M_3 = 2M_1\ \cos\frac{\pi}{30}\,, \quad M_4 = 2M_2\ \cos\frac{7\pi}{30}\,, \quad \text{etc}$$

where the "inelastic terms" include all other possible products of the reaction,

$$\text{inelastic terms} = \sum_{OUT \neq A_1+A_1} (2\pi)^2\, \delta\big(P^\mu(\theta_1) + P^\mu(\theta_2) - P^\mu_{OUT}\big) \times$$

$$\times\, S_{A_1+A_1\to OUT}\ \mid OUT\,\rangle\,. \tag{10.5}$$

where OUT denote the the states like

$$OUT = \big\{A_1 + A_2,\ A_1 + A_1 + A_1,\ \text{etc}\big\}\,. \tag{10.6}$$

I will call the coefficient S_{11} in (10.4) the $2 \to 2$ elastic scattering amplitude. By the Lorentz invariance it depends on the difference of the rapidities,

$$S_{11} = S_{11}(\theta)\,, \qquad \theta = \theta_1 - \theta_2\,. \tag{10.7}$$

The analytic properties of the amplitudes as the functions of the variable θ are discussed extensively in the context of integrable field theories and associated factorizable S-matrices. So, I will be schematic here, and just emphasize the difference in the analytic structure of $S_{11}(\theta)$ in generic theory from that of the integrable one.

If one expresses the amplitude S_{11} in terms of the Mandelstam invariant s,

$$s = \big(p^\mu(\theta_1) + p^\mu(\theta_2)\big)^2 = 2M_1^2\,\big(1 + \cosh\theta\big)\,, \quad (t = 0\,, \quad u = 4M_1^2 - s) \tag{10.8}$$

then according to usual arguments the function $S_{11}(s)$ is analytic in the whole s-plane with the exception of the real axis where the poles associated with the bound states, and branch cuts due to the scattering states are located. The function $S_{11}(s)$ obeys the cross-invariance property

$$S_{11}(s) = S_{11}(4M_1^2 - s)\,, \tag{10.9}$$

and typical picture of the poles and cuts is shown in the **Fig.1**

$$xxxxxx{*}{*}{*}{*}{*}{*}{*}{*}{*}{*}{*}{*}{*}---\bullet-----\bullet---{*}{*}{*}{*}{*}{*}{*}{*}{*}{*}{*}{*}{*}xxxxxxx \quad (Fig.1)$$

The branch cut from $4M_1^2$ to infinity is due to the two-particle states $A_1 + A_1$, the cut from $(M_1 + M_2)^2$ reflects the states $A_1 + A_2$, further cuts come from the multi-particle states.

In an integrable theory all inelastic processes are forbidden, so the branching points at the $A_1 + A_1$ threshold $4M_1^2$, and its cross-image at 0, are all branching points $S_{11}(s)$ has in the finite part of the s-plane. Then it is possible to show that in terms of the rapidity variable θ,

$$\theta \; : \qquad s = 4M_1^2 \cosh^2 \frac{\theta}{2} \tag{10.10}$$

the amplitude S_{11} is a meromorphic function,

$$S_{11}(\theta) \;\; : \quad \text{meromorphic function}\,.$$

The cross-symmetry (10.9) and the two-particle unitarity condition $SS^{\dagger} = I$ lead to the equations

$$\text{crossing} : \quad S(\theta) = S(i\pi - \theta)\,, \tag{10.11}$$

$$\text{unitarity} : \quad S(\theta)S(-\theta) = 1\,. \tag{10.12}$$

When changing to the rapidities, the s-plane is mapped onto the "physical strip" $0 < \Im m\, \theta < \pi$

$$\text{physical strip} \;\; : \quad 0 < \Im m\, \theta < \pi\,,$$

while the edges of the branch cuts from $4M_1^2$ to ∞ and from 0 to $-\infty$ become the positive and negative parts of the axes $\Im m\, \theta = 0$ and $\Im m\, \theta = \pi$, respectively. The segment $0 < s < 4M_1^2$ of the real axis is mapped on the segment $0 < \Im m\, \theta < \pi\,,\ \Re e\, \theta = 0$; here the poles associated with the bound states are located. In fact, it is possible to show that the poles with positive residues in the variable $-i\theta$ correspond to the s-channel bound states, while the poles with negative residues in that variable correspond to the u-channel images of these bound states. Finally, according to (10.11) and (10.12) the amplitude $S_{11}(\theta)$ is $2\pi\, i$-periodic function of θ,

$$S_{11}(\theta) = S(\theta + 2\pi\, i)\,. \tag{10.13}$$

The situation is depicted in **Fig.2**, where the complex plane of the variable

$$\alpha = -i\theta \tag{10.14}$$

is shown

$$- - | - - - \diamond - - - - - \diamond - - - | - - - \bullet - - - - - \bullet - - - | - - \qquad (Fig.2)$$

The strip $-\pi < \Im m\,\theta < 0$ $(-\pi < \Re e\,\alpha < 0)$ corresponds to the "second shit" of the s-surface. In Fig.2 the bullets represent possible poles, and the diamonds denote corresponding zeros of the amplitude $S_{11}(\theta)$.

In general, non-integrable case the inelastic channels are present, which is reflected in the appearance of the additional "inelastic" branch cuts in Fig.1. Such cuts are not eliminated by passing to the rapidity variable, therefore the $2 \to 2$ amplitude is no longer a meromorphic function, but it can have branch cuts along the axes $\Im m\,\theta = 0,\ \pi \mod 2\pi$, as is shown in Fig.3

$$
\begin{array}{ccccccccc}
 & 1/S & & 1/S^* & S & & S^* & 1/S & \\
------| & ---\diamond--- & ---\diamond--- & ---| & ---\bullet--- & ---\bullet--- & ---| & --- & (Fig.3) \\
 & 1/S^* & & 1/S & S^* & & S & 1/S^* &
\end{array}
$$

The relations (10.11) and (10.12) represent the two-particle unitarity, and hence they remain valid in the general non-integrable case. The discontinuities across the cuts in Fig.3 are related to inelastic elements of the S-matrix. The complete unitarity condition is very complicated.

To get insight at the properties of the model we are going to follow evolution of the poles and zeros of the amplitude $S_{11}(\theta)$ as η changes from 0 to $-\infty$. I am going to describe what I believe is the most natural scenario of such evolution. In the following drawings the inelastic branch cuts are not shown, but their presence is implied unless stated otherwise.

As was already mentioned, at $\eta = 0$ the IFT is integrable, with factorizable S-matrix. The basic $2 \to 2$ amplitude S_{11} can be found explicitly,

$$
S_{11}(\theta) = \frac{\sinh\theta + i\sin(2\pi/3)}{\sinh\theta - i\sin(2\pi/3)}\ \frac{\sinh\theta + i\sin(2\pi/5)}{\sinh\theta - i\sin(2\pi/5)}\ \frac{\sinh\theta + i\sin(\pi/15)}{\sinh\theta - i\sin(\pi/15)}\,, \qquad (10.14)
$$

It shows presence of six poles in the physical strip, at

$$\theta_1 = 2\pi i/3\,, \qquad \theta_2 = 2\pi i/5\,, \qquad \theta_3 = \pi/15\,, \tag{10.15}$$

and

$$\bar{\theta}_1 = \pi i/3\,, \qquad \bar{\theta}_2 = 3\pi i/5\,, \qquad \bar{\theta}_3 = 14\pi/15\,. \tag{10.16}$$

The poles (10.15) have positive residues in $-i\theta$; they correspond to the particles A_1, A_2, A_3 (with the masses M_1, M_2, M_3 respectively) appearing in the s-channel, while (10.16) are the u-channel poles (they have negative residues in $-i\theta$). The situation at $\eta = 0$ is shown in **Fig.4**.

−|−◇−−−−−◇−−◇−−◇−−◇−−−−−◇−|−•−−−−−•−−•−−•−−•−−−−−•−|−

$(Fig.4)$

$$\eta = 0\,.$$

Consider now what happens when η is shifted away from zero in the negative direction. The theory is no longer integrable, the inelastic cuts appear, but just like I said for the most part I'll not pay them attention.

When η is shifted away from zero in the negative direction, the poles $-i\theta_2$ and $-i\theta_3$ start moving to the left, towards the two-particle threshold $-i\theta = 0$. The pole $-i\theta_1 = 2\pi/3$ does not move - it corresponds to the particle A_1 itself appearing as the bound state in the s-channel, and it remains at this position

$$-i\theta_1(\eta) = 2\pi/3\,, \tag{10.17}$$

independent of the value of η.

The pole at $-i\theta_3 = \pi/15$, which is originally rather close to the threshold, crosses the threshold at certain relatively small negative η,

$$-i\theta_3(\eta) = 0 \qquad \text{at} \quad \eta = \eta_3 \approx -0.1371\,, \tag{10.18}$$

At this value of η the particle A_3 leaves the spectrum. Its mass $M_3(\eta)$ however can be analytically continued below this value, where it for awhile remains real. Anyway, for $\eta < \eta_3$ the spectrum of IFT contains only two particles, A_1 and A_2.

The locations of the poles and zeros of $S_{11}(\theta)$ at η slightly below η_3 is shown in **Fig.5**

−|−•−−−−−◇−◇−−−−−◇−◇−−−−−•−|−◇−−−−−•−•−−−−−•−•−−−−−◇−|−

$(Fig.5)$

η slightly below η_3 .

When η further decreases, the pole $-i\theta_2$ continues to move to the left, approaching the pole $-i\bar{\theta}_1 = \pi/3$. At the same time the zero at $-i\theta = i\theta_3$ moves to the right approaching the same pole $-i\bar{\theta}_1 = \pi/3$. It is easy to see that this moving pole and moving zero have to hit the pole $-i\bar{\theta}_1$ simultaneously; otherwise the A_1 pole $-i\theta_1$ would acquire wrong sign of the residue. The pole $-i\theta_2$ and the zero $i\theta_3$ hit the pole at $-i\bar{\theta}_1$ at certain

$$\eta_* \approx -0.47822 \tag{10.19}$$

At this point

$$-i\theta_2(\eta_*) = i\theta_3(\eta_*) = \pi/3\,, \tag{10.20}$$

i.e.

$$M_2(\eta_*) = M_3(\eta_*) = \sqrt{3}\; M_1(\eta_*)\,. \tag{10.21}$$

In (10.21) of course $M_3(\eta)$ stands for the analytic continuation of the mass $M_3(\eta)$ below η_3. Again, the locations of the poles and zeros of $S_{11}(\theta)$ at η slightly below η_* is shown in **Fig.6**

−|−−−◇−−−◇−−•−−•−−−◇−−−◇−−−|−−−•−−−•−−−◇−−−◇−−•−−−•−−−|−

$(Fig.6)$

η slightly below η_* .

Next event occurs at some

$$\eta_{**} \approx -0.513 \tag{10.22}$$

where the zeros at $-i\theta = i\theta_3$ and $-i\theta = \pi - i\theta_3$ collide at $-i\theta = i\pi/2$ and then below η_{**} become complex-conjugate pair in the complex plane of $-i\theta$. Note that at this point

$$M_3(\eta_{**}) = \sqrt{2}\ M_1(\eta_{**})\,. \tag{10.23}$$

The poles and zeros of $S_{11}(\theta)$ at η slightly below η_{**} is shown in **Fig.7**

−|−−◇−−−◇−−−−−−−−−−◇−−−◇−−|−−●−−−●−−−−−−−−−−●−−−●−−|−

$(Fig.7)$

η slightly below η_{**} .

When η further decreases the complex-conjugate zeros in the physical strip depart further away from the real axis, while the pole $-i\theta = -i\theta_2$ corresponding to the particle A_2 approaches the threshold $-i\theta = 0$; it hits the threshold at $\eta = \eta_2$,

$$\eta_2 \approx -2.09\,, \tag{10.24}$$

and at $\eta < \eta_2$ only one particle remains in the spectrum of the IFT (but again, analytic continuation of $M_2(\eta)$ exists and is real below η_2.). The picture is in **Fig.8**

−|−●−−−−−◇−−−−−−−−−−◇−−−−−●−|−◇−−−−−●−−−−−−−−−−●−−−−−◇−|−

$(Fig.8)$

η slightly below η_2 .

Finally, when $\eta \to -\infty$ the complex-conjugate zeros speed away to infinity, and the zeros $-i\theta = i\theta_2$ and $-i\theta = \pi - i\theta_2$ approach the poles $-i\bar{\theta}_1 = \pi/3$ and $-i\theta_1 = 2\pi/3$, respectively, as in **Fig.9**

−|−−−−●−◇−−−−−−−−−−◇−●−−−−−|−−−−−◇−●−−−−−−−−−−●−◇−−−−−|−

$(Fig.9)$

$$\eta \to -\infty\,,$$

so that at $\eta = -\infty$ all poles and zeros disappear and we have $S_{11} = -1$ as it should be for free particles.

It is not difficult to estimate how fast the zero at $-i\theta = \pi - i\theta_2$ approach the pole at $2\pi/3$ as $\eta \to -\infty$. When the zeros approach the poles, the residues at those poles become small. These residues can be found using perturbation theory in h.

At $h = 0$ we have free particles A_1 and $S_{11} = -1$. Recall that in the high-T regime the operator σ always converts even number of particles to odd number, and vice versa. Hence the first non-trivial contribution to the $2 \to 2$ elastic amplitude appears in the order $\sim h^2$. We have by definition we have

$$_{out}\langle\, A(\theta_1')A(\theta_2') \mid A(\theta_1)A(\theta_2)\,\rangle_{in} = (2\pi)^2\,\delta(\theta_1 - \theta_1')\delta(\theta_2 - \theta_2')\,S_{11}(\theta_1 - \theta_2)\,, \quad (10.25)$$

where I have assumed that $\theta_1' \neq \theta_2$ and $\theta_2' \neq \theta_1$, so that the term with $\delta(\theta_1' - \theta_2)\delta(\theta_2' - \theta_1)$ is ignored. On the other hand, by usual perturbation theory

$$_{out}\langle\, A(\theta_1')A(\theta_2') \mid A(\theta_1)A(\theta_2)\,\rangle_{in} = -(2\pi)^2\,\delta(\theta_1 - \theta_1')\delta(\theta_2 - \theta_2') + \mathcal{M}_{2\to 2}\,, \quad (10.26)$$

with

$$\mathcal{M}_{2\to 2} = -\frac{1}{2}\,(2\pi)^2\,\delta^{(2)}(P_f - P_i)\,\times$$
$$\times\, h^2 \int {}_{out}\langle\, A(\theta_1')A(\theta_2') \mid T\sigma(x,t)\sigma(0,0) \mid A(\theta_1)A(\theta_2)\,\rangle_{in}\,dxdt\,, \qquad (10.27)$$

where all the matrix elements are from the unperturbed theory with $h = 0$. The momentum delta-function came from the integration over overall space-time shifts. Using

$$(2\pi)^2\,\delta^{(2)}(P_f - P_i) = (2\pi)^2\,\delta(\theta_1 - \theta_1')\delta(\theta_2 - \theta_2')\,\frac{1}{m^2\,\sinh(\theta_1 - \theta_2)} \qquad (10.28)$$

we have

$$S_{11}(\theta) = -\left(1 + \frac{iA(\theta)}{m^2\,\sinh\theta} + ...\right)\,, \qquad (10.29)$$

where

$$A(\theta_1 - \theta_2) = i\,h^2 \int_{t>0} {}_{in}\langle\, A(\theta_1)A(\theta_2) \mid \sigma(x,t)\sigma(0,0) \mid A(\theta_1)A(\theta_2)\,\rangle_{in}\,dxdt\,. \quad (10.30)$$

The matrix element here can be expanded in terms of the intermediate states. Obviously, the pole at $\theta_1 - \theta_2 = 2\pi i/3$ we are comes from the intermediate state with one particle $A(\theta')$, as in the diagram in **Fig.10**

We have

$$A_{\text{pole}}(\theta_1 - \theta_2) = \frac{i\, h^2}{m^2} \int \frac{2\pi i\, \delta(\sinh\theta' - \sinh\theta_1 - \sinh\theta_2)}{\cosh\theta' - \cosh\theta_1 - \cosh\theta_2} \times$$

$$\left| \langle\, A(\theta') \mid \sigma(0,0) \mid A(\theta_1)A(\theta_2)\,\rangle \right|^2 \frac{d\theta'}{2\pi}\,. \tag{10.31}$$

As usual, the spatial momentum delta-function and the energy denominator came from the integrations over x and t, respectively.

The θ integration is eliminated by the delta-function. Since the residue we are looking for is expected to be Lorentz-invariant, it is convenient to assume the center of mass frame where $\theta_1 = -\theta_2 = \theta/2$ (as before, $\theta = \theta_1 - \theta_2$), and $\theta' = 0$ by the delta-function. We find

$$A_{\text{pole}}(\theta) = -\frac{h^2}{m^2} \frac{\left|\langle A(0) \mid \sigma(0) \mid A(\theta/2)A(-\theta/2)\rangle\right|^2}{1 - 2\,\cosh\theta/2}\,. \tag{10.32}$$

As expected, this has a pole at $\theta = 2\pi i/3$, and we find that at θ close to $2\pi i/3$

$$S_{11}(\theta) \to \frac{\frac{4}{3}\,\Gamma^2}{\theta - 2\pi i/3} \qquad \text{as} \quad \theta \to 2\pi i/3\,, \tag{10.33}$$

where

$$\Gamma^2 = \frac{h^2}{m^4} \left|\langle\, A(0) \mid \sigma(0) \mid A(i\pi/3)A(-i\pi/3)\,\rangle\right|^2 = 27 \left(\frac{\bar\sigma h}{m^2}\right)^2\,, \tag{10.34}$$

where I have taken into account that in the high-T theory with $h = 0$

$$\langle\, A(0) \mid \sigma(0) \mid A(i\pi/3)A(-i\pi/3)\,\rangle = 3\sqrt{3}\;\bar\sigma\,. \tag{10.35}$$

From this we find that $S_{11}(\theta)$ has zeros in the physical strip at $\theta = 2\pi i/3 + i\Delta$ and $\theta = i\pi/3 - i\Delta$ with

$$\Delta = 9\,\lambda^2 + O(\lambda^4)\,, \qquad \text{where again} \quad \lambda = \frac{2\bar{\sigma}\,h}{m^2}\,. \tag{10.36}$$

Let us say few words about inelastic processes. Generally at $\eta \neq 0$ and $\eta \neq \pm\infty$ the IFT is not integrable, and all kinds of inelastic processes are allowed. For instance, at sufficiently large negative η, where only one particle $A = A_1$ exists, one can study the two-particle scattering which in general leads to the production of any number of particles A. One can write [2]

$$\begin{aligned} \mid A(\theta_1)A(\theta_2)\,\rangle_{in} &= S(\theta_1-\theta_2)\ \mid A(\theta_1)A(\theta_2)\,\rangle_{out} + \\ &+ \sum_{n=3}^{\infty}\int_{\beta_1,\ldots,\beta_n} S(\theta_1,\theta_2|\beta_1,\ldots,\beta_n)\ \mid A(\beta_1)\cdots A(\beta_n)\,\rangle_{out}\,, \end{aligned} \tag{10.37}$$

where

$$S(\theta_1,\theta_2|\beta_1,\ldots,\beta_n) = (2\pi)^2\,\delta^{(2)}(P_f - P_i)\,A(\theta_1,\theta_2|\beta_1,\cdots,\beta_n) \tag{10.38}$$

are the amplitudes of the $2\to n$ reactions. In (10.37) and below I use the short-hand notation $S(\theta) = S_{11}(\theta)$.

If the center of mass energy $E = E_{\rm COM}$ of the incoming particles is less then $3M_1$, i.e.

$$E = 2\,M_1\,\cosh\frac{\theta}{2} < 3\,M_1 \quad \Rightarrow |\theta| < \Theta_3 = 2\,\log\frac{3+\sqrt{5}}{2} \tag{10.39}$$

(where again $\theta = \theta_1 - \theta_2$) all inelastic channels are closed, and it is simple consequence of unitarity that $S(\theta)$ is pure phase factor,

$$|S(\theta)|^2 = 1 \quad \text{for real } \theta, \quad \text{such that} \quad |\theta| < \Theta_3\,. \tag{10.40}$$

When $|\theta|$ exceeds the $2\to 3$ threshold Θ_3 this is no longer true. One can define the total inelastic cross-section as

$$\sigma(\theta) = 1 - |S(\theta)|^2\,. \tag{10.41}$$

[2] Here as usual we use the notation $\int_{\beta_1,\ldots,\beta_n} = \frac{1}{n!}\int\frac{d\beta_1}{2\pi}\cdots\frac{d\beta_1}{2\pi}$.

This quantity is the probability of production of $n > 2$ particles in the two-particle collision. It can also be written as the sum of $2 \to n$ particle terms, $\sigma(\theta) =$

$$\frac{1}{\sinh(\theta_1 - \theta_2)} \sum_{n=3}^{\infty} \int_{\beta_1,\ldots,\beta_n} (2\pi)^2 \, \delta^{(2)}(P(\theta_1,\theta_2) - P(\{\beta\})) \, |A(\theta_1,\theta_2|\beta_1,\ldots,\beta_n)|^2 \, . \tag{10.42}$$

Since $\sigma(\theta)$ is probability, we have

$$0 \leq \sigma(\theta) \leq 1 \quad \text{for all real} \quad \theta \, . \tag{10.43}$$

Interesting open problem is the large-energy behavior of the cross-section $\sigma(\theta)$. The answer is not obvious even in the weak-coupling domain of high-T with $h << |m|^{15/8}$. It is possible to study the leading term $\lambda^2 \, \sigma^{(2)}(\theta)$ in the expansion

$$\sigma(\theta) = h^2 \, \sigma^{(2)}(\theta) + O(h^4) \, . \tag{10.44}$$

This term can be determined by using the leading term

$$A(\theta_1,\theta_2|\beta_1,\ldots,\beta_n) = i \, h \, \langle \, A(\theta_1)A(\theta_2) \mid \sigma(0) \mid A(\beta_1) \cdots A(\beta_n) \, \rangle + O(h^2) \tag{10.45}$$

in the equation (10.42). Then the term $\sim h^2$ in (10.42) is expressed through the integrated matrix element

$$\int \langle \, A(\theta_1)A(\theta_2) \mid \sigma(x)\sigma(0) \mid A(\theta_1)A(\theta_2) \, \rangle_{\text{irred}} \, d^2x \, , \tag{10.46}$$

where the connected one-particle irreducible matrix element is taken.

Although the matrix element in (10.46) can be written down explicitly in terms of the functions $\Psi_{\pm}(x,\theta)$. The integral (10.46) is complicated, but it is possible to extract its leading high-energy behavior, which yields

$$\sigma^{(2)}(\theta) = 8 \, G_2 \, \log\left(E^2/m^2\right) + O(1) \qquad \text{as} \quad E \to \infty \, , \tag{10.47}$$

where the coefficient in front of the log term is related to certain moment of the two-point correlation function

$$G_2 = \frac{1}{2\pi} \int |z|^2 \, G(|z|) \, d^2z = \int_0^{\infty} r^3 \, G(r) \, dr \, , \tag{10.48}$$

where

$$G(r) = \langle \, \sigma(r)\sigma(0) \, \rangle_{h=0,m<0} \, . \tag{10.49}$$

At sufficiently large E the h^2 term $h^2\,\sigma^{(2)}(E)$ exceeds the unitarity bound (10.43); hence the higher-order terms in h^2 must become significant at large E. True large E behavior of the inelastic cross-section $\sigma(E)$ remains open question. Obviously, there are three possibilities, i) $\sigma(E) \to 0$, ii) $\sigma(E) \to 1$ and iii) $\sigma(E) \to \sigma_\infty;\ 0 < \sigma_\infty < 1$. Which of these possibilities is realized?

Among other interesting questions which appear in concern with the IFT as the particle theory are:

What is the fate of the particles $M_4, M_5, M_6, ...$ etc after they disappear from the spectrum (i.e. become heavier then $2M_1$)? General answer is that they likely become resonance states, but we still may be interested in how their masses and widths change with the parameter η.

What can be said about the scattering theory at $\eta \to +\infty$? Although at $\eta = +\infty$ is the free theory, the limit $\eta \to +\infty$ must be very subtle. Is there a classical description of this theory, something like the classical description of the meson spectrum at large positive η.

Many interesting questions appear when one considers analytic continuation of the IFT to complex values of the scaling parameter η. Among those let me mention just one related to the Yang-Lee critical point.

Consider the high-T domain $m < 0$, and let us consider the scaling parameter

$$\xi = \frac{h}{|m|^{15/8}} \tag{10.50}$$

as a complex variable. At this point we think of m as being real and negative, therefore this is equivalent as taking the external magnetic field h as a complex variable.

Analytic properties of the thermodynamic quantities of the Ising model, say its free energy

$$F = -\log Z/V\,, \qquad V \to \infty\,, \tag{10.51}$$

as the function of complex H, are relatively well understood. It follows from the theorem due to Yang and Lee that the free energy $F(H)$ is analytic in both right and left half-planes $\Re e\, H > 0$ and $\Re e\, H < 0$. In the hight-T domain $T < T_c$ there are singularities located at the imaginary axis. In the thermodynamic limit $V = \infty$

we have two brunching points at

$$H = \pm i\, H_{\rm YL}(T)\,. \tag{10.52}$$

These singularities appear as the result of accumulation of the Yang-Lee zeros of Z, which leads to two branch cuts, from $i\, H_{\rm YL}$ to $+i\infty$, and from $-i\, H_{\rm YL}$ to $-i\infty$, see **Fig.11**

$$- - - - - - - - - - -| - - - - - - - - - - - \quad \Re e\, H \qquad (Fig.11)$$

As T approaches T_c from above the points $H_{\rm YL}(T)$ approach the real axis as

$$H_{\rm YL}(T) \sim (T - T_c)^{15/8}\,, \tag{10.53}$$

and at $T = T_c$ they pinch the real axis leading to the critical singularity at $H = 0$, $T = T_c$.

In the scaling domain $T \to T_c$, $H \to 0$ with

$$H/(T - T_c)^{15/8} \sim \xi = h/|m|^{15/8} \quad \text{fixed} \tag{10.54}$$

this means that the singular part of the free energy $F_{\rm sing}$ is analytic function of ξ with two brunching points at pure imaginary $\xi = \pm\, i\, \xi_*$, see **Fig.12**

$$- - - - - - - - - - -| - - - - - - - - - - - \quad \Re e\, \xi \qquad (Fig.12)$$

Unlike the Fig.11 the positions $\pm\, i\, \xi_*$ are not functions of any parameters but are pure numbers,

$$\xi_* = 0.1893... \tag{10.55}$$

The YL point(s) $i\,\xi_*$ is critical, in the sense that the mass of the lightest particle $M_1(\xi)$, being analytically continued to complex ξ, vanish at $\pm\, i\,\xi_*$,

$$M_1(\xi = \pm\, i\,\xi_*) = 0\,, \tag{10.56}$$

i.e. the correlation length $R_c \sim M_1^{-1}$ diverges at these points. As the result, all thermodynamic quantities, including $R_c(\xi)$, are singular at the Yang-Lee point(s).

The Yang-Lee criticality in 2D, i.e. the associated RG fixed point, is known to be described by certain CFT, namely the

$$c = -22/5 \qquad \text{minimal CFT} \tag{10.57}$$

which has two primary fields

$$I \quad \text{with } \Delta = 0\,, \qquad \Phi_{-1/5} \quad \text{with } \Delta = -1/5\,. \tag{10.58}$$

The character of the singularity at $\pm\, i\,\xi_*$ can be predicted from these dimensions. For instance, the mass $M_1(\xi)$ vanishes as

$$M_1(\xi) \simeq b_0\, m\, \big(\xi^2 + \xi_*)^2\big)^{5/12}\,, \tag{10.59}$$

The leading singularities at $\xi^2 + \xi_*^2 = 0$ are described by the perturbed Yang-Lee CFT,

$$\mathcal{A}_{\rm IR} = \mathcal{A}_{c=-22/5\ \rm CFT} + \lambda(\xi) \int \Phi_{-1/5}(x)\, d^2x\,, \tag{10.60}$$

where λ vanishes at the YL points in a non-singular way,

$$\lambda(\xi) \simeq i\, a_1\, |m|^{12/5}\, \big(\xi^2 + \xi_*^2\big) + O(\delta\xi^2)\,, \qquad a_1 \approx 3.1 \tag{10.61}$$

The imaginary axis in the ξ plane corresponds to the ray

$$\eta = y\, e^{\frac{4\pi i}{15}} \tag{10.62}$$

in the η-plane. One can observe the behavior of M_1 along this ray.

The picture in Fig.9, i.e.

—|————•—◇—————————◇—•————|————◇—•—————————•—◇————|—

$(Fig.9)$

$$\Delta \simeq 36\,\bar{s}^2\,\xi^2 \qquad \xi^2 \to 0\,.$$

can be further developed to the imaginary axis in the ξ-plane by taking negative ξ^2. For small negative ξ^2 the zeros pass the poles,

−|−−−−−−◇−•−−−−•−◇−−−−−−−|−−−−−−−•−◇−−−−◇−•−−−−−−−|−

($Fig.$13)

$$\xi^2 < 0\,, \qquad y < -4.2$$

Note that in this situation the residue of the pole $\theta = 2\pi i/3$ becomes negative, making it manifest that the theory is no longer unitary.

When ξ increases along the imaginary axis, i.e. y increases from $-\infty$, the zeros approach each other and then collide, becoming after that a complex-conjugate pair, as in **Fig.14**

−|−−−−−−◇−−−−−−−◇−−−−−−−|−−−−−−−•−−−−−−−•−−−−−−−|−

($Fig.$14)

$$\xi^2 < 0\,, \qquad y < -4.2$$

When ξ^2 approaches $-\xi_*^2$ the zeros depart to infinity, leaving behind the pattern in **Fig.15**

−|−−−−−−◇−−−−−−−◇−−−−−−−|−−−−−−−•−−−−−−−•−−−−−−−|−

($Fig.$15)

$$\xi^2 = -\xi_*^2\,, \qquad y = -2.4295...$$

This corresponds to known S-matrix of the integrable Yang-Lee theory (10.60)

$$S_{\mathrm{YL}}(\theta) = \frac{\sinh\theta + i\,\sin(2\pi/3)}{\sinh\theta - i\,\sin(2\pi/3)}\,. \tag{10.63}$$

At $\xi^2 + \xi_*^2$ the theory is massless