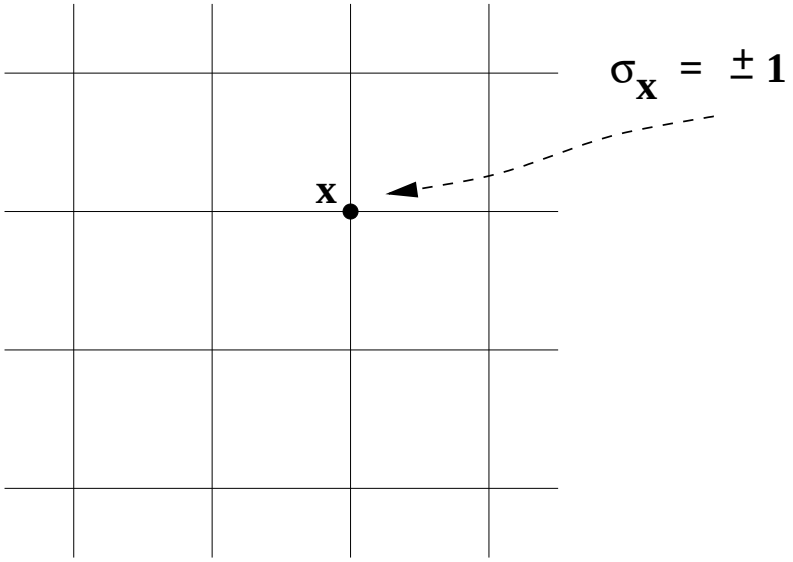
L1

Ising field theory (IFT) is continuous (or "scaling") limit of the 2D Ising model. Its significance is that it describes the most basic universality class of 2D criticality. It is also important model of 2D (or 1+1) quantum field theory (QFT), with rich content of particles and interesting phenomena.

It is possible to define the IFT in intrinsically field-theoretic terms, without reference to the lattice Ising model and its scaling limit. I will introduce such formulation later. However, to understand the nature of local observables (such as order and disorder parameters) it is useful to remember the lattice formulation as well. Therefore I will start with reminding basic properties of the 2D Ising model. This is familiar subject (see e.g. the monograph of Wu and McCoy), and I'll be brief.



2D Ising model is a lattice model of classical statistical mechanics. In its simplest version, it is formulated as follows. Consider (infinite) square lattice with the lattice cites labelled by $\mathbf{x} = (n_1, n_2)$. The degrees of freedom are "spins" $\sigma_{\mathbf{x}}$, associated with the cites, and taking two values,

$$\sigma_{\mathbf{x}} = \pm 1 \,. \tag{1.1}$$

The configuration space of the system thus is the collection $\{\sigma_{\mathbf{x}},\ \mathbf{x} \in \text{lattice}\}$ of all these spins.

Statistical mechanics is defined by the Gibbs distribution

$$P\{\sigma_{\mathbf{x}}\} = Z^{-1} \exp\left(-\frac{1}{kT} E\{\sigma_{\mathbf{x}}\} \right),$$

where $E\{\sigma_{\mathbf{x}}\}$ is the energy functional, and Z is the partition function. The latter is chosen to account for the *nearest neighbor* interactions only,

$$E\{\sigma_{\mathbf{x}}\} = -J \sum_{\mathbf{xy}=\text{nn}} \sigma_{\mathbf{x}}\sigma_{\mathbf{y}} - B \sum_{\mathbf{x}} \sigma_{\mathbf{x}}\,, \tag{1.2}$$

where J and B are parameters. This can be regarded as a primitive "ferromagnet". If J is positive (which I assume), the first term in E makes it energetically favorable for the neighboring "spins" to aline (i.e. take the same values), wile the last term describes interaction with external field B. Since J and B enter only in certain combinations with the temperature T, it is convenient to introduce shorter notations:

$$P\{\sigma_{\mathbf{x}}\} = Z^{-1} \exp\left(K \sum_{\mathbf{xy}=nn} \sigma_{\mathbf{x}}\sigma_{\mathbf{y}} + H \sum_{\mathbf{x}} \sigma_{\mathbf{x}} \right), \tag{1.3}$$

where $K = J/kT$, $H = B/kT$. The theory depends on two parameters, K and H.

Remark 1. Perhaps more useful interpretation of the model is in terms of the "lattice gas". Consider a gas of molecules which are hard but sticky - that means the the interaction between two molecules has a "hard core" (so that that they cannot sit one on top of another) and it also has a short-range attraction, so that the molecules like to sit close to each other. This is rather realistic interaction, and on general grounds one could expect to observe liquid-gas phase transition in such system. Of course, exact calculation of the partition sum is exceedingly difficult problem for any potential, so one can start making simplifications. To simplify things let us replace integrations over the coordinates of the molecules by discrete sums, namely assume that possible positions of the molecules are restricted to the cites $\mathbf{x}$ of 2D lattice. Having in mind the grand canonical ensemble, let us introduce the "occupation numbers" $n_{\mathbf{x}} = 0, 1$; it takes the value 0 at the empty cites and 1 at the cites occupied by the molecules. Obviously, the total number of molecules is

$$N = \sum_{\mathbf{x}} n_{\mathbf{x}}\,.$$

The "sticky" nature of the interaction is captured by the following simple form of the energy functional

$$E\{n_{\mathbf{x}}\} = -w \sum_{\mathbf{xy}=nn} n_{\mathbf{x}}\, n_{\mathbf{y}}$$

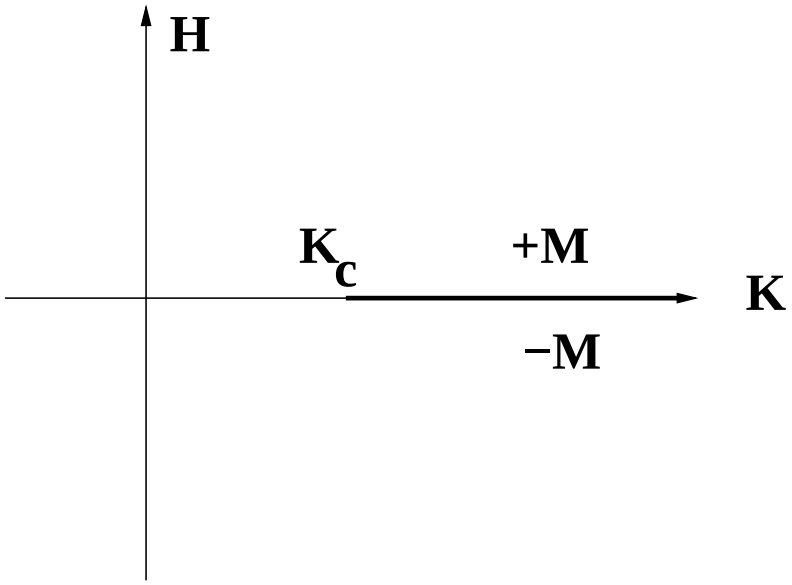
which adds attraction energy w to any pair of molecules which sit next to each other. It is not difficult to check that

$$E\{n_{\mathbf{x}}\} - \mu N$$

is equivalent to the energy functional (1.2) if one relates the variables

$$n_{\mathbf{x}} = \frac{1}{2}\left(1 - \sigma_{\mathbf{x}}\right). \tag{1.4}$$

and impose proper relation between the energy w and μ here and the parameters K and H in (1.2).



Qualitatively, thermodynamic properties of the Ising model (1.3) are well understood. In the (K, H) plane the system has a line of first-order phase transitions which is located at $H = 0$ and extends from some finite K_c to $+\infty$ (see Fig.1). The magnetization

$$M = \langle\, \sigma_{\mathbf{x}} \,\rangle$$

is discontinuous across this line, i.e.

$$M(K > K_c, H = +0) = -M(K > K_c, H = -0) \neq 0\,.$$

(but $M(K < K_c, H = 0) = 0$). From the point of view of the lattice gas this is the liquid-gas transition (see Eq.(1.4)). The transition line ends at the critical point $(K = K_c, H = 0)$. If we restrict attention to the case $H = 0$ (zero external field)

the point $K = K_c$ corresponds to the Curie point of the ferromagnet, the point where spontaneous magnetization first appears at sufficiently low temperatures. In terms of the variable K (or T) it is the second order phase transition.

To make quantitative analysis, one would like to find the partition function

$$Z(K,H) = \sum_{\{\sigma_{\mathbf{x}}\}} \exp\left(K \sum_{\mathbf{xy}=nn} \sigma_{\mathbf{x}}\sigma_{\mathbf{y}} + H \sum_{\mathbf{x}} \sigma_{\mathbf{x}} \right), \tag{1.5}$$

and the correlation functions

$$\langle\, \sigma_{\mathbf{x}_1} \cdots \sigma_{\mathbf{x}_n} \,\rangle = \sum_{\{\sigma_{\mathbf{x}}\}} \sigma_{\mathbf{x}_1} \cdots \sigma_{\mathbf{x}_n}\, P\{\sigma_{\mathbf{x}}\}\,. \tag{1.6}$$

The problem (for the partition function) was solved exactly by Onsager in the case $H = 0$. The theory with $H = 0$ reduces to the problem of free fermions on the lattice. I am not going to reproduce full solution of the lattice model here; some version of the solution can be found in virtually any textbook on statistical mechanics. My aim here will be to show how the free-fermion structure emerges, and on the way to introduce important concept of the "disorder parameter" (which belongs to Kadanoff and Cheva).

So, let us restrict attention to the case of $H = 0$. It is instructive to take first a quick look at the high- and low- temperature expansions of the partition function.

At high temperatures T we have $K << 1$ and it is meaningful to expand Z in the powers of K. It is convenient to use the identity

$$e^{K\,\sigma_{\mathbf{x}}\sigma_{\mathbf{y}}} = \cosh K \left(1 + \sigma_{\mathbf{x}}\sigma_{\mathbf{y}}\, \tanh K \right) \tag{1.7}$$

to write

$$Z(K, H=0) = \cosh^{2N} K \sum_{\sigma_{\mathbf{x}}} \prod_{\mathbf{xy}=nn} \left(1 + \sigma_{\mathbf{x}}\sigma_{\mathbf{y}}\, \tanh K \right), \tag{1.8}$$

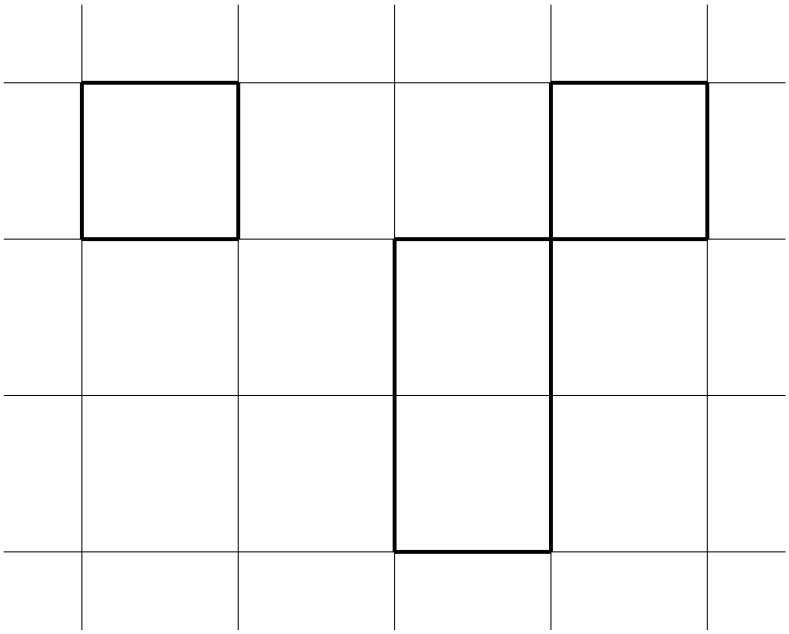
where N is the total number of the lattice sites ($N \to \infty$ for infinite lattice). Each factor in (1.8) corresponds to certain link of the lattice. If $K \to 0$ the first terms in these factors dominate and Z reduces to the trivial factor $2^N \cosh^{2N} K$. The high-temperature expansion in powers of

$$t = \tanh K$$

is obtained by taking the second term $t\,\sigma_{\mathbf{x}}\sigma_{\mathbf{y}}$ from some of the factors (i.e. for some of the lattice links) in (1.8). When the second term is taken, let us mark the associated link by a bold line. The t-expansion then is expressed in terms of graphs which are built from such bold links on the lattice. Since $\sigma_{\mathbf{x}}^2 = 1$ and $\sum_{\sigma=\pm 1} \sigma_{\mathbf{x}} = 0$, the summation over $\sigma_{\mathbf{x}}$ exterminates all odd powers of $\sigma_{\mathbf{x}}$ at the same cite $\mathbf{x}$. Hence only even numbers, i.e. 0, 2 or 4, of the bold links can meet at any lattice cite. The result is the sum of graphs which consist of continuous bold lines on the lattice which are allowed to cross at the "four-vertices", the cites where four of the bold links meet. The graphs are not necessarily connected, but each graph brings contribution

$$t^L \tag{1.9}$$

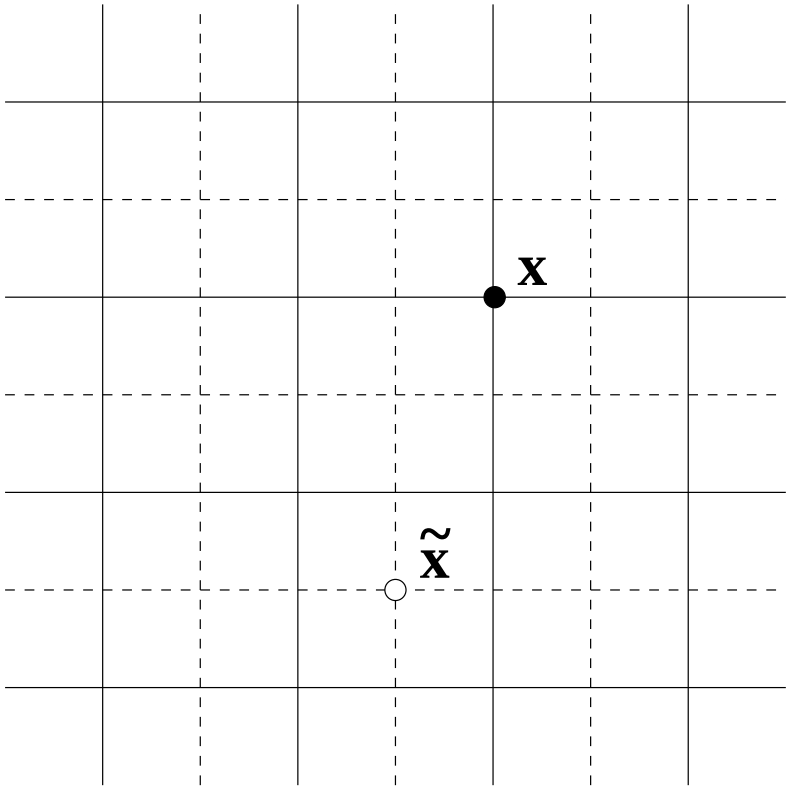
to the "renormalized" partition function $Z/2^N \cosh^{2N} K$, where L is the total length of the bold lines in the graph. Examples of such loops are shown in Fig.2.



The above analysis can be repeated for the case of the correlation functions (1.6); the result is that each spin insertion $\sigma_{\mathbf{x}}$ generates the end-point for the bold lines at the cite $\mathbf{x}$, as is depicted in the Fig.3 in the case of the two-point function.

The bold lines in the graphs can be thought of the the Euclidean-space trajectories of particles. At the first glance, these particles appear to be interacting ones. Indeed, the vacuum trajectories (i.e. the trajectories with no endpoints) of free particles are combinations of closed loops, with the important property that the statistical weight of any combination of the loops is the product of the weights of the individual loops, no matter if they intersect (or self-intersect) or not. This seems

not to be the case for the Ising graphs - the "four vertices" seem to represent non-trivial interactions between the particles. Fortunately, the sum of the Ising graphs can be transformed to the sum of non-interacting loops, at the price of giving some of these loops negative statistical weights. As the result the theory reduces to the free *fermionic* particles. I am not going to describe here combinatorics which leads to this result (see e.g. Landau&Lifshitz book on Statistical mechanics). Instead, later on I will derive equivalent result by different approach.



Now consider the case of low temperature, i.e. $K >> 1$. When T is strictly zero, there are two degenerate ground states of the energy functional (1.2) (with $H = 0$), the one with all spins equal $+1$ and the one with all spins equal -1. These two states have the same statistical weight $\left(e^K\right)^N$, but they have opposite spontaneous magnetization $M = \pm 1$. Let us concentrate attention at the state with all spins $+1$. If the temperature is small but not exactly zero (i.e. K is large but not infinite) some contributions to the partition function come from configurations with the majority of the spins equal to $+1$ but with some small fraction of the spins being equal to -1, as illustrated in Fig.4. These configurations can be given representations in terms of graphs if one introduces the "dual" lattice. The dual lattice is the lattice whose cites are the centers of the faces of the original lattice. For the case of the square lattice (which we stick to) the dual lattice is also a square

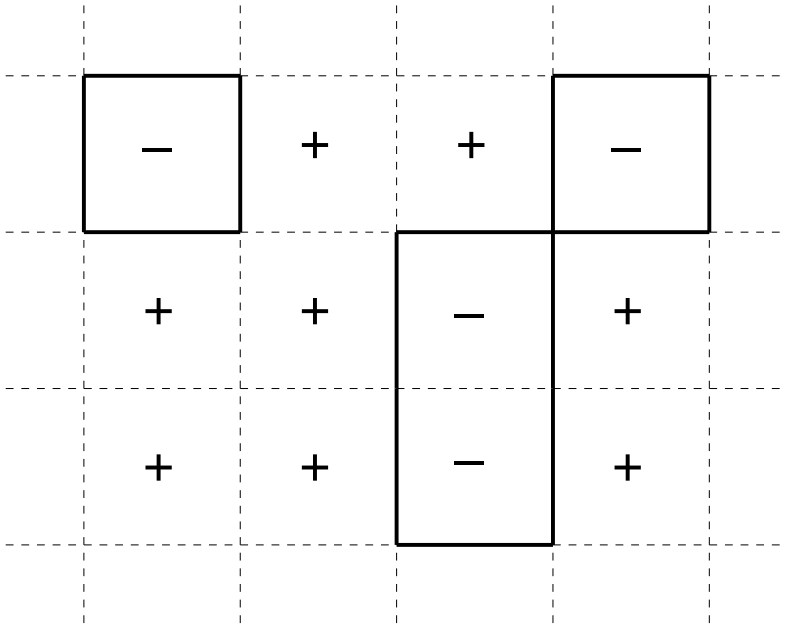
lattice, see Fig.4. The cites of the original lattice are the faces of the dual lattice and vice versa. I will denote $\tilde{\mathbf{x}} = (k_1, k_2)$ (with half-integer k) the cites of the dual lattice. For any spin configuration $\{\sigma_{\mathbf{x}}\}$ on the original lattice, one can draw a graph *on the dual lattice* by applying the following rule. Take any link of the dual lattice. There are two neighboring cites $\mathbf{x}$ and $\mathbf{y}$ of the original lattice immediately at the sides of this link. If the spins $\sigma_{\mathbf{x}}$ and $\sigma_{\mathbf{y}}$ have opposite signs, i.e. if $\sigma_{\mathbf{x}}\sigma_{\mathbf{y}} = -1$, then mark this link bold. On the other hand, if $\sigma_{\mathbf{x}}\sigma_{\mathbf{y}} = +1$, then leave the link blank. Thus all possible spin configurations of the original lattice generate graphs on the dual lattice, and it is easy to see that these graphs are exactly of the same type as the graphs we encountered in the high-temperature expansion. Namely, only even number of bold links can meet at any cite $\tilde{\mathbf{x}}$ of the dual lattice, and hence the graphs consist of continuous bold lines with "four vertices". Moreover, given spin configuration brings contribution

$$\tilde{t}^L \tag{1.10}$$

to the modified partition Z/e^{NK}, where L is the length (i.e. the number of bold links) of the associated graph, and

$$\tilde{t} = e^{-2K} \, . \tag{1.11}$$

Indeed, each bold link separates opposite spins, so its statistical weight differs by the factor (1.11) from the statistical weight of the empty link (representing parallel neighboring spins.



We observe remarkable "duality" of the Ising model: Its thermodynamic properties at low T are related to those at low T, namely

$$\frac{Z(K)}{2^N \cosh^{2N} K} = \frac{Z(\tilde{K})}{e^{N\tilde{K}}} ,$$

where $\tilde{K}$ relates to K as

$$e^{-2K} = \tanh \tilde{K} . \tag{1.12}$$

Note that from (1.12) follows $e^{-2\tilde{K}} = \tanh K$, so that the relation (1.12) is inversion. Note also that (1.12) relates values of K in the low-T regime to its values in the high-T regime. Assuming that the critical point K_c is unique, it must satisfy $K_c = \tilde{K}_c$, i.e.

$$K_c = \frac{1}{2} \log \left(\sqrt{2} + 1\right) . \tag{1.13}$$

The graphs appearing in the low-T expansion also can be viewed as the Euclidean space-time histories of some particles. Although the graphs have the same structures and the same weights (in terms of the "dual" parameters, that is) as the high-T graphs, the interpretations of the particles are quite different. While the particles appearing in the high-T graphs can be called the spin particles (since the particle can be emitted by a single $\sigma_{\mathbf{x}}$ insertion, the lines in the low-T graphs have different relation to the spin configurations. The low-T graph lines separate domains of the lattice which are "filled" with the spins of the same sign - the "drops". Therefore the particles represented by the lines in the low-T graphs are rather the "kinks" separating degenerate vacua of the opposite magnetization. It is remarkable that despite this very different interpretations the low-T and the high-T particles have the same dynamical properties (they are free fermions, as we will see little later).

Let me present more formal derivation of this duality relation, since it naturally introduces important notion of the *disorder parameter*. Consider again the partition function (with $H = 0$)

$$Z(K) = \sum_{\{\sigma_{\mathbf{x}}\}} e^{K \sum_{nn} \sigma_{\mathbf{x}}\sigma_{\mathbf{y}}} . \tag{1.14}$$

The expression in the sum factorizes in terms of variables

$$g_{\mathbf{xy}} = \sigma_{\mathbf{x}}\sigma_{\mathbf{y}} . \tag{1.15}$$

These variables are associated with the links $(\mathbf{xy})$ of the lattice, so that there are $2N$ of them. The complication is that these variables are not all independent, they

must satisfy the constraints ("zero curvature conditions")

$$\prod_{\text{polygon}} g_{\mathbf{xy}} = +1\,,$$

where the product involves $g_{\mathbf{xy}}$ associated with all links of any polygon on the lattice - this follows from the definition (1.15). Of course, all these constraints follow from the elementary constraints, associated with elementary polygons - the lattice faces. The latter are labelled by the cites of the dual lattice, so for every $\tilde{\mathbf{x}}$ we have

$$\prod_{\text{around } \tilde{\mathbf{x}}} g_{\mathbf{xy}} = 1 \qquad \text{for} \quad \text{every} \quad \tilde{\mathbf{x}}\,. \tag{1.16}$$

Then the summation in 91.14) can be performed over independent $g_{\mathbf{xy}}$ if we also insert the delta-symbol to enforce all the constraints (1.16),

$$\prod_{\tilde{\mathbf{x}}} \delta\Big(\prod_{\text{around } \tilde{\mathbf{x}}} g_{\mathbf{xy}} = 1 \Big)\,. \tag{1.17}$$

Now, each delta-function in (1.17) can be written as the sum

$$\delta\Big(\prod_{\text{around } \tilde{\mathbf{x}}} g_{\mathbf{xy}} = 1 \Big) = \frac{1}{2} \sum_{n=0,1} \Big[\prod_{\text{around } \tilde{\mathbf{x}}} g_{\mathbf{xy}} \Big]^n\,. \tag{1.18}$$

Indeed, if the product is equal to $+1$, the two terms in the sum add up to 1, but if the product equals -1 these two terms cancel each other. Since there are N delta-functions in (1.17), we will need N additional summation variables $n_{\tilde{\mathbf{x}}}$, one for each cite of the dual lattice.

With this we can write

$$Z(K) = \frac{1}{2^N} \sum_{\{g_{\mathbf{xy}}\}} e^{K \sum_{nn} g_{\mathbf{xy}}} \sum_{n_{\tilde{\mathbf{x}}}} \Big[\prod_{\tilde{\mathbf{x}}} g_{\mathbf{xy}} \Big]^{n_{\tilde{\mathbf{x}}}}\,. \tag{1.19}$$

For fixed configuration of $\{n_{\tilde{\mathbf{x}}}\}$ the expression (1.19) factorizes in terms of the link variables g, i.e. the summation over $\{g_{\mathbf{xy}}\}$ reduces to $2N$ identical sums

$$\sum_{g_{\mathbf{xy}}=\pm 1} e^{K\, g_{\mathbf{xy}}} \big[g_{\mathbf{xy}} \big]^{n_{\tilde{\mathbf{x}}}+n_{\tilde{\mathbf{y}}}} = e^K + \big(-1\big)^{n_{\tilde{\mathbf{x}}}+n_{\tilde{\mathbf{y}}}}\, e^{-K}\,, \tag{1.20}$$

where $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ are two cites of the dual lattice adjacent to the link $(\mathbf{xy})$. this expression can be brought to a nicer form if one introduces new variables $\mu_{\tilde{\mathbf{x}}} = \pm 1$ instead of $n_{\tilde{\mathbf{x}}} = 0, 1$,

$$n_{\tilde{\mathbf{x}}} = \frac{1}{2}\left(1 - \mu_{\tilde{\mathbf{x}}}\right). \tag{1.21}$$

It is easy to check that (1.20) can be written as

$$e^K + \left(-1\right)^{\frac{1-\mu_{\tilde{\mathbf{x}}}}{2} + \frac{1-\mu_{\tilde{\mathbf{y}}}}{2}} e^{-K} = e^K + \mu_{\tilde{\mathbf{x}}}\mu_{\tilde{\mathbf{y}}}\ e^{-K}. \tag{1.22}$$

And it further transforms with the use of the dual parameter $\tilde{K}$ from (1.12) $e^{-2K} = \tanh \tilde{K}$,

$$\frac{e^K}{\cosh \tilde{K}}\ e^{\tilde{K}\,\mu_{\tilde{\mathbf{x}}}\mu_{\tilde{\mathbf{y}}}}. \tag{1.23}$$

We finally obtain for (1.19)

$$Z(K) = \frac{e^{2NK}}{2^N\ \cosh^{2N} \tilde{K}} \sum_{\{\mu_{\tilde{\mathbf{x}}}\}} e^{\tilde{K} \sum_{nn}\ \mu_{\tilde{\mathbf{x}}}\mu_{\tilde{\mathbf{y}}}}. \tag{1.24}$$

Thus the duality transformation can be understood as certain non-local change of variables in the partition sum, which brings it to the sum over the dual variables $\mu_{\tilde{\mathbf{x}}}$, while the energy functional takes the original Ising form with K replaced by $\tilde{K}$. The variables $\mu_{\tilde{\mathbf{x}}}$ are usually called the "disorder parameter".

I would like to stress that the possibility to make this duality transformation strongly depends on the global Z_2 (spin reversal) symmetry of the theory with $H = 0$. If this symmetry is broken, as in the case of non-zero H, the transformation leads to theory involving special Z_2 gauge fields, and the dual theory becomes much more complicated. I will discuss some of related topics later on.

Exercise 1: Perform the duality transformation of the two-point correlation function

$$\langle\, \sigma_{\mathbf{x}_1}\, \sigma_{\mathbf{x}_2}\, \rangle = Z^{-1} \sum_{\{\sigma_{\mathbf{x}}\}} \sigma_{\mathbf{x}_1}\, \sigma_{\mathbf{x}_2}\ P\{\sigma_{\mathbf{x}}\},$$

i.e. express it as the sum over the dual variables $\{\mu_{\tilde{\mathbf{x}}}\}$.

To understand the nature of the disorder parameter, consider again the partition sum (1.14). Take arbitrary simple closed contour $\tilde{\Gamma}$ on the *dual* lattice. It splits the original lattice into two parts, the part Λ which consists of the sites inside Γ,

and the part $\bar{\Lambda}$ composed of the cites outside Γ. Let us make in (1.14) the following change of variables

$$\sigma_{\mathbf{x}} \to \sigma_{\mathbf{x}} \qquad \text{if} \quad \mathbf{x} \in \bar{\Lambda}\,,$$
$$\sigma_{\mathbf{x}} \to -\sigma_{\mathbf{x}} \qquad \text{if} \quad \mathbf{x} \in \Lambda\,, \tag{1.25}$$

The only terms in the energy functional affected by this transformation are those corresponding to the links $(\mathbf{xy})$ which cross the contour Γ; I will denote such links by perhaps clumsy symbol

$$(\mathbf{xy}) \times \Gamma\,.$$

These terms change sign in the exponentials in (1.14). Thus we have

$$Z(K) \equiv \sum_{\{\sigma_{\mathbf{x}}\}} P\{\sigma_{\mathbf{x}}\} = \sum_{\{\sigma_{\mathbf{x}}\}} P\{\sigma_{\mathbf{x}}\}\; T\{\sigma_{\mathbf{x}}\sigma_{\mathbf{y}}; (\mathbf{xy}) \times \Gamma\}\,, \tag{1.26}$$

where the insertion T takes into account this change of the signs,

$$T\{\sigma_{\mathbf{x}}\sigma_{\mathbf{y}}; (\mathbf{xy}) \times \Gamma\} = e^{-2K \sum_{\mathbf{xy}\times\Gamma} \sigma_{\mathbf{x}}\sigma_{\mathbf{y}}}\,. \tag{1.27}$$

I will often abbreviate (1.27) as $T\{\Gamma\}$. The Eq.(1.26) shows that, as the consequence of the global Z_2 symmetry, insertion of $T\{\Gamma\}$ with any closed Γ does not change the partition sum. One can insert $T\{\Gamma\}$ into more complicated correlation function which involves also some σ insertions. One finds

$$\langle\, \sigma_{\mathbf{x}_1} \cdots \sigma_{\mathbf{x}_n}\, T\{\Gamma\}\,\rangle = \big(-1\big)^{n_\Gamma}\, \langle\, \sigma_{\mathbf{x}_1} \cdots \sigma_{\mathbf{x}_n}\, T\{\Gamma\}\,\rangle\,, \tag{1.28}$$

where n_γ is the number of the σ insertions surrounded by Γ.

Now, consider some *open* contour $\Gamma_{\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2}$ on the dual lattice, with the end points at $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$. Define $T\{\Gamma_{\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2}\}$ as in (1.27) with Γ replaced by $\Gamma_{\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2}$. The above analysis shows that the expectation value

$$\langle\, T\{\Gamma_{\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2}\}\,\rangle = Z^{-1} \sum_{\{\sigma_{\mathbf{x}}\}} P\{\sigma_{\mathbf{x}}\}\; T\{\Gamma_{\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2}\} \tag{1.29}$$

does not depend on the exact form of the contour $\Gamma_{\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2}$, but only on the positions of its end-points $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_1$. In fact, as the result of the Exercise 1 shows, this expectation value coincides with the two-point correlation function of the disorder variables $\mu_{\tilde{\mathbf{x}}}$,

$$(1.29) = \langle\, \mu_{\tilde{\mathbf{x}}_1}\, \mu_{\tilde{\mathbf{x}}_2}\,\rangle \equiv Z^{-1} \sum_{\{\mu_{\tilde{\mathbf{x}}}\}} \mu_{\tilde{\mathbf{x}}_1}\, \mu_{\tilde{\mathbf{x}}_2}\; e^{\tilde{K} \sum_{nn} \mu_{\tilde{\mathbf{x}}}\mu_{\tilde{\mathbf{y}}}}\,. \tag{1.30}$$

For some further insight, consider expectation value of T together with some σ insertions,

$$\langle\, \sigma_{\mathbf{x}_1} \cdots \sigma_{\mathbf{x}_n}\; T\{\Gamma_{\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2}\}\,\rangle = Z^{-1} \sum_{\{\sigma_{\mathbf{x}}\}} P\{\sigma_{\mathbf{x}}\}\; \sigma_{\mathbf{x}_1} \cdots \sigma_{\mathbf{x}_n}\; T\{\Gamma_{\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2}\} \tag{1.31}$$

This quantity does depend on the form of the contour Γ, but the dependence is "weak". The expectation value (1.31) does not change under deformations of Γ as long as the contour does not cross any of the points $\mathbf{x}_1, \cdots, \mathbf{x}_n$, and it changes sign when such crossing occurs. Another way of stating the same is to say that (1.31) is not a single-valued but a double-valued function of the points $\tilde{\mathbf{x}}_1$, $\tilde{\mathbf{x}}_2$ and $\mathbf{x}_1, \cdots, \mathbf{x}_n$, which change sign every time $\tilde{\mathbf{x}}_1$ or $\tilde{\mathbf{x}}_2$ goes around any of the points $\mathbf{x}_1, \cdots, \mathbf{x}_n$. The expectation value (1.31) defines the mixed correlation function involving both order and disorder parameters

$$(1.31) \equiv \langle\, \sigma_{\mathbf{x}_1} \cdots \sigma_{\mathbf{x}_n}\; \mu_{\tilde{\mathbf{x}}_1}\, \mu_{\tilde{\mathbf{x}}_2}\,\rangle\,. \tag{1.32}$$

This correlation function is double-valued function with the monodromy properties described above.

The above construction admits straightforward generalization to include more insertions of the disorder field. In fact, it is convenient to introduce contours (on the dual lattice) $\Gamma_{\tilde{\mathbf{x}}}$ ending at the cite $\tilde{\mathbf{x}}$ of the dual lattice, with the other end brought away to infinity. Although it usually does not much matter, we will assume that the contour extends to the left horizontal infinity. We define

$$\mu_{\tilde{\mathbf{x}}} = T\{\Gamma_{\tilde{\mathbf{x}}}\}\,. \tag{1.33}$$

This definition is understood in terms of the insertion in the sum over the lattice spin configurations. This allows to define arbitrary mixed correlation functions

$$\langle\, \sigma_{\mathbf{x}_1}\; \cdots\; \sigma_{\mathbf{x}_n}\; \mu_{\tilde{\mathbf{x}}_1}\; \cdots\; \mu_{\tilde{\mathbf{x}}_m}\,\rangle\,. \tag{1.34}$$

In fact, this explicit construction is not very important it is seldom used in practice. What is important is our observation about the monodromy properties of the correlation functions (1.34). To recapitulate, the mixed correlation functions are double-valued functions of the points $\mathbf{x}_i$ and $\tilde{\mathbf{x}}_k$ involved. They change sign when any of the $\tilde{\mathbf{x}}_k$ is brought around any of the points $\mathbf{x}_i$. It does not change when any of $\mathbf{x}_i$ is brought around any other $\mathbf{x}_j$, and the same is true for the $\tilde{\mathbf{x}}$'s. This property will be brought out to the continuous field theory arising in the scaling limit of the Ising model.

To finish our discussion of the duality transformation, let us consider somewhat different model. Consider again the $H = 0$ Ising model, which i now choose to write in terms of the disorder variables

$$P\{\mu_{\tilde{\mathbf{x}}}\} = Z^{-1} \exp\left(\tilde{K} \sum_{nn} \mu_{\tilde{\mathbf{x}}}\mu_{\tilde{\mathbf{y}}}\right). \tag{1.35}$$

The low-T domain of the original model correspond to small $\tilde{K}$ in (1.35). Recall that the particles which appeared in the low-T expansions were interpreted as kinks in terms of the order variables σ. In the form (1.35) the same particles appear as the μ-particles, i.e. the insertion of μ emits the kink.

Note that the form (1.35) exhibits explicit "dual" Z_2 symmetry corresponding to the sign reversal of all μ's. This is a global symmetry, but we can make it local by adding the Z_2 "gauge connections" $A_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}}$, where $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ are neighboring cites of the dual lattice, i.e. the connections are associated with the links of the dual lattice. If we replace

$$\mu_{\tilde{\mathbf{x}}}\, \mu_{\tilde{\mathbf{y}}} \to \mu_{\tilde{\mathbf{x}}}\, A_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}}\, \mu_{\tilde{\mathbf{y}}} \tag{1.36}$$

in (1.35), the theory acquires local Z_2 gauge symmetry

$$\mu_{\tilde{\mathbf{x}}} \to \epsilon_{\tilde{\mathbf{x}}}\, \mu_{\tilde{\mathbf{x}}}\,, \qquad A_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}} \to \epsilon_{\tilde{\mathbf{x}}}\, A_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}}\, \epsilon_{\tilde{\mathbf{y}}}\,, \tag{1.37}$$

where $\epsilon_{\tilde{\mathbf{x}}} = \pm 1$ are the elements of Z_2. By usual routine of the lattice gauge theory, one can add the self-interaction term

$$\tilde{G} \sum_{\mathbf{x}} F_{\mathbf{x}} \tag{1.38}$$

for the gauge field A; here $F_{\mathbf{x}}$ is the "field strength" (or the curvature of the connection A),

$$F_{\mathbf{x}} = \prod_{\text{around } \mathbf{x}} A_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}}\,, \tag{1.39}$$

which itself is gauge invariant, and $\tilde{G}$ is the gauge coupling constant. We can now define the "gauge Ising model"

$$P\{\mu_{\tilde{\mathbf{x}}}, A_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}}\} = Z^{-1} \exp\left(\tilde{K} \sum_{nn} \mu_{\tilde{\mathbf{x}}}\, A_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}}\, \mu_{\tilde{\mathbf{y}}} + \tilde{G} \sum_{\mathbf{x}} F_{\mathbf{x}}\right). \tag{1.40}$$

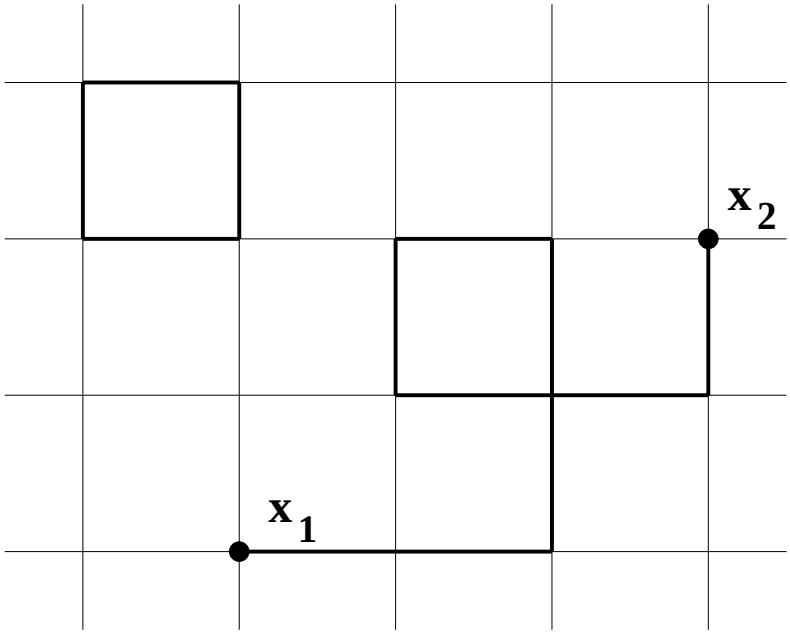
The distribution function of course is defined on the orbits of the gauge group. We can always fix the gauge by the condition

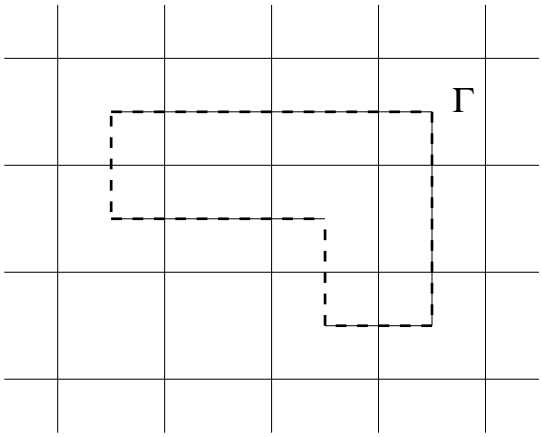
$$\mu_{\tilde{\mathbf{x}}} = +1\,. \tag{1.41}$$

Then the partition function takes the form of the sum

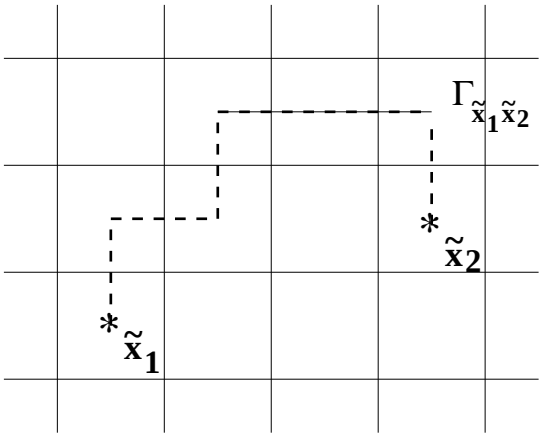
$$Z = \sum_{\{A\}} \exp\left(\tilde{K} \sum_{\text{links}} A_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}} + \tilde{G} \sum_{\mathbf{x}} F_{\mathbf{x}}\right). \tag{1.42}$$

Now we can make a duality transform along the same lines as were applied to the usual Ising model.





(a)



(b)