Main Symbols

$a(\boldsymbol{k},t)$, $a_{\boldsymbol{k}}$, $b(\boldsymbol{k},t)$, $b_{\boldsymbol{k}}$, $c(\boldsymbol{k},t)$, $c_{\boldsymbol{k}}$	wave amplitudes
d	dimension (of the space)
E	energy
$\varepsilon(\boldsymbol{k})$	energy density (in the $\boldsymbol{k}$-space)
$E(k)$	energy density (in the k-space)
g	acceleration of gravity
g_m	magnetic-to-mechanical momentum ratio
$\boldsymbol{H}$, H	magnetic field
$\mathcal{H}$	Hamiltonian
$\hbar$	Planck constant
$I_{\boldsymbol{k}}\{n(\boldsymbol{k}',t)\}$, $I(\boldsymbol{k})$, $I_{\boldsymbol{k}}$	collision integral
$\boldsymbol{k}$	wave vector
k	wave number
m	scaling index of the interaction coefficient
$n(\boldsymbol{k},t)$, $n_{\boldsymbol{k}}$	wave density (in the $\boldsymbol{k}$-space)
N	total number of waves
$\boldsymbol{P}$	energy flux (in the $\boldsymbol{k}$-space)
$P(k)$, P_k, P	energy flux (in the k-space)
Π	total wave momentum
$\boldsymbol{R}$	momentum flux (in the $\boldsymbol{k}$-space)
S	entropy
$T(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) = T_{1234}$	coefficient of four-wave interaction
$V(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = V_{123}$	coefficient of three-wave interaction
α	index of wave frequency
$\Gamma(k)$	external increment (or decrement)
$\delta(x)$	(Dirac) delta-function
$\omega(\boldsymbol{k})$, ω_k	wave frequency
ξ	dimensionless variable
$\propto$	proportional
$\approx$	approximately equal
$\simeq$	of the same order

0. Introduction

A central problem of theoretical physics is the construction of a theory of turbulence. This book is the outcome of our continued efforts to solve this problem. The aim of this book is to present several major ideas, which we shall outline in this Introduction.

The basic idea of the book is that turbulence is a general physics problem which requires a comprehensive approach. The problem of turbulence goes far beyond the limits of hydrodynamics and the Navier-Stokes equation. In our view, turbulence is a highly excited state of a system with many degrees of freedom (in most cases a continuous medium) to be described statistically. This excited state is extremely far away from thermodynamic equilibrium and is accompanied by intensive energy dissipation. Such states can be found or created in plasmas, magnets, or nonlinear dielectrics by applying strong electromagnetic fields to them. Quite a few types of turbulence can be observed in hydrodynamics.

Our book is devoted to *developed turbulence*, i.e., to situations with turbulence involving many degrees of freedom. In hydrodynamics this corresponds to extremely large Reynolds numbers. We will not elaborate on the genesis of turbulence and about turbulence described by a low-dimensional attractor. We believe that these topics have already been extensively covered in the literature.

Developed turbulence in general refers to a system where the scales of pumping and effectively damping motions (modes) differ dramatically. The nonlinear interaction allows for an energy redistribution between different modes. The fundamental problem of the theory is therefore to find the stationary spectrum of the turbulence, i.e., the law of energy distribution over the different scales. Since theoretical physics is mainly concerned with universal distributions, the stationary spectrum is sought inbetween the scales of pumping and damping modes (source and sink). In this range (the so-called inertial interval), one can expect the formation of a universal distribution independent of the specific characteristics of the source and sink.

One can figure out two qualitatively different pictures of universal turbulence. The first one, usually associated with the names of Richardson, Kolmogorov and Obukhov, presupposes that the major physical process is a continuing fragmentation that provides a relay energy transfer from the source to the sink. As a result of the large number of fragmentation acts, the distribution "forgets" the details of the energy source. Hence, the turbulence spectrum depends on a single characteristic of pumping: the energy dissipated per unit time and unit volume P. Another picture of universal turbulence, called the structural one, appeared more recently. It is based on the concept of the generation of spatio-temporal structures of universal form (solitons, collapses, etc.)

The main part of this book deals with the first of these two pictures of turbulence. The central idea is the locality of the interaction, as proposed by Kolmogorov in describing turbulence of incompressible fluids: only vortices of spatial extensions of the same order strongly interact with each other. This presumes a constant energy flux P in k-space coinciding with the rate with which the turbulent system dissipates energy. The locality ensures that the stationary spectrum of the energy E_k in the inertial interval may be expressed in terms of the flux P and the current wave vector k only. Dimensional analysis yields

$$E_k = \lambda P^{2/3} k^{-5/3} , \qquad (0.1)$$

where λ is a constant. Expression (0.1) is called *the Kolmogorov-Obukhov spectrum* of turbulence of an incompressible fluid [0.1].

Kolmogorov's hypothesis lead to a lot of activities and to an immense scientific literature. Experimenters essentially confirm the validity of (0.1). A rigorous theoretical substantiation of the Kolmogorov spectrum is not yet available. This is mainly due to the absence of a small parameter in the theory of hydrodynamic turbulence. Vortex interaction in incompressible fluids is strong and there exists no small parameter in the hydrodynamic equations. So if one tries to obtain the equations for correlation functions, then some of the mathematical objects in the theory lead not just to asymptotic series, but even to divergent ones. This is reflected for by the fact that it is impossible to perform a consistent linearization of the hydrodynamic equations for an incompressible fluid against a stationary homogeneous background (it yields only the trivial result zero). Except for the free boundary case, there are no homogeneous-background waves in hydrodynamics, whose amplitudes may be taken to be sufficiently small. This is not the case with other media: neither in plasma turbulence and waves on a fluid surface, nor with intensive laser pulses propagating in a nonlinear dielectric, i.e., in all cases in which the system has a consistent linear approximation that describes small-amplitude waves with dispersion. In that case one can consider a situation in which the level of wave excitation is small and effects of the nonlinear interaction are smaller than the linear effects caused by the wave propagation velocity dispersion. Such a turbulence is called a *weak turbulence* and allows a quite efficient theoretical description.

Thus, this book is divided into two volumes. The first volume - which you are reading just now - describes weak wave turbulence; the second is dedicated to *strong turbulence*, essentially to the vortex turbulence of incompressible fluids.

The first volume contains a consistent description of the theory of the developed weak turbulence in different media: fluids, gases, plasmas, and magnets. In this volume we use the terms "wave" and "weak" turbulence as synonyms.

In the theory of weak turbulence, the series that yield the equations for correlation functions contain a small parameter (the nonlinearity level) and are asymptotic, which substantiates the theory sufficiently well. In particular, it may be reliably established that at a low nonlinearity level, turbulence is a set of waves whose phases are close to random. This makes it possible to express higher correlators in terms of lower ones and to ignore higher equations in the chain of equations for correlation functions. As a result, weak turbulence may be described in terms of a closed kinetic equation for the pair correlator, which is the mean square of the wave amplitude. Having established an effective language for describing the phenomenon, Kolmogorov's ideas are very efficiently applied to construct a theory of weak turbulence. For stationary kinetic equations *V.E. Zakharov* found exact power-type solutions identified with the Kolmogorov spectra [0.2]. These solutions correspond to a constant k-space flux of one of the integrals of motion of the system, which often turns out to be the energy. The theory of such spectra including the problems of their stability, formation and matching with sources is quite comprehensive and well advanced. It has not been systematically presented (except for the outline contained in the review by *Zakharov* [0.3]). It has already been applied to the theory of wave turbulence on fluid surfaces, the theory of "optical" turbulence and is expected to find many more physical applications. We believe that acquaintance with this theory is indispensable to everyone who is seriously interested in the theory of turbulence.

In the limit of small nonlinearity, formation of the dynamic structures of the soliton or collapse type is impossible. Therefore universal weak turbulence is most frequently of the Kolmogorov type, and most of its manifestations may be explained in terms of macroscopic characteristics, i.e., fluxes of the integrals of motion. Such an approach has the same relation to the "microscopic" description (in terms of pair correlators) as thermodynamics has to statistical physics. The very possibility of a quasi-thermodynamic description in terms of mean values arises from the fact that the statistics of a weakly turbulent wave field is close to a Gaussian, the major contribution to the average characteristics stemming from the set of most probable events.

The theory of weak turbulence involves a large variety of specific types of turbulence. In order to study this variety from a unified viewpoint, one should adopt a general approach to the description of various nonlinear media. Indeed, according to the aforementioned principle of considering developed turbulence as a universal phenomenon, we ought to invent a universal "language" for its description. Such a language, in our opinion, is the Hamiltonian formalism, which reveals the Hamiltonian structure concealed in the equations of the medium [0.4]. The Hamiltonian theory of equations describing continuous media is an interesting topic of modern mathematical physics.

Chapter 1 of this book can be treated as an elementary introduction to the theory and addresses in particular physicists: our approach being rather pragmatic we intend to show that the dynamic equations for very different media written in normal variables (complex wave amplitudes) acquire a standard form which is quite convenient for application of statistical averaging methods.

Chapter 2 is devoted to the derivation of the averaged kinetic equations. There are several averaging methods. In this volume we shall use only elementary ones based on the hypothesis of phase randomness and on euristically decoupling correlators. [In the second volume we shall describe a more sophisticated diagram technique; we shall pay much attention to substantiating the kinetic equation and shall give two different derivations. We shall treat carefully the question of the range of applicability of the kinetic equation, bearing in mind that it is easy to make an error there. In particular, we shall discuss in detail the problem of acoustic turbulence, i.e., of the statistical description of waves with a linear dispersion law, and clarify the rather sophisticated applicability conditions of the kinetic equation in this case. The derivation of the kinetic equation demonstrates most convincingly the advantages of the Hamiltonian: the kinetic equation has a standard form, the structural functions entering it are simply expressed via the Hamiltonian coefficients.] Classical and quantum kinetic equations are the main mathematical objects of study in the first volume, their general properties being discussed in Chap. 2.

Chapter 3 is the central one of this volume. The exact stationary solutions of the kinetic equations are obtained and shown to be just the Kolmogorov-like spectra referred to in the title of the book. Thus, the Kolmogorov-Obukhov hypothesis is converted into a strict theorem in the theory of weak turbulence. The aforementioned property of interaction locality can be easily verified in that theory for every specific case by calculating a single integral. A general locality criterion (which is the condition for the existence of the Kolmogorov spectrum) is obtained. Solutions of the Kolmogorov type are obtained, not only for scale-invariant isotropic media, but also for anisotropic media and for those close to scale-invariant ones. Boundary conditions for Kolmogorov solutions, i.e., for the matching with sources and sinks are also given.

Chapter 4 deals with the stability problem and the formation of the Kolmogorov spectra. It is interesting that two absolutely different types of instabilities can be dealt with. First, usual instability results in the exponential growth of perturbations; all known spectra are stable with respect to such an instability. Second, there may be a "structural instability" first predicted by *L'vov* and *Falkovich* [0.5]. In the structurally unstable case, a small anisotropy of the pumping caused the stationary spectrum to be substantially anisotropic in the inertial interval. Such an instability can be treated as a manifestation of self-organization in the nonlinear systems. Thus the hypothesis about the local isotropy of the developed-turbulence spectrum (suggested by *Taylor* [0.6] for hydrodynamics) may be incorrect in the case of wave turbulence while the Kolmogorov hypothesis about interaction locality may still be valid. We elaborate in detail on the general stability theory for Kolmogorov spectra of weak turbulence as developed by *Balk and Zakharov* [0.7]. A substantial part of Sect. 4.2 is a translation of the Russian paper [0.7]. We also discuss the different regimes of the nonstationary behavior of wave turbulence systems.

Chapter 5 deals with physical applications of the general theory developed in the preceding chapters. Due to the large variety of such applications it is impossible to discuss every physical system with satisfactory completeness. However, we give answers to the main questions:

– existence of stationary spectra,
– connection between the flux and pump characteristics,
– behavior in the damping region,
– spectrum stability,
– nonstationary regimes.

Throughout this first volume the material is developed in detail since it is addresses students and junior researchers. Some issues which we considered to be rather specialized are printed in small letters. In a first reading these places may be ignored. The first volume is to serve as a simple, yet comprehensive introduction to the general theory of developed turbulence. As far as wave turbulence itself is concerned, we briefly summarize the derivation of the theory in *Chap. 6, the Conclusion*, by giving the recipe of investigation of any new wave system. The reader will see that the recipe is fairly simple. By now the theory has been elaborated to such an extent that answers to most questions may be expressed in terms of the characteristics of a wave system obtained from dimensional analysis or simple asymptotic estimates. Thus, the *Conclusion* contains a methodological guide for the first volume.

Coming to the end of the *Introduction* of this first volume, we briefly expose the prospective contents of the second volume of our monograph.

Volume 2 will be devoted to *strong turbulence* and will be a natural continuation of the first volume, yet containing a new formalism and new ideas. It consistently elaborates on the necessary diagram technique (in doing so, it gives a sufficiently rigorous derivation of the kinetic equation). For hydrodynamics, this technique is developed both in the canonical Clebsch variables and in "natural" variables. The major part of the second volume is devoted to a consistent statistical theory of turbulence of incompressible fluids. This theory proceeds from the Navier-Stokes equation and the diagram approach to perturbations, in which every term of an infinite series is matched with a certain diagram. Thus, two essentially different types of vortex interaction are identified.

The first one is the *sweeping interaction* corresponding to the transfer of a small k-vortex (with dimension $1/k$) as a whole by the spatially homogeneous part of the field of large vortex velocities. This interaction is characterized by the Doppler frequency kv_t, where v_t is the mean-square velocity of turbulent pulsations associated with the spectrum (0.1)

$$v_t^2 \simeq \int_{k_0}^{\infty} E(\boldsymbol{k})\, d\boldsymbol{k} \simeq (PL)^{2/3} . \tag{0.2}$$

Here $k_0 = 1/L$ is an "external" boundary of the inertial interval; L, the "energy-containing" or external scale of turbulence coinciding in order of magnitude with the characteristic dimension of the flown-over body. At $k \lesssim k_0$, the spectrum $E(\boldsymbol{k})$ is nonuniversal.

The second type is the *dynamic interaction* of vortices of about the same scale which leads to energy exchange between them and which is responsible for the formation of the turbulence spectrum. This interaction is characterized by the frequency

$$\gamma(k) \simeq P^{1/3} k^{2/3} , \tag{0.3}$$

which we shall call the Kolmogorov frequency. This frequency, as well as the spectrum (0.1), may be determined from dimensional analysis. In the inertial interval the Doppler frequency kv_t is seen to be larger than the Kolmogorov frequency $\gamma(k)$:

$$kv_t \simeq \gamma(k)(kL)^{1/3} .$$

The existence of two types of interaction with different k-dependences implies that the theory of turbulence is not scale-invariant; not even in the inertial interval with $kL \gg 1$. The theory explicitly contains the external scale L. The presence of a large kL-parameter does not simplify the theory, since the weaker dynamic $\gamma(k)$-interaction cannot be discarded. Indeed, it is exactly this interaction that determines the turbulence spectrum. At the same time, the stronger sweeping interaction, which manifests itself as a real physical effect, is of purely kinematic character and bears no relationship to the problem of energy distribution over the scales. This makes the search for scale-invariant energy spectra in the inertial interval rather difficult. Thus, to construct a consistent theory of hydrodynamic turbulence, one has to overcome two major difficulties. The first is associated with the strong interaction and the absence of a small parameter in the theory. The second is due to the existence of two types of interaction and the absence of scale invariance.

To our understanding the second difficulty was overcome by *Belinicher and L'vov* [0.8]. They constructed the statistical theory of developed homogeneous turbulence of incompressible fluids in a coordinate system moving with the velocity of the fluid (in a spatial point r). The transition to this new variable, the so-called quasi-Lagrangian velocity, eliminates any transfer of k-vortices in the region of $1/k$ scale around the reference point r. Elimination of sweeping in a limited region only, turns out to suffice to completely eliminate from the theory its "masking" effect on the dynamic interaction of vortices in the cascade process of energy transfer to small scales. We shall present an analysis of expressions for diagrams of the perturbation theory to an arbitrary order and shall show that the integrals converge both in the infrared and the ultraviolet regions. This proves the Kolmogorov-Obukhov hypothesis about the locality of the dynamic interaction of vortices: the main contribution to the change in the k-vortex energy is made by k_1-vortices of the same scale (k_1 of the order of k) and localized in the $1/k$-region in the vicinity of the given vortex [0.8]. In the limit $kL \to \infty$, one can obtain the scale-invariant solution of the Dyson-Wyld diagram equations which corresponds to the known Richardson-Kolmogorov-Obukhov picture of developed universal turbulence.

As far as the first difficulty is concerned it has not yet been overcome. For this reason, the mathematical objects of the diagram perturbation theory are formal series depending on an external parameter.

The problem of the unambiguous correspondence of the observed physical values to the formal series in the theory of developed hydrodynamic turbulence is, however, open for discussion. Many important problems remain uninvestigated, such as uniqueness and stability of the obtained solution, transition of a nonuniversal solution in the energy-containing interval to the scale-invariant solution in the inertial interval, etc. Thus, there are ample opportunities for further research.

It should be noted, however, that the traditional descriptions of turbulence, including the diagram technique, suggest that the statistics of strong turbulence does not differ much from Gaussian statistics. By now we know that even weak turbulence can contain a non-Gaussian component. Strong turbulence may be essentially non-Gaussian which implies that some rather specific spatio-temporal configurations may contribute unproportionally large to various mean values. This property of turbulence has to do with the known "intermittency" phenomenon. Thus, the energy dissipation density at each moment in time may be distributed in space in a rather inhomogeneous manner, in contrast to the implicit assumption of dissipation homogeneity made in the Richardson-Kolmogorov-Obukhov picture. Numerical and laboratory experiments show that in many physical situations (both in hydrodynamics and in the strong turbulence of plasmas), there are clear-cut short-living zones of dissipation. We associate these zones of higher energy emission with collapses, i.e., the points where the solutions of the original equations describing the medium have singularities.

Let us illustrate this by an example. Quite a universal physical model is the nonlinear Schrödinger equation

$$\frac{\partial\Psi(\boldsymbol{r},t)}{\partial t}+\triangle\Psi+T|\Psi|^2\Psi=0\ , \tag{0.4}$$

which describes, in particular, the propagation of intensive quasi-monochromatic wave packets in nonlinear dielectrics and the resulting "optical" turbulence. Solutions of (0.4) at $T>0$ (implying mutual wave attraction) may become singular with time. This corresponds to self-focusing of light in a nonlinear dielectric.

It should be noted that in the case of repulsion ($T<0$), no singularities can evolve, and the qualitative picture of turbulence must be absolutely different. Meanwhile, in weak turbulence description in the low nonlinearity limit, one can obtain from (0.4) the kinetic equation which contains the quantity $|T|^2$. Thus, in the weak turbulence limit, the sign of the interaction coefficient is insignificant.

The high-frequency part of the spectrum is determined just by the structure of the singularities formed. Therefore the theory of structure turbulence should obviously be built up in an absolutely different way rather than in terms of correlation functions in k-space. In that case, it seems natural to return to treating the different dynamic processes in r-space. From a set of realizations one should choose those making the major contribution to the mean characteristics, and average over this subset of spatio-temporal structures to obtain the wanted statistical characteristics. We shall present a sketch of such a theory using acoustic turbulence as an example. The question which remains open is, how relevant are the collapsing structures to the classical turbulence of incompressible fluids?