Trace formulae for counting nodal domains at the boundaries of 2D quantum billiards

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Abstract. Given a Dirichlet eigenfunction of a 2D quantum billiard, the boundary domain count is the number of intersections of the nodal lines with the boundary. An integer sequence can be defined by these numbers, sorted according to the energy of the eigenfunction. We show that these sequences store information about the periodic orbits of the underlying classical problem, in a manner similar to the well known trace formulae of the spectral counting function. We describe the analytical derivation of such formulae for integrable systems. For chaotic billiards, we derive a trace formula which is based on a random waves hypothesis, and verify its correctness numerically.

1. Introduction

In recent years, there has been a renewed interest in the nodal structure of solutions of the Helmholtz equation $-\Delta \psi = k^2 \psi$. For a real continuous function $\psi(r)$ on some domain $\Omega$, the nodal domains are connected components on which $\psi(r) \neq 0$. The nodal set (nodal lines in the 2D case) is the zero set of $\psi(r)$. For 2D boundary value problems (quantum billiards), $\nu_n$, the number of nodal domains of the $n$th solution (ordered by increasing $k^2$) has the Courant upper bound $\nu_n \leq n$, but this limit is only reached for a finite number of eigenfunctions [1]. In 2002, Blum et al. [2] have shown that the nodal count sequence $\{\nu_n\}_{n=1}^{\infty}$ of quantum billiards contain fingerprints of the underlying classical system. The information in this sequence differs from the information stored in the spectrum, and it was shown to resolve several known cases of isospectral systems [3, 4]. It was also shown that the sequence uniquely determines a single problem within certain classes (inverse nodal problem) [5, 6]. Like the spectrum, the nodal count sequence clearly distinguishes between separable and chaotic dynamics [2]. While for the separable case the nodal pattern is well understood, current understanding of the chaotic case relies on statistical models. Away from boundaries, the eigenfunctions are modelled by Berry’s random wave model [7]. For statistics like the nodal count, which depend on the detailed topology of the function, Bogomolny and Schmit [8] proposed a critical percolation model. This model successfully predicts the asymptotic spectral averages of $\nu_n$, as well as other nodal statistics. In particular, it predicts that in the high
energy limit, the nodal lines should behave like SLE$_6$ curves, which was also numerically confirmed [9]. However, it fails to predict some other statistical measures of the random wave model [10, 11].

Formulating an analytical expression for the exact nodal count (which could be used as basis for semi-classical analysis) seems to be a hard problem, even in two dimensions. In one dimension the situation is much simpler, because in this case the number of nodal points uniquely determine the number of nodal domains, without requiring further topological information. Hence it can be expressed in terms of purely local quantities. Explicitly, the number of nodal domains for a $C^1$ function $u(s)$ on a closed curve (where $s = 0$ is identified with $s = L$) is given by

$$\eta = \int_0^L \delta(u(s))|\dot{u}(s)|ds.$$ 

There is no equivalent formula for the number of nodal domains in more than one dimension. In [11] we restricted 2D fields to a reference curve, and used the resulting nodal statistics to characterize the fields. In the case of billiard eigenfunctions, restriction to the boundary is particularly attractive. The boundary function (which in the Dirichlet case is defined as the normal derivative of $\psi_n$ at the boundary) completely determines the eigenfunction. Hence, it should hold all the relevant information. In particular, the intersections of nodal lines with the boundary (which are just the nodal points of the boundary function) provide restrictions on the possible configurations of 2D nodal domains. Toth and Zelditch [12] have recently shown that the number of boundary domains is $O(\sqrt{n})$. This was used by Polterovich [13] to obtain an upper limit to the number of nodal domains of Neumann billiards. Further than the upper limit of [12], the average intersections count is believed to be proportional to $\sqrt{n}$. This is based on random wave models [2] and analysis of random combinations of eigenfunctions [14].

In [2] it was shown that the distribution of the spectrally averaged boundary intersections count (BIC) of Dirichlet billiards clearly distinguishes between separable and chaotic systems, much like corresponding distributions of the spectral density. For the case of the spectral density, the distribution in the chaotic case is conjectured to conform to RMT ensembles [15]. Progress towards directly validating this conjecture [16, 17, 18] was based on the Gutzwiller trace formula [19], which provides a connection between classical periodic orbits and the quantum spectrum. In analogy, a semi-classical formula expressing the nodal intersections count in terms of classical parameters would provide a powerful theoretical tool for understanding the nodal statistics of billiards. In this paper we present evidence supporting the existence of such a connection.

In section 2 and section 3 we derive analytical semi-classical expressions for the BIC of integrable systems. The resulting formula, like the corresponding formula for the density of states [20], consists of a smooth part depending on geometrical parameters, and a lower order oscillating part expressed as a sum over periodic orbits. In section 4 we demonstrate numerical evidence supporting the existence of similar formula for the chaotic billiards. For two chaotic systems (the Sinai and the Africa billiard), we subtracted the smooth part from the boundary intersections count sequence, and applied
Fourier transform to the remaining oscillating part. The results show prominent peaks at positions matching the lengths of specific periodic orbits of the corresponding billiard. To provide theoretical support to these results, we introduce a conjecture about the distribution of high energy eigenfunctions of chaotic billiards. Namely, that the limiting distribution of the boundary function is Gaussian in the chaotic case. This can be supported numerically by comparing fourth order cumulants to the second order ones. Combining this conjecture with the Rice formula \cite{21} for zeros of Gaussian fields, we derive a trace formula for the BIC. Furthermore, this derivation predicts that the contributions of individual orbits can be controlled by restricting the count to specific sections of the boundary (based on the bounce points of the orbit). This can be directly observed from the numerics.

2. Trace formula for the boundary intersections count of separable systems

In the case of integrable systems, we often have exact analytical expressions for both the wavenumber $k$ and the BIC $\eta$ in terms of the quantum numbers. We seek an expression for $\eta$ in terms of $n$, the position in the energy sorted sequence of eigenfunctions. We start by computing the wavelength-density of the BIC $d_\eta(k) = \sum_n \delta(k - k_n)\eta_n$. Similarly to the method used in \cite{22}, we write this as a sum over the quantum numbers, and evaluate the summation using the Poisson summation formula (PSF). Then, the known trace formula for $n(k)$ is inverted and substituted, getting a formula for the “ordinal density” $d_\eta(n) = \sum_m \delta(n - m)\eta_m$, which is equivalent to $\eta(n)$.

In this section we consider separable systems. In this case, each coordinate is a 1 dimensional Sturm-Liouville problem, and has a corresponding quantum number. The overall function, formed by multiplying the partial solutions, will have a checker-board-like nodal pattern, and the number of nodal domains can be computed by multiplying the quantum numbers. The BIC is also simple to compute, and will generally be a linear combination of the quantum numbers, with coefficients chosen according to the specific form of the boundary. The derivation of the trace formula uses the quantum conditions of Einstein, Brillouin and Keller (EBK), and follows the footsteps of \cite{20}.

2.1. BIC density as a function of $k$

We consider the class of integrable 2D systems which are also separable, i.e. the eigenfunctions can be written as $\psi_{l,m}(q_1,q_2) = \phi_l(q_1)\phi_m(q_2)$. The nodal set is a mesh of “$q_1$ lines” (curves on which $\phi_l(q_1) = 0$) and “$q_2$ lines” (satisfying $\phi_m(q_2) = 0$). In the case of Dirichlet billiards, the boundary $\partial\Omega$ is composed of one or more $q_i$ lines. Furthermore, each $q_i$ line intersects the boundary exactly $\tau_i$ times. The derivation given in this section holds for any set satisfying this condition. Hence it can also be applied to problems that have no natural boundary, such as tori and surfaces of revolution.

As in \cite{2}, we assume that $k^2 = H(I_1,I_2)$ where $H$ is an homogeneous function of degree $2$, and $I_1,I_2$ are the actions corresponding to $q_1,q_2$ respectively. We also
assume that the energy surface specified by \( H(I_1, I_2) = 1 \) in the positive \((I_1, I_2)\) plane is monotonic.

EBK quantization specifies that the actions are given by \( I_1 = l + \alpha_1, I_2 = m + \alpha_2 \) where \( l, m \) are the integer quantum numbers, and the \( \alpha_i \) are quarter-integer topological indices. Using the fact that each coordinate is a 1 dimensional Sturm-Liouville problem and our assumptions regarding the boundary, we find that \( \eta_{l,m} = l\tau_1 + m\tau_2 \), and

\[
d_{\eta}(k) = \sum_{l,m \geq 0} (l\tau_1 + m\tau_2) \delta(k - \sqrt{H(l + \alpha_1, m + \alpha_2)}).
\]

Applying the Poisson summation formula, we get

\[
d_{\eta}(k) = \sum_{M,N} \int_{\alpha_1}^{\infty} dI_1 \int_{\alpha_2}^{\infty} dI_2 \delta(k - \sqrt{H}) \eta(I_1 - \alpha_1, I_2 - \alpha_2) e^{i2\pi(M(I_1-\alpha_1)+N(I_2-\alpha_2))}
+ \frac{1}{2} \sum_{M} \int_{\alpha_1}^{\infty} dI_1 \delta(k - \sqrt{H(I_1, \alpha_2)}) \eta(I_1 - \alpha_1, 0) e^{i2\pi M(I_1-\alpha_1)}
+ \frac{1}{2} \sum_{N} \int_{\alpha_2}^{\infty} dI_2 \delta(k - \sqrt{H(\alpha_1, I_2)}) \eta(0, I_2 - \alpha_2) e^{i2\pi N(I_2-\alpha_2)}
+ \frac{1}{4} \delta(k - \sqrt{H(\alpha_1, \alpha_2)}) \eta_{0,0}.
\] (1)

The main contributions for \( d_{\eta}(k) \) come from the first term of this expression (which is, as we will see below, \( O(k^2) \)). The second and third terms give rise to \( O(k) \) corrections, corresponding to “special” orbits, which are the limiting case of a continuous family. The last term is zero for energies larger than the base level.

We use the homogeneity of \( H \) to reduce the integral to the unit energy surface. Let \( s \) be the arc length on the surface and parametrize it as \((\xi_1(s), \xi_2(s))\), so that \( \dot{\xi}_1^2 + \dot{\xi}_2^2 = 1 \) (where the dots specify derivative by \( s \)). Let \( \omega_i = \frac{\partial H}{\partial \dot{r}_i} \) be the angular frequencies at \((\xi_1, \xi_2)\), and \( \omega = \sqrt{\omega_1^2 + \omega_2^2} \). From the fact that \( H \) is constant on the surface \((\dot{H} = 0)\), we get \((\omega_1, \omega_2) = \omega(\dot{\xi}_2, \dot{\xi}_1)\). Changing the coordinates from \((I_1, I_2)\) to \((\lambda, s)\):

\[
I_1 = \lambda \xi_1(s); \quad I_2 = \lambda \xi_2(s),
\]

we find that \( dI_1 dI_2 = d\lambda ds (\lambda W(s)) \) and \( \delta(k^2 - H(I_1, I_2)) = \delta(k - \lambda)/(\lambda \omega W(s)) \), where \( W = \xi_1 \dot{\xi}_2 - \xi_2 \dot{\xi}_1 \) is the Wronskian. Substituting these in the integral, and ignoring corrections of order \( \alpha_i/k \), we get

\[
d_{\eta}(k) \sim 2k^2 \sum_{M,N} \int_0^A d\tau \frac{\tau_1 \dot{\xi}_1 + \tau_2 \dot{\xi}_2}{\omega} e^{i2\pi k(M \xi_1 + N \xi_2)} e^{-i2\pi(M,N)\cdot(\alpha_1, \alpha_2)},
\] (2)

where \( A \) is the area of the unit energy surface. The term with \( M = N = 0 \) gives the smooth part of the formula: \( 2k^2(\tau_1 A_1 + \tau_2 A_2) \), where \( A_i = \int_{\omega} d\xi \dot{\xi}_i \). For the rest of the terms, we use the stationary phase approximation (SPA), which gives the condition \((\dot{\xi}_1, \dot{\xi}_2) \cdot (M, N) = 0\), or \((\omega_1, \omega_2) \parallel (M, N)\). If we write \((M, N) = r(\mu, \nu)\) where \( r \) is the GCD of \( M, N \), we get that the motion corresponding to the point \((\xi_1, \xi_2)\) is periodic, with period \( T = 2\pi \mu/\omega_1 = 2\pi \nu/\omega_2 \). So \((\mu, \nu)\) corresponds to the topology of the relevant
torus and we identify $r$ as the number of repetitions over the orbit. The phase of the exponent in (2) is proportional to the action of the orbit

$$2\pi(\mu \xi_1 + \nu \xi_2) = \oint (p_1 dq_1 + p_2 dq_2) = S,$$

and the second derivative of the phase is proportional to

$$\mu \ddot{\xi}_2 + \nu \ddot{\xi}_2 = -\kappa \sqrt{\mu^2 + \nu^2} = -\kappa \frac{\omega T}{2\pi},$$

where $\kappa$ is the curvature of the unit energy surface. Substituting these results in equation (2), we finally get

$$d_\eta(k) = 2k^2(\tau_1 A_1 + \tau_2 A_2)$$

$$+ \sqrt{32\pi} k^{3/2} \sum_{p \in \mathcal{P}, O, r=1}^\infty \frac{\xi_1^{(p)}}{\sqrt{r}} \frac{\xi_2^{(p)}}{|\kappa_p| T_p} \cos \left( r(kS_p - \frac{\pi}{2} \mu_p \cdot \beta) - \sigma_p \frac{\pi}{4} \right),$$

where $p$ enumerates periodic tori, $\xi_i^{(p)}$, $\omega_p$, $\kappa_p$, $T_p$ and $S_p$ are the relevant parameters of a representative orbit at unit energy, $\mu_p = (\mu_p, \nu_p)$ is the topological index of the torus, $\beta = 4(\alpha_1, \alpha_2)$ are integer indices, and $\sigma_p$ is the sign of $\kappa_p$.

### 2.2. Example: Rectangle Billiard

For a rectangle of width $a$ and height $b$, the actions are given by $I_x = \frac{a}{\pi} p_x$, $I_y = \frac{b}{\pi} p_y$. The hamiltonian is

$$H = \frac{\pi^2}{2} \left( \frac{I_x^2}{a^2} + \frac{I_y^2}{b^2} \right).$$

Hence, $\omega_x = 2\pi^2 I_x / a^2$, $\omega_y = 2\pi^2 I_y / b^2$, and $\omega = 2\pi^2 \sqrt{I_x^2 / a^4 + I_y^2 / b^4}$. The angle $\theta$ is specified by $\tan(\theta) = p_y / p_x = a I_y / (b I_x)$.

$$L_{\mu,\nu} = 2\sqrt{(\mu a)^2 + (\nu b)^2}$$

$$d_\eta(k) = \frac{2ab(a + b)}{\pi^3} k^2$$

$$+ k^{3/2}(ab) \left( \frac{2}{\pi} \right)^{5/2} \sum_{r,\mu,\nu}^\infty \frac{2\mu a^2 + 2\nu b^2}{L_p^{3/2} \sqrt{r}} \cos \left( i(rkL_p - \frac{\pi}{4}) \right) \ldots$$

Finish this...

### 2.3. Spectral inversion $k(n)$

For the calculation of $d_\eta(n)$, we need the inverse function $k(n)$. As was done in [22], we compute this by formally inverting $n(k)$. The derivation given here is done in a more detailed way, and yields higher order terms. These higher order corrections are not required for this section, but we use them in section 3 where the theoretical trace formula for the right triangle billiard is numerically tested to high accuracy.
We will assume that the spectrum of the integrable billiard satisfies a trace formula of the following form

\[
n(k) = \frac{A}{4\pi} k^2 - \frac{L}{4\pi} k + D_0 + \sqrt{k} \sum_{p \in \text{PPO}} C_p \sin(kL_p + \phi_p) + \sum_{p \in \text{SPO}} D_p \sin(kL_p + \varphi_p) + O(k^{-1/2}),
\]

where \( \text{PPO} \) is the set of periodic tori (which are continuous families of Parabolic Periodic Orbits), \( L_p \) the length of the orbit and \( C_p \) and \( \phi_p \) are other parameters which depend on classical features of the orbit. Similarly, \( \text{SPO} \) is a set of “special”, isolated orbits, with \( D_p \) and \( \varphi_p \) depending on classical features of the orbit. An example for such a formula is the trace formula for the right isosceles triangle, equation (12), given in section 3.

In that example, the periodic tori are parametrized by \( p = (M,N) \in \mathbb{Z}^2 \) (where \( \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\} \)), with \( L_{M,N} = 2a \sqrt{M^2 + N^2} \), \( C_{M,N} = 2A(2\pi LM_{M,N})^{-3/2} \) and \( \phi_{M,N} = -\pi/4 \). The \( \text{SPO} \) in that example consist of repetitions of two special orbits, of lengths \( 2a \) and \( \sqrt{2a} \), with \( N \geq 1 \) the number of repetitions, \( D_N = 1/(2\pi N) \) and \( \varphi_N = \pi \).

Since \( n(k) \) is not really invertible, we introduce a smoothed function \( n_\sigma(k) \) which is monotonously increasing. We choose some positive and symmetric smoothing kernel \( \rho_\sigma(q) \) of typical width \( \sigma \) (decaying fast enough for \( q \gg \sigma \)), and such that the Fourier transform \( \hat{\rho}_\sigma(x) \) decays exponentially for large \( x \). Convolving both sides of (5) with \( \rho_\sigma \), we get

\[
n_\sigma(k) = \frac{A}{4\pi} k^2 - \frac{L}{4\pi} k + D_0^\sigma + \sqrt{k} \sum_{p \in \text{PPO}} C_p^\sigma \sin(kL_p + \phi_p) + \sum_{p \in \text{SPO}} D_p^\sigma \sin(kL_p + \varphi_p) + O(k^{-1/2})
\]

where \( n_\sigma = n * \rho_\sigma \), \( D_0^\sigma = D_0 + \int_{-\infty}^{\infty} k^2 \rho_\sigma(k) dk \cdot A/(4\pi) \), \( C_p^\sigma = C_p \hat{\rho}_\sigma(L_p) \) and \( D_p^\sigma = D_p \hat{\rho}_\sigma(L_p) \). The smoothed step function \( n_\sigma \) is now monotonously increasing and invertible. We try to derive an asymptotic trace formula for the inverse function \( k_\sigma \equiv (n_\sigma)^{-1} \) based on (6). To simplify notation, we will use \( q = \sqrt{4\pi n/A} \) as the parameter of the inverted function, instead of \( n \). Denote also \( f^\sigma(k) = \sum_{\text{PPO}} C_p^\sigma \sin(kL_p + \phi_p) \) and \( g^\sigma(k) = \sum_{\text{SPO}} D_p^\sigma \sin(kL_p + \varphi_p) \).

The exponential decay of \( \hat{\rho}_\sigma \) ensures that \( M_C^\sigma = |\sum_{\text{PPO}} C_p^\sigma| \) and \( M_D^\sigma = |\sum_{\text{SPO}} D_p^\sigma| \) converge, so they can be used as a uniform bounds to \( |f^\sigma(k)| \) and \( |g^\sigma(k)| \) respectively. Hence, the fourth and fifth terms of equation (6) are \( O(\sqrt{k}) \) and \( O(1) \) respectively. With this, we can formally invert the expansion, and get

\[
\frac{A}{2\pi} k_\sigma(q) = \frac{A}{2\pi} q + \frac{L}{4\pi} + \left( \frac{L^2}{16\pi A} - D_0^\sigma \right) q^{-1} - q^{-1/2} f^\sigma(k) - q^{-1} g^\sigma(k) + O(q^{-3/2}).
\]

To eliminate the \( k \) dependence from the right hand side of equation (7), we will use the fact that the \( k \) dependant terms in (7) vanish for high \( q \). Denoting \( \bar{q} \equiv q + L/(2A) \) and \( \delta \equiv -q^{-1/2}(2\pi/A) f^\sigma(k) \), we have from (7) \( k_\sigma = \bar{q} + \delta + O(q^{-1}) \). Choose a proper
truncation length $L_M$, so that $\sum_{L_p > L_M} C_p^\sigma \ll 1$. For $q$ large enough, $q \gg (M_c^2 L_M / A)^2$, we have $|\delta| < q^{-1/2} 2\pi M_c^2 / A \ll 2\pi / L_p$, which is smaller than the period $2\pi / L_p$ of the sine term corresponding to orbit $p$, for all orbits that have a significant contribution to $f^\sigma(k)$. Hence we can approximate the sine by the first order Taylor expansion in $\delta$:

$$\sin(kL_p + \varphi_p) \sim \sin(qL_p + \varphi_p) - q^{-1/2} \frac{2\pi L_p}{A} \sum_{p' \in \text{PPO}} C_p^\sigma \cos(qL_p + \varphi_p) \sin(qL_{p'} + \varphi_{p'}).$$

Inserting this into (7), we finally get

$$\frac{A}{2\pi} k_\sigma(q) = \frac{A}{2\pi} q + \frac{L}{4\pi} + \left( \frac{L^2}{16\pi A} - D_0^\sigma \right) q^{-1} - q^{-1/2} \sum_{p \in \text{PPO}} C_p^\sigma \sin(qL_p + \varphi_p) - q^{-1} \sum_{p' \in \text{PPO}} C_p^\sigma C_{p'}^\sigma (L_p - L_{p'}) \sin[q(L_p - L_{p'}) + \varphi_p - \varphi_{p'}] + q^{-1} \sum_{p, p' \in \text{PPO}} C_p^\sigma C_{p'}^\sigma (L_p + L_{p'}) \sin[q(L_p + L_{p'}) + \varphi_p + \varphi_{p'}] - q^{-1} \sum_{p \in \text{PPO}} D_p^\sigma \sin(qL_p + \varphi_p) + O(q^{-3/2}). \quad (8)$$

While equations (6) and (7) are correct for any $\sigma$, we have only shown the correctness of (8) for values of $q$ which are large compared to a $\sigma$-dependant lower bound, $q_m \equiv (M_c^2 L_M / A)^2$. The bound might increase to infinity as $\sigma \to 0$ (for example, in the case of the triangle, with Lorentzian smoothing $\hat{\rho}_\sigma(x) = \exp(-2\sigma|x|)$, we get $M_c \sim \sigma^{-1/2}$ and $q_m \sim \sigma^{-1}$). This is not sufficient for our case, where $k_\sigma(n)$ is used as an approximation for $k(n)$. To keep the error small, $n_\sigma$ must be close to $n$, so the influence of neighbouring steps of $n(k)$ must be suppressed by the convolution, i.e. the smoothing length $\sigma$ should be smaller than the level spacing. For $q \gg 2\pi / (A\sigma)$, the above derivation of (8) works, but the step structure is wiped out by the smoothing, and $n_\sigma$ loses accuracy as an approximation of $n$. To remedy this, one needs to find a tighter bound for $f^\sigma(k)$. In actual problems, we expect such a bound to exist. For example, in the case of the triangle, $f^\sigma(k)$ converges at $\sigma = 0$ for every $k$, and numerics suggest that $f^0(k)$ is $O(1)$ in $k$. One should be able to find a lower limit $q_m$ which does not depend on $\sigma$, and show (8) for smoothing tight enough to keep the step structure. However, instead of doing that, we demonstrate the validity of equation (8) numerically, for several values of $\sigma$.

The $q^{-1/2}$ term of (8) contains $f^\sigma(\hat{q})$, the usual sum over periodic tori. Define

$$k_\sigma^{\text{smooth}} = q + \frac{L}{2A} + \frac{L^2}{16\pi A D_0^\sigma} q^{-1}, \quad k_\sigma^{\text{osc}}(q) = k_\sigma(q) - k_\sigma^{\text{smooth}}(q).$$

For numerical verification of the $q^{-1/2}$ term, we calculate $f_1(q) \equiv q^{1/2} k_\sigma^{\text{osc}}(q) \cdot W_\Delta(q - q_0)$, where $k_\sigma^{\text{osc}}$ is calculated from the numerical spectrum (using $k_\sigma = (\rho_\sigma * n(k))^{-1}$), and $W_\Delta(q - q_0)$ is a smooth spectral window of width $\Delta$, centred around $q_0$. We used a
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Figure 1: Fourier transform of $k_{\sigma}^{osc}(q)$—periodic orbit contribution. $q_0 = 840$, $\Delta = 280$ (Gaussian window $W_{\Delta}$), $\sigma = 0.225$ (Lorentzian smoothing $\rho_\sigma$).

Gaussian, which has a more localized effect on $\hat{f}_1$ than the naive cutoff $\Theta(q_{\text{max}} - q)$.

From the right hand side of equation (8), we see that the Fourier transform of $f_1$ can be approximated by

$$\hat{g}_1(x) = \frac{\pi}{iA} \sum_{p \in \text{PPO}} C_p e^{-i [\phi_p + L_p \Delta/(2A)]} \cdot \hat{W}_\Delta(x - L_p) e^{i q_0 (x - L_p)}$$

(where $g_1(q) \equiv -(2\pi/A) f^\sigma(q) \cdot W_\Delta$ is the $O(q^{-1/2})$ term of the right hand side, cut off with the window). As can be seen in figure \ref{fig1a} we get an accurate match between the numerics and this prediction. In figure \ref{fig1a} we see that the heights and positions of the peaks match the periodic tori as predicted by equation (8). Lower order peaks are also observable there: in particular, the SPO peak at $L_p = 2$, and the pair-sum peak at $L_p + L_{p'} = 2 + 2\sqrt{2}$. Figure \ref{fig1b} shows a complex-valued magnification of the peak corresponding to the torus with $L_p = 2$. This demonstrates the accuracy of the match and the validity of the phase $\phi_p$.

To verify the $O(q^{-1})$ terms, we subtract the known sum of periodic tori, and proceed in a similar manner. Define $f_2(q) \equiv q[k_{\sigma}^{osc}(q) - q^{-1/2} g_1(q)] \cdot W_\Delta(q - q_0)$, then by equation (8) this can be estimated by $g_2(q) = g_2^{\text{SPO}} + g_2^{\text{diff}} + g_2^{\text{sum}}$, with
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In figure 2, we see a good match for the predictions of the $O(q^{-1})$ terms as well. In the magnified plot figure 2b we see two peaks: the one at $x = 8$ is composed of contributions from $g^\text{sum}_2$, $g^\text{diff}_2$ and $g^\text{SPO}_2$, while the one at $2(\sqrt{2}^2 + 5^2 - \sqrt{2}) \sim 7.94$ is purely due to an orbit difference. A similar analysis was performed with lower values of $\sigma$. As can be seen in figure 2b, an increasing amount of orbit-pair differences from $g_2^\text{diff}$ become observable. However, equation 8 still has predictive power at $\sigma = 0.00$, which is sufficiently low to preserve the step structure around $q_0 = 0.00$, the centre of the spectral window, as can be seen in figure 2b.

2.4. Eliminating the $k$ dependence—$\eta(n)$

3. Trace formula for the boundary intersections count of the isosceles right triangle

Integrable systems which are not separable do not have the simple checker-board-like nodal pattern that characterizes separable ones (see, for example, the sample eigenfunctions which are shown in figure 3). Nevertheless, we expect that nodal statistics would still have common features with the separable case, which would enable us to differentiate between integrable and chaotic systems. The isosceles right triangle is one of the few cases where despite being non-separable we still have analytical expressions for the eigenvalues, eigenfunctions, and the BIC (the number of nodal domains can also be computed using a closed algorithm [23]). This allows us to follow the same procedure that was used in section 2 and derive a trace formula.
Consider the triangle bounded by the lines $x = a$, $y = 0$ and $x = y$. It has an area $A = \frac{a^2}{2}$, and boundary length $L = a(\sqrt{2} + 2)$. The eigenfunctions are exactly the anti-symmetric combinations of degenerate pairs of eigenfunctions of the $a \times a$ square:

$$\psi_{l,m}(x, y) = \frac{\sqrt{2}}{a} \left[ \sin\left(\frac{l\pi}{a}x\right) \sin\left(\frac{m\pi}{a}y\right) - \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{l\pi}{a}y\right) \right]$$

(9)

for all $l > m > 0$. The eigenvalues are given by

$$k_{l,m} = \frac{\pi}{a} \sqrt{l^2 + m^2}.$$ 

It is not hard to see (see Appendix A) that the BIC is given by

$$\eta_{l,m} = \begin{cases} 
  l + m - 3 & \text{if } l + m = 1 \pmod{2} \\
  l + m - 2 & \text{if } l + m = 0 \pmod{2}
\end{cases}$$

(10)

or equivalently $\eta_{l,m} = l + m - \frac{5}{2} + \frac{1}{2} \exp[i\pi(l + m)]$.

The trace formula for the spectral counting function can be found directly by applying the Poisson summation formula to the definition

$$n(k) = \sum_{l > m > 0} \Theta(k - k_{l,m}) = \sum_{l > m > 0} \Theta\left(k - \frac{\pi}{a} \sqrt{l^2 + m^2}\right).$$

(11)

This evaluates to

$$n(k) = \frac{A}{4\pi} k^2 - \frac{L}{4\pi} k + \frac{3}{8} + \sum_{(M,N) \in \mathbb{Z}^2} \frac{2A\sqrt{k}}{2\pi L_{M,N}} \sin(kL_{M,N} - \frac{\pi}{4})$$

$$- \sum_{N \in \mathbb{N}^*} \frac{1}{2\pi N} \left[ \sin(2Nak) + \sin(\sqrt{2}Nak) \right] + o(1),$$

(12)

where $L_{M,N} = 2a\sqrt{M^2 + N^2}$. The $O(\sqrt{k})$ term may be interpreted as a sum over periodic tori, where the torus corresponding to $(M, N)$ consists of orbits that bounce from the bottom edge ($y = 0$) at angle $\psi$ with $\tan(\psi) = N/M$, and whose length is $L_{M,N}$. Note that if $r = \gcd(M, N)$ and $(M, N) = (r\mu, r\nu)$, then such an orbit is in fact
r repetitions of the orbit \((\mu, \nu)\) (this is just a special case of the generic semi-classical formula by Berry and Tabor [20]). The O(1) term corresponds to isolated orbits. \(\sqrt{2}Na\) is the length of the orbit hitting the corner \((x = a, y = 0)\) at 45\(^\circ\) (with \(N\) repetitions), and \(2Na\) is the length of the orbits that lie on the catheti \((y = 0)\) and \((x = a)\).

3.1. BIC as a function of \(k\)

In a similar way to the approach used above, we now combine equation (10) with the Poisson summation formula, to get a trace formula for

\[ C(k) \equiv \sum_{l > m > 0} (|l| + |m| - \frac{5}{2} + \frac{1}{2} \epsilon i (l + m)) \Theta \left( k - \frac{\pi}{a} \sqrt{l^2 + m^2} \right) \]

(13)

The difference is that here we enumerate half integer values rather than integer ones.

\[ \frac{1}{8} \sum_{M,N} \int dl \, dm \, \eta_{l,m}^R \Theta \left( k - k_{l,m} \right) e^{2\pi i (l + m N)} \]

\[ = \frac{1}{16} \sum_{M,N} \int dl \, dm \, \Theta \left( k - \frac{\pi}{a} \sqrt{l^2 + m^2} \right) e^{2\pi i \left[ (l + \frac{1}{2}) + m(N + \frac{1}{2}) \right]} \]

\[ = \frac{1}{2} \sum_{M,N} N_{M+\frac{1}{2},N+\frac{1}{2}}, \quad \text{where } \]

\[ N_{M,N} \text{ is the O}(\sqrt{k}) \] expression that leads to periodic tori term of equation (12).

The rest of the calculations, as well as the computation of the bulk term \(C^B\) are explained in Appendix B. The highest order terms of the result are given by

\[ C(k) = \frac{(ak)^3}{3\pi^3} - \frac{5\pi + 8}{16\pi^2} (ak)^2 + \frac{7 + 3\sqrt{2}}{6\pi} ak \]

\[ + a^2 k^\frac{3}{4} \sum_{(M,N) \in \mathbb{Z}^2} \frac{2\lambda_{M,N}}{(2\pi L_{M,N})^{5/2}} \sin \left( \frac{L_{M,N}k}{2} - \frac{\pi}{4} \right) \]

\[ - a^2 k^\frac{3}{4} \sum_{N \in \mathbb{N}, \pi^2 2aN} \left[ \frac{1}{\pi N} \cos + \sin \right] (2Nak) \]

\[ - a^2 k^\frac{3}{4} \sum_{N \in \mathbb{N}, \pi^2 2aN} \sin \left( \frac{\sqrt{2}Nak}{\pi} \right) + O(\sqrt{k}), \quad \text{(15)} \]

where \(L_{M,N} = 2a\sqrt{M^2 + N^2}\) is the length of the orbit (as before), and \(\lambda_{M,N} = 2a(|M| + |N|)\). As in equation (12), we can see the contributions of the periodic tori and
the isolated orbits. The next order corrections are \( \mathcal{O}(\sqrt{k}) \), and they contain (among other terms, depending on the PPO and SPO orbits) the sum

\[
\sum_{M,N \in \mathbb{Z}} \frac{A \sqrt{k}}{(2\pi \tilde{L}_{M,N})^{3/2}} \sin(k\tilde{L}_{M,N} - \frac{\pi}{4}),
\]

where \( \tilde{L}_{M,N} \equiv a \sqrt{[(2M + 1)^2 + (2N + 1)^2]} \). This introduces oscillations whose frequencies do not correspond to lengths of periodic orbits, but rather to “semi periodic” ones. After moving a length of \( \tilde{L}_{M,N} \) (which is half the length of some periodic orbit), the particle reaches a point which is the reflection (along the symmetry axis of the triangle) of the starting point, and from there follows a path which is the reflection of the first half of the orbit. These terms arise due to the contribution of the round-off term \( \eta^R \) (as shown in equation (14)), and they do not appear in the spectral trace formula, nor in the BIC formulae of separable billiards. In figure ?? the contribution of such a semi-periodic orbit is observed at \( \tilde{L}_{0,1} = \sqrt{10} \).

### 3.2. Combining with the spectral inversion

To get the final trace formula for \( C(q) \), one needs to substitute the spectral inversion formula \[8\] in equation (15) (the relevant parameters \( C_p \) and \( D_p \) are listed in section 2.3). Before we combine the formulae, we will assume that the step function \( C(k) \) is smoothed using a convolution kernel \( \rho_\varsigma(k) \) of width \( \varsigma \). Define \( C^\varsigma(k) = (\rho_\varsigma \ast C)(k) \). We will later use \( C^{\varsigma,\sigma}(k) \equiv C^\varsigma(k^{\sigma}(q)) \) as an approximation for \( C(q) \). Note that the limiting form of \( C^{\varsigma,\sigma}(q) \) as \( \varsigma,\sigma \to 0 \) depends on the order of the limits. If \( \varsigma \ll \sigma \), the limiting form starts at 0, and jumps to \( C_n \) at \( q = \sqrt{[(n - \frac{1}{2})4\pi/A]} \). If \( \sigma \ll \varsigma \), the limiting form starts at \( C_1/2 \) and jumps to \( \frac{1}{2}(C_n + C_{n+1}) \) at \( q = \sqrt{n \cdot 4\pi/A} \). If they go to 0 together (keeping \( \sigma = \varsigma \ll 1 \)), then the limiting form is composed of linear segments connecting the points \((\sqrt{n \cdot 4\pi/A}, C_n)\).

A trace formula for \( C^\varsigma \) is easily obtained by applying the convolution on equation (15). As usual, each trigonometric function \( \sin(L^p k + \phi_p) \) on the right hand side gains a factor of \( \hat{\rho}_\varsigma(L^p) \). Next, a formula for \( C^{\varsigma,\sigma}(q) \) is obtained by formally substituting [8] in the formula for \( C^\varsigma(k) \). When substituting [8] for \( k \) in the arguments of the trigonometric functions, we repeat the reasoning of section 2.3. We choose \( q \) high enough to make the oscillating part of [8] smaller than the relevant period, and then expand the sine to first order around \( \tilde{q} \). In particular, for the \( \mathcal{O}(k^{3/2}) \) terms of (15) we get

\[
\hat{\rho}_\varsigma(L^p) \frac{2\lambda_p}{(2\pi L^p)^{3/2}} \sin(L^p k - \frac{\pi}{4}) \sim \hat{\rho}_\varsigma(L^p) \frac{2\lambda_p}{(2\pi L^p)^{3/2}} \sin(L^p \tilde{q} - \frac{\pi}{4})
\]

\[-q^{-1/2} \sum_{p' \in \mathbb{Z}^2} \hat{\rho}_\varsigma(L^p) \hat{\rho}_\sigma(L^{p'}) \lambda_p L^{p/2} \cos(L^p \tilde{q} - \frac{\pi}{4}) \sin(L^{p'} \tilde{q} - \frac{\pi}{4}).\]

Summing the \( \mathcal{O}(q^{-1/2}) \) correction over \( p \in \mathbb{Z}^2 \), we get, after symmetrization with respect to exchange of the dummy indices \( p \leftrightarrow p' \)

\[
q^{-1/2} \sum_{p,p' \in \mathbb{Z}^2} \frac{\hat{\rho}_\sigma \hat{\rho}_p'}{8\sqrt{\lambda_p L^p L^{p'/2}}} (\lambda_p \rho_\varsigma + \lambda_{p'} \rho_\sigma) \cos((L^p + L^{p'}) \tilde{q})
\]
intersections are the points where the boundary function changes sign, we start by

\[
+q^{-1/2} \sum_{p,p' \in \mathbb{Z}^2} \frac{\hat{\rho}_\sigma \hat{\rho}'_\sigma}{8\pi^3 L_p^{3/2} L_{p'}^{3/2}} (\lambda_p \hat{\rho}_\sigma - \lambda_p' \hat{\rho}'_\sigma) \sin((L_p - L_{p'})q),
\]

where we have used the shortcut notations \( \hat{\rho}_\sigma \equiv \hat{\rho}_\sigma(L_p), \hat{\rho}'_\sigma \equiv \hat{\rho}_\sigma(L_{p'}), \hat{\rho}_\sigma \equiv \hat{\rho}_\sigma(L_p), \) and \( \hat{\rho}'_\sigma \equiv \hat{\rho}_\sigma(L_{p'}). \)

After combining this with the rest of the terms that arise from the substitution of \( k^\sigma(q) \) in \( C^\sigma(k) \), we finally get

\[
C^\sigma(q) = \frac{(aq)^3}{3\pi^3} + \alpha_2 \left( \frac{aq}{\pi} \right)^2 + \alpha_1 \frac{aq}{\pi}
+ a^3 q^{3/2} \sum_{p \in \mathbb{Z}^2} \frac{2(\hat{\rho}_\sigma \pi \lambda_p - \hat{\rho}_\sigma 4L_p)}{\pi(2\pi L_p)^{5/2}} \sin(L_pq - \frac{\pi}{4})
- a^2 q \sum_{N \in \mathbb{N}^*} \frac{1}{\pi^3 2aN} \left[ \frac{\hat{\rho}_\sigma}{N} \cos(\hat{\rho}_\sigma \pi - \hat{\rho}_\sigma 4) \right] (2aNq)
- a^2 q \sum_{N \in \mathbb{N}^*} \frac{\hat{\rho}_\sigma \pi - \hat{\rho}_\sigma 2\sqrt{2}}{\pi^3 \sqrt{2aN}} \sin(\sqrt{2aNq})
+ a^2 q \sum_{p,p' \in \mathbb{Z}^2} \frac{\pi(\lambda_p \hat{\rho}'_\sigma + \lambda_p' \hat{\rho}'_\sigma)}{8\pi^3 L_p^{3/2} L_{p'}^{3/2}} \cos((L_p + L_{p'})\hat{\rho}_\sigma)
+ a^2 q \sum_{p,p' \in \mathbb{Z}^2} \frac{\pi(\lambda_p \hat{\rho}'_\sigma + \lambda_p' \hat{\rho}'_\sigma)}{8\pi^3 L_p^{3/2} L_{p'}^{3/2}} \sin((L_p - L_{p'})\hat{\rho}_\sigma)
+ O(\sqrt{q}),
\]

where

\[
\alpha_2 = \frac{2 + \sqrt{2}}{\pi} - \frac{5\pi + 8}{16}
\]

\[
\alpha_1 = \frac{3}{\pi^2} (3 + 2\sqrt{2} - \frac{\pi}{2}) - \frac{(5\pi + 8)(2 + \sqrt{2})}{8\pi} + \frac{7 + 3\sqrt{2}}{6}
+ \left( \frac{a}{\pi} \right)^2 \int_{-\infty}^{\infty} \rho_\sigma(q)q^2 dq.
\]

Note that the “semi-periodic” orbit lengths described in section \[\text{3.1}\] are of lower order than the contribution of the orbit-differences, which is badly behaved for small \( \sigma \). Hence they can not be observed in the numerical data for \( C(q) \). In figure \[\text{??} \ldots \]

4. The BIC of chaotic billiards

Appendix A. Counting boundary intersections for the right equilateral triangle

In this section, we show that the boundary intersections count for the eigenfunction \( \psi_{l,m} \) (with \( l > m > 0 \)) of equation \[\text{[2]}\] is given by equation \[\text{[10]}\]. Since the boundary intersections are the points where the boundary function changes sign, we start by
calculating the boundary function \( u_{l,m} = (1/k_{l,m})(\mathbf{n} \cdot \nabla)\psi_{l,m} \). With proper choice of boundary parameter \( s \), we get

\[
\begin{align*}
\sqrt{2} \pi a l \sin(mx) & - m \sin(lx) \quad & \text{if } s < a \\
\sqrt{2} \pi a (-1)^{l+m+1}l \sin(my) & - m \sin(ly) \quad & \text{if } a < s < 2a \\
2 \pi a l \cos(lz) \sin(mz) & + m \cos(mz) \sin(lz) \quad & \text{if } s > 2a,
\end{align*}
\]

(A.1)

where \( x = \pi s/a, y = \pi (2 - s/a) \) and \( z = \pi (2 + \sqrt{2} - s/a)/\sqrt{2} \) are linear transformations of \( s \), normalized to vary through the range \((0, \pi)\) on the respective edges of the triangle.

The number of zeros of \( u \) in each of the segments \((0, a)\) and \((a, 2a)\) is the number of solutions for

\[
\frac{\sin(lx)}{l} = \frac{\sin(mx)}{m}
\]

with \( 0 < x < \pi \). We will show that this is exactly \( m - 1 \). Denote \( z_i = \pi i/m \) for \( i = 1, \ldots, m - 1 \) (the zeros of \( \sin(mx) \)). Let \( \delta = \sin^{-1}(m/l)/m < \pi/(2m) \). Equation (A.2) can only have solutions in one of the (non overlapping) ranges \([z_i - \delta, z_i + \delta] \), because out of these ranges we have \(|f_1(x)| \equiv |\sin(mx)/m| > 1/l\), whereas \(|f_2(x)| \equiv |\sin(lx)/l| \leq 1/l\) for all \( x \). For odd \( i \), we find that on the higher edge \( x_+ = z_i + \delta \) we have \( f_1(x_+) = -1/l \leq f_2(x_+) \) and on the lower edge \( x_- = z_i - \delta \) we have \( f_1(x_-) = 1/l \geq f_2(x_-) \). Hence, the interval must contain at least one solution of (A.2). For even \( i \), the argument repeats with reversed signs.

We will now show that each of these intervals can contain at most one solution. Consider the interval around \( z_i \) for some even \( i \) (the proof for odd \( i \) is practically the same, and will not be repeated here). Let \( y = x - z_i \) (so on the interval we have \(-\delta < y < \delta\), then \( f_1(x) = \sin(my)/m \), and \( f_2(x) = \sin(ly + \phi)/l \), where \( \phi = lz_i \mod 2\pi \). Assume that there are two solutions in the interval, located at \( x_1 = z_i + y_1 \) and \( x_2 = z_i + y_2 \), with \(-\delta \leq y_1 < y_2 \leq \delta \). Since \( f_1 - f_2 = 0 \) at these two points, there must exist \( y^* \in (y_1, y_2) \) satisfying \( f_1'(z_i + y^*) - f_2'(z_i + y^*) = 0 \). Hence we can write

\[
\cos(ly^* + \phi) = \cos(my^*) \Rightarrow ly^* + \phi = \pm(my^*).
\]

(A.3)

We will now assume that \( y^* > 0 \) and reach contradiction by showing that \( f_1(x_2) > f_2(x_2) \), contrary to the assumption that \( x_2 \) is a solution. If we assume that \( y^* < 0 \), an equivalent argument would show that \( f_1(x_1) < f_2(x_1) \) reaching contradiction again (this part will not be described here). Since \( 0 < y^* < \delta \), we have \( \sin(my^*) > 0 \), and

\[
f_1(x^*) = \frac{\sin(my^*)}{m} > \left| \frac{\sin(ly^* + \phi)}{l} \right| = |f_2(x^*)|.
\]

We distinguish two cases:

(i) If \( f_2(x^*) > 0 \), we have from equation (A.3) \( ly^* + \phi = my^* \). For points larger than \( x^* \), we have \( f_1'(x^* + d) = \cos(my^* + md) \), and \( f_2'(x^* + d) = \cos(my^* + ld) \). Since \( \cos \) is decreasing in this region and since \( f_2' \) gains phase faster, we find that for \( d \ll 1 \)
\[ f'_1(x^* + d) > f'_2(x^* + d). \] As we increase \( d \), the difference \( f_1 - f_2 \) (which is positive at \( x^* \)) only increases. The functions can not intersect as long as the derivative difference remains positive. The first point after \( x^* \) where \( f'_1 - f'_2 = 0 \) again is at \( d = 2\pi/(l + m) \). However, since for any \( 0 < x < \pi/2 \) the inequality \( \sin^{-1} x < \pi x/2 \) holds, we have

\[ y_2 < \delta = \frac{1}{m} \sin^{-1} \frac{m}{l} < \frac{\pi}{2l} < \frac{2\pi}{l + m}, \]

so \( f'_1 - f'_2 \) is still positive at \( x_2 \), and \( f_1(x_2) > f_2(x_2) \) as stated.

(ii) If \( f_2(x^*) < 0 \), equation (A.3) gives \( ly^* + \phi = -my^*\), \( f'_2(x^* + d) = \cos(my^* - ld) \) and \( f_2(x^* + d) = \sin(ld - my^*)/l \). For \( d < my^*/l \) (i.e. \( y^* < x - z_i < y_1^* \equiv y^*(l + m)/l \)), this is negative, and certainly smaller than \( f_1(x^* + d) \). In this region we also have \( f'_2(x) > \cos(my^*) > f'_1(x) \). At \( x = z_i + y_i^* \) the derivative \( f'_2 \) is 1, but it decreases faster than \( f'_1 \) until they become equal at \( x_2^* = z_i + y_i^* \), where \( y_i^* \equiv y^*(l + m)/(l - m) \).

At \( x_2^* \), we have

\[ f_1(x_2^*) = \frac{1}{m} \sin \left( my^* \frac{l + m}{l - m} \right) > \frac{1}{l} \sin \left( my^* \frac{l + m}{l - m} \right) = f_2(x_2^*). \]

Hence, since \( f'_1 - f'_2 \) is negative in the range \( (x^*, x_2^*) \) and \( f_1(x_2^*) - f_2(x_2^*) > 0 \) we get that \( f_1 > f_2 \) throughout this region. Now, if \( x_2 < x_2^* \) we get \( f_1(x_2) > f_2(x_2) \) as required. If \( x_2 > x_2^* \) then we have found a new point \( x_2^* \), with positive \( f_2 \), where \( f'_1 = f'_2 \). This brings us back to case (i) and we can proceed as described there, with \( x_2^* \) taking the place of \( x^* \).

This establishes the uniqueness of solutions inside each interval. Next, we will show that these \( m - 1 \) solutions are all of odd multiplicity (which implies that \( u \) changes sign in these points, and therefore they are boundary intersections). Let \( f(x) \equiv f_2(x) - f_1(x) \) and suppose that \( f(x_0) = 0 \) for some \( 0 < x_0 < \pi \). If \( x_0 \) is degenerate, then

\[ 0 = f'(x_0) = \cos(lx_0) - \cos(mx_0) \Rightarrow (l \mp m)x_0 = 2\pi N \]

for some integer \( N \). Substituting this in \( f \), we get

\[ 0 = f(x_0) \Rightarrow \sin(lx_0) = \pm m \frac{\sin(mx_0)}{m} = \pm \frac{m}{l} \sin(lx_0). \]

But since \( 1 > m/l > 0 \), this can only happen if both \( lx_0 \) and \( mx_0 \) are multiples of \( \pi \). It is easy to verify that in this case \( f''(x_0) = 0 \) and \( f^{(3)}(x_0) = \pm (l^2 - m^2) \neq 0 \). Thus, all roots of \( f \) in the interval are of multiplicity 1 or 3.

For the range \( 2a < s < a(2 + \sqrt{2}) \), we define \( \mu = l + m, \nu = l - m \), and use (A.1) to write

\[ u_{bm} = 2 \frac{\pi}{a} \left[-l \cos(lz) \sin(mz) + m \cos(mz) \sin(lz)\right] \]
\[ = \frac{\pi}{a} [\mu \sin(\nu z) - \nu \sin(\mu z)], \quad \text{(A.4)} \]

so, repeating the previous argument, the number of boundary intersections in this range is \( \nu - 1 = l - m - 1 \).
The total number of boundary intersections in the 3 intervals is \(2(m-1) + (l-m-1) = m + l - 3\). However, we still need to check the three corners. At all the corners \(u = 0\), but \(u\) is not smooth there, so we need to check whether or not it changes sign at these points. Returning to equation (A.1) and expanding for values of \(s\) approaching \(0, a, 2a\) and \(a(\sqrt{2} + 2)\) from both sides, we get that at \(s = 0\) the sign is positive for both sides, for \(s = 2a\) the sign is \((-1)^{l+m-1}\) on both sides, so these points are never BI. For \(s = a\), the sign is \((-1)^{l+m-1}\) if approaching from below and \((-1)^l\) if approaching from above. Hence it is a boundary intersection if and only if \(l + m = 0 \pmod{2}\). Combining this result with the count inside the intervals, we get the final result, equation (10).

Appendix B. Computing the trace formula for \(C(k)\)

In this section we compute the accumulated boundary intersections count for the right isosceles triangle, described by equations (13) and (B.1). We split the expression for \(F_{l,m}\) into three parts \(F_{l,m} = F_{l,m}^B + F_{l,m}^C + F_{l,m}^R\). the "bulk" term \(F_{l,m}^B = |l| + |m|\), the constant term \(F_{l,m}^C = -\frac{5}{2}\) and the "round off" term \(F_{l,m}^R = \frac{1}{2} \exp[i \pi (l + m)]\). Correspondingly, we split \(C(k) = C^B(k) + C^C(k) + C^R(k)\) and compute each in turn. Note that \(C^C(k) = -\frac{5}{2} n(k)\).

\[
\ldots
\]

Denote
\[
F_{l,m} = (|l| + |m| - \frac{5}{2} + \frac{1}{2} \exp[i \pi (l + m)]) \Theta(k - \frac{\pi}{a} \sqrt{2 + m^2}).
\]

\(F_{l,m}\) does not depend on the signs of \(l\) and \(m\), and is also symmetric \(F_{l,m} = F_{m,l}\). Hence, we can write
\[
\sum_{l>m>0} F_{l,m} = \frac{1}{8} \sum_{l,m} F_{l,m} - \frac{1}{4} \sum_{m} F_{m,0} - \frac{1}{4} \sum_{m} F_{m,m} + \frac{3}{8} F_{0,0},
\]
where the sums on the right side are over all \(Z\). Finally we can apply the Poisson summation formula:
\[
\sum_{l>m>0} F_{l,m} = \frac{1}{8} \sum_{M,N} \int dl \, dm \, F_{l,m} e^{2\pi i (lM + mN)}
- \frac{1}{4} \sum_{M} \int dm \, F_{m,0} e^{2\pi i mM} - \frac{1}{4} \sum_{M} \int dm \, F_{m,m} e^{2\pi i mM} + \frac{3}{8} F_{0,0}.
\] (B.1)

Next, we split \(\eta_{l,m}\) into three parts \(\eta_{l,m} = \eta_{l,m}^B + \eta_{l,m}^C + \eta_{l,m}^R\). the "bulk" term \(\eta_{l,m}^B = |l| + |m|\), the constant term \(\eta_{l,m}^C = -\frac{5}{2}\) and the "round off" term \(\eta_{l,m}^R = \frac{1}{2} \exp[i \pi (l + m)]\). Correspondingly, we split \(C(k) = C^B(k) + C^C(k) + C^R(k)\) and compute each contribution in turn. Comparing to equation (11), we immediately get \(C^C(k) = -\frac{5}{2} n(k)\). The contribution of the round-off term is also easy to compute in terms of the spectral trace formula. For example, the contribution to the first term of equation (B.1) becomes
\[
\frac{1}{8} \sum_{M,N} \int dl \, dm \, F_{l,m}^R e^{2\pi i (lM + mN)}
\]
\[
\frac{1}{16} \sum_{M,N} \int dl \, dm \, \Theta(k - \frac{\pi}{a} \sqrt{l^2 + m^2}) e^{2\pi i [l(M + \frac{1}{2}) + m(N + \frac{1}{2})]}
\]

\[
= \frac{1}{2} \sum_{M,N} \mathcal{N}_{M+\frac{1}{2},N+\frac{1}{2}},
\]

where \( \mathcal{N}_{M,N} \) is \( O(\sqrt{k}) \), and was already calculated as the corresponding term of equation (12) (which we associate with the contributions of the periodic tori).

**Appendix C. Scratchpad**

Todo: check the \( 2N \) and \( \sqrt{2}N \) “special” orbits. They are probably bifurcation points or something like that.

Following was extracted from the inversion section. I am not sure I understand the last sentence (it probably refers to the exponential prefactor):

Notice, however that when we use \( \eta(k(n_\sigma)) \) as an approximation for \( \eta(n) \), we will have to ensure that the difference between \( n \) and \( n_\sigma \) is small enough, and to ensure this, \( \sigma \) must be smaller than the difference between neighbouring levels. If we consider the spectrum up to \( k_M \), then we need \( \sigma < \min \{ \Delta k \} \sim 1/k_M \). So, when evaluating the sums in the right-hand side of (6), we have to go up to orbit lengths proportional to \( k_M \), before we can neglect the tail of the series.

Testing some formulae:

\[
n(k) = \sum_{l > m > 0} \Theta(k - k_{l,m})
\]

\[
= \sum_{M,N} \int_{l > m > 0} dl \, dm \, \Theta\left(k - \frac{\pi}{a} \sqrt{l^2 + m^2}\right) \exp(i2\pi(Ml + Nm)).
\]

\[
a = \int_{a > b} \int_{b > c > 0} a \cdot b
\]

dimensionally nicer (5):

\[
n(k) = \frac{\mathcal{A}}{4\pi} k^2 - \frac{\mathcal{L}}{4\pi} k + \mathcal{D}_0
\]

\[
+ \sqrt{k} \sum_p \frac{A\hat{C}_p}{(L_p)^{3/2}} \sin(kL_p + \phi_p) + \sum_{p'} \frac{L\hat{D}_{p'}}{L_{p'}} \sin(kL_{p'} + \phi_{p'}) + o(1),
\]

\[
\frac{\mathcal{A}}{2\pi} k(q) = \frac{\mathcal{A}}{2\pi} q + \frac{\mathcal{L}}{4\pi} + \left( \frac{\mathcal{L}^2}{16\pi\mathcal{A}} - \mathcal{D}_0 \right) q^{-1}
\]

\[
- q^{-1/2} \sum_p C_p \sin(qL_p + \phi_p)
\]

\[
- \frac{\pi}{2\mathcal{A}} q^{-1} \sum_{i,j} C_i C_j (L_i - L_j) \sin(q(L_i - L_j) + \phi_i - \phi_j)
\]
\[ + \frac{\pi}{2A} q^{-1} \sum_{i,j} C_i C_j (L_i + L_j) \sin(q(L_i + L_j) + \phi_i + \phi_j) \]

\[ - q^{-1} \sum_{\nu} D_\nu \sin(qL_\nu + \varphi_\nu) + O(q^{-3/2}), \quad (C.2) \]

Check the following against the triangle formulae:
\[
\frac{A}{2\pi} k(q) = \frac{A}{2\pi} q + \frac{L}{4\pi} + \left( \frac{L^2}{16\pi A} - D_0 \right) q^{-1}
\]

\[ - A q^{-1/2} \sum_p \frac{\hat{C}_p}{(L_p)^{3/2}} \sin(q L_p + \phi_p) \]

\[ - \frac{\pi A}{2} q^{-1} \sum_{i,j} \frac{\hat{C}_i \hat{C}_j (L_i - L_j)}{(L_i L_j)^{3/2}} \sin(q(L_i - L_j) + \phi_i - \phi_j) \]

\[ + \frac{\pi A}{2} q^{-1} \sum_{i,j} \frac{\hat{C}_i \hat{C}_j (L_i + L_j)}{(L_i L_j)^{3/2}} \sin(q(L_i + L_j) + \phi_i + \phi_j) \]

\[ - L q^{-1} \sum_{\nu} \frac{D_\nu}{L_\nu} \sin(qL_\nu + \varphi_\nu) + O(q^{-3/2}), \quad (C.3) \]

Stuff from the derivation for separable billiards:
\[ I_1 = q\xi_1, \ I_2 = q\xi_2(\xi_1), \] we get (with \( \beta_1 = \xi_2^{-1}(\alpha_2/k) \) and \( \beta_2 = \xi_2(\alpha_1/k) \)):
\[
\eta(k) = k^2 \sum_{M,N} \int_{\alpha_1/k}^{\beta_1} d\xi_1 (\xi_1 \xi_2' - \xi_2) \left[ a(\xi_1 - \frac{\alpha_1}{k}) + b(\xi_2 - \frac{\alpha_2}{k}) \right] e^{i2\pi k(M\xi_1 + N\xi_2)}
\]

\[
+ \frac{1}{2} \sum_M \int_0^{\infty} dI_1 \delta(\sqrt{H(I_1, \alpha_2) - k}) \eta(I_1, \alpha_2) e^{i2\pi M I_1}
\]

\[
+ \frac{1}{2} \sum_N \int_0^{\infty} dI_2 \delta(\sqrt{H(\alpha_1, I_2) - k}) \eta(\alpha_1, I_2) e^{i2\pi N I_2}
\]

\[
+ \frac{1}{4} \delta(\sqrt{H(\alpha_1, \alpha_2) - k}) \eta(\alpha_1, \alpha_2) \quad (C.4)
\]

\[
\eta(k) = k^2 \sum_{M,N} \int_{\alpha_1/k}^{\beta_1} d\xi_1 (\xi_1 \xi_2' - \xi_2) \left[ a(\xi_1 - \frac{\alpha_1}{k}) + b(\xi_2 - \frac{\alpha_2}{k}) \right] e^{i2\pi k(M\xi_1 + N\xi_2)}
\]

\[
+ \frac{1}{2} ak \sum_M (\beta_1 - \frac{\alpha_2}{k}) (\beta_1 - \frac{\alpha_1}{k}) e^{i2\pi M k \beta_1}
\]

\[
+ \frac{1}{2} bk \sum_N (\beta_2 - \frac{\alpha_1}{k}) (\beta_2 - \frac{\alpha_2}{k}) e^{i2\pi N k \beta_2} \quad (C.5)
\]

In the cases where \( \Gamma \) is not parallel to the axes, we get for \( k \gg 1 \)
\[
\eta(k) = k^2 \sum_{M,N} \int_0^{\beta_1} d\xi_1 (\xi_1 \xi_2' - \xi_2) (a\xi_1 + b\xi_2) e^{i2\pi k(M\xi_1 + N\xi_2)}
\]
Trace formulae for nodal domains at billiard boundaries

\[ + \frac{1}{2} a_1 \beta_1 k \sum_M e^{i2\pi Mk\beta_1} + \frac{1}{2} b_2 \beta_2 k \sum_N e^{i2\pi Nk\beta_2} \]  

(C.6)

integrals evaluated using SPA . . .

\[ \eta(k) = k^2 \alpha_0 + k^{3/2} \sum_{M,N \neq (0,0)} \alpha_{M,N} e^{ik\phi_{M,N}} \]

\[ + \frac{1}{2} a_1 \beta_1 k \sum_M e^{i2\pi Mk\beta_1} + \frac{1}{2} b_2 \beta_2 k \sum_N e^{i2\pi Nk\beta_2} \]  

(C.7)

We see that there is an \(O(k^2)\) smooth part, \(O(k^{3/2})\) oscillatory contributions due to the periodic orbits, plus \(O(k)\) corrections due to orbits which are parallel to one of the coordinates.

Appendix C.1. Spectral inversion \(k(n)\)—older version

For an integrable billiard, the asymptotic expansion of the spectral counting function is of the following form

\[ n(k) = \frac{A}{4\pi} k^2 - \frac{L}{4\pi} k + D_0 \]

\[ + \sqrt{k} \sum_p C_p \sin(kL_p + \phi_p) + \sum_{p'} D_{p'} \sin(kL_{p'} + \phi_{p'}) + o(1), \]  

(C.8)

Where \(p\) goes over periodic tori (continuous families of parabolic periodic orbits), \(L_p\) the length of the orbit and \(C_p\) a parameter which depends on some classical properties of the orbit. Similarly, \(p'\) goes over “special”, isolated orbits, whose length is \(L_{p'}\) and \(D_{p'}\) is a parameter which depends on some classical properties.

If we consider only finite sums over \(p\) and \(p'\) (which can be justified if we choose a proper smoothing function \(\rho\) and consider \(\rho \ast n(k)\) instead of \(n(k)\)), then the \(O(\sqrt{k})\) terms are bounded and the formula can be inverted using standard iterative methods. To simplify notation, we will use \(q = \sqrt{4\pi n/A}\) as the parameter of the inverted function, instead of \(n\). Naive inversion of equation (C.8) (treating the \(p\) and \(p'\) summations as if they were constants) yields

\[ k(q) = q + \frac{L}{2A} + \frac{L^2 - 16\pi AD_0}{8A^2} q^{-1} \]

\[ - \frac{2\pi}{A} q^{-1/2} \sum_p C_p \sin(kL_p + \phi_p) - \frac{2\pi}{A} q^{-1} \sum_{p'} D_{p'} \sin(kL_{p'} + \phi_{p'}) + O(q^{-3/2}). \]  

(C.9)

To express the \(k\) dependant functions in terms of \(q\), we will use the fact that the \(k\) dependant terms in this expansion vanish for high \(q\). Choose \(q\) large enough so that \(MNq^{-1/2} \ll \tau\) where \(\tau = \min_p\{2\pi/L_p\}\) is the smallest period appearing in the sum, \(M\) is an upper bound for \(C_p/A\) and \(N\) is the number of non-negligible terms in the \(p\) summation. Then, denoting \(\tilde{q} = q + \frac{L}{2A}\) we have

\[ |\delta| \equiv |q^{-1/2} \frac{2\pi}{A} \sum_p C_p \sin(kL_p + \phi_p)| \leq 2\pi MNq^{-1/2} \ll \tau. \]
For all \( L_p \) in the sum we have \(|\delta|L_p \ll \tau L_p \leq 2\pi\), so the correction to the argument of the sine \( k L_p = (\tilde{q} - \delta + O(q^{-1}))L_p \sim \tilde{q}L_p \) is small compared to the period, and we can approximate it by the first order expansion in \( \delta \):

\[
\sin(kL_p + \varphi_p) \sim \sin(\tilde{q}L_p + \varphi_p) - q^{-1/2}L_p \frac{2\pi}{A} \sum_{p'} C_{p'} \cos(\tilde{q}L_p + \varphi_p) \sin(\tilde{q}L_{p'} + \phi_{p'}). 
\]

Inserting this into equation (C.9), we get

\[
k(q) = q + \frac{L}{2A} + \frac{L^2 - 16\pi A D_0}{8A^2} q^{-1} - \frac{2\pi}{A} q^{-1/2} \sum_p C_p \sin(\tilde{q}L_p + \phi_p) + \frac{2\pi^2}{A^2} q^{-1} \sum_{i,j} C_i C_j L_j [\sin(\tilde{q}(L_i - L_j) + \phi_i - \phi_j) + \sin(\tilde{q}(L_i + L_j) + \phi_i + \phi_j)] - \frac{2\pi}{A} q^{-1} \sum_{p'} D_{p'} \sin(\tilde{q}L_{p'} + \varphi_{p'}) + O(q^{-3/2}),
\]

(C.10)

Where the indices \( i \) and \( j \) enumerate periodic tori, like the index \( p \). We see that in addition to the isolated orbits, the \( O(q^{-1}) \) term contains contributions from pairs of periodic tori, and the appropriate “lengths” are the sums and the differences of the orbit lengths of the individual tori.

It should be noted that equation (C.10) requires a different kind of smoothing than the “standard” kind of trace formulae. The standard kind of trace formulae, such as equation (C.8), acquires validity when we smooth both sides of the equation using some convolution kernel. This has the effect of suppressing the high oscillations, i.e. effectively multiplying each \( C_p \) by a factor that decays fast enough for large \( L_p \) and ensuring convergence of the right hand side of the equation. However, this method would not work in the present case. In equation (C.10), the \( O(q^{-1}) \) term contains contributions from orbit differences, with oscillation frequency of \(|L_i - L_j|\). Convolving with some function of \( q \) would give a decay factor depending on that frequency. But since there are infinitely many orbit pairs with \(|L_i - L_j|\) in any given interval, the right hand side of (C.10) would not converge. To overcome this problem, one should apply the smoothing on the y-axis of the step function, i.e. convolve the inverse function \( q(k) \) with some function of \( k \). The corresponding right hand side of (C.10) acquires a decay factor for each periodic torus \( p \) depending on its orbit length \( L_p \) (the terms corresponding to pairs of tori acquire two decay factors), and we get an effectively finite number of terms, provided that the factor decays fast enough.
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