Thesis For a Degree “Doctor of Philosophy”

Topics in String Field Theory
and
Super-Virasoro Algebra in Superconformal Mechanics

Submitted by
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September 2003
Acknowledgments

The author wishes to express sincere appreciation to Prof. Jacob Sonnen-schein, and Prof. Shimon Yankielowicz for their guidance and patience which made this thesis possible and to Prof. Yaron Oz and Dr. Nadav Dukker for many fruitful discussions.

It is a pleasure to thank Prof. Alon Faraggi, Prof. Michael Green, Prof. Chris Hull, Dr. Betti Hartmann, Prof. Antal Jevicki, Dr. Daniel Waldram and Prof. Barton Zwiebach for their hospitality during a tour where parts of this thesis were presented, and for their comments. I am grateful to my colleagues and friends, Michael Kroyter and Udi Fuchs, who made this research so stimulating and enjoyable.

Last but not least, I would like to thank my parents, my partner and my daughter for their invaluable support and encouragements.
Abstract

This thesis discusses the structure of bosonic String Field Theory (SFT) in the continues basis, and the conditions on extending the $osp(2|2) \cong su(1,1|1)$ algebra of superconformal mechanics to super-Virasoro algebra.

We find a new subalgebra of SFT star product, which consist of squeezed states whose matrices commute with $(K_1)^2$. This subalgebra contains a large set of projectors. The states are represented by their eigenvalues and we find a mapping between the eigenvalues representation and other known representations.

An analytical expression for the finite part of the spectral density $\rho_{\text{fin}}$ and the form of the matter part of the Virasoro generators $L_n$ in the $\kappa$ basis is given. These generators construct string field theory’s derivation $Q_{BRST}$. We find that the Virasoro generators are given by one dimensional delta functions with complex arguments.

We give the explicit form of the half-string representation in the continuous kappa basis. We show the comma structure of the three-vertex, when expanded around an arbitrary projector, and demonstrate the simplicity of this formalism with some applications, such as gauge transformations and identification of subalgebras.

For $\mathcal{N} = 1,2$ superconformal mechanics in $0+1$ dimensions the invertability of the Hamiltonian is used to construct half of the super Virasoro algebra. The full algebra is derived when the special conformal generator is also invertible. The generators are quantized and a general prescription is given for the construction of the $\mathcal{N} = 1$ algebra independently of the specific details of the superconformal mechanics provided that in addition its quadratic Casimir operator vanishes.
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Chapter 1

Introduction

Present modern physics is well described in the framework of quantum mechanics and general relativity, the greatest scientific achievements of the 20th century. The so-called Standard Model of particle physics is based on two quantum field theories which summarize our knowledge on the fundamental laws of physics. The first is the electroweak theory [1, 2, 3] which describes the electromagnetic and the weak interactions, while the second is Quantum Chromo Dynamics (QCD) which describes strong interactions (see e.g [4]). For large scales, from the stars up to the universe, we have general relativity [5, 6] which describes the classical physics of gravitation. Though both theories have been tested experimentally to a very high accuracy it is their theoretical frameworks that are incompatible.

The unsuccessful attempts to reconcile quantum field theory with general relativity, has lead to the recognition that what was previously taken for granted, should be reconsidered. One of the fundamental concepts that has been questioned was that of 3 + 1 dimensional space-time populated with a variety of fundamental point-like particles. Abandoning four dimensions has opened exciting possibilities of higher dimensional space-time hosting a variety of extended objects such as strings. In string theory elementary particles are taken as different modes of a vibrating string whose typical size is of the order of the Plank length (about $10^{-33}$ cm), i.e for all foreseen experiments these objects will look like point-like particles. However, once we give up the notion of point like particles in favor of extended objects we immediately face revolutionary consequences.
<table>
<thead>
<tr>
<th>Name</th>
<th># of Q’s</th>
<th>Bosonic spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type IIA</td>
<td>32</td>
<td>NS-NS $g_{µν}$, $b_{µν}$, $\phi$, $C_µ$, $C_{µνρ}$</td>
</tr>
<tr>
<td>Type IIB</td>
<td>32</td>
<td>NS-NS $g_{µν}$, $b_{µν}$, $\phi$, $C_{µνρσ}$, $C_{µν}$</td>
</tr>
<tr>
<td>Type I $SO(32)$</td>
<td>16</td>
<td>NS-NS $g_{µν}$, $C_{µν}$</td>
</tr>
<tr>
<td>Heterotic $E_8 \times E_8$</td>
<td>16</td>
<td>$g_{µν}$, $b_{µν}$, $\phi$, $A^a_µ$</td>
</tr>
<tr>
<td>Heterotic $SO(32)$</td>
<td>16</td>
<td>$g_{µν}$, $b_{µν}$, $\phi$, $A^a_µ$</td>
</tr>
</tbody>
</table>

Table 1.1: List of the 5 superstring theories. The column of # of Q's denote the number of supersymmetric charges.

- A consistent quantum string theory is most likely UV divergent free.

- The spectrum of the theory contains massless particles, among them is a spin two state with properties of the graviton.

  - String theory is consistent only in ten dimensional space-time \(^1\) and not in four as we experience.

  - There is no unique string theory but five different types, each with a large vacuum degeneracy, as is elaborated in table 1.

The first two corollaries are very encouraging. They show that string theory might be a finite quantum theory of gravity. The third and fourth problematic corollaries are resolved, to a certain extent, using the powerful ideas of compactification and duality.

The idea of compactification dates back to the work of Kaluza and Klein almost 80 years ago (see e.g [7]). It is based on the notion that in a higher dimensional world some of the dimensions are compact, i.e not infinitely long, and are too small for us to observe. This is analogous to a two dimensional
tube with a very small microscopic radius which appears as a one dimensional line to a macroscopic observer, a tube to a mesoscopic observer and a flat two dimensional space to a microscopic observer. In string theory we need to compactify six dimensions out of the ten in order to obtain a realistic four dimensional theory. There are many ways to introduce such compactifications but none of them is more fundamental then the other.

While compactification is a well defined procedure, duality symmetries in general are stated as conjectures. Since these conjectures states that a system has several descriptions in terms of different degrees of freedom, it is clear that proving it will require a full understanding of the system beyond perturbation theory. Nevertheless dualities conjectures pass several non trivial tests that lead us to hope that uniqueness of string theory might be achieved through dualities relations. The degeneracy of the string vacuum is also expected to be lifted by non-perturbative effects, which are accessible through strong-weak dualities.

We give here a brief account on some of the main dualities considered in string theory based on [8, 9, 10]. The class of $T$-dualities describes perturbative dualities which consist of two principle subclasses. The first, as in the case of toroidal compactification, relates different points in the space of the parameters of the compactification to equivalent theories, by discrete symmetry. The other subclass relates two different theories as different descriptions of the same theory in different regions of the moduli space. For example the $E_8 \times E_8$ heterotic string compactified on $S^1$ and the $SO(32)$ heterotic string compactified on $S^1$ describes the same theory. In contrast to the perturbative $T$ dualities the $S$ dualities refer to the equivalence of two perturbatively different theories. These dualities map the strong coupling limit of one theory into the weak coupling limit of the other and vice versa. The duality between type I and $SO(32)$ heterotic string theories is a good example of $S$ duality. The relation between their dilatons is $\phi_I = -\phi_H$ which means in terms of the string coupling constant $g = e^{\phi}$ a strong-weak duality. This duality conjecture relies on the fact that both theories have the same low energy effective action up to a rescaling of the metric, which is $N = 1$ supergravity coupled to
SO(32) super Yang-Mills. In general the product of the $T$ duality group and $S$ duality group is contained in a bigger group which corresponds to a whole set of transformations on the fields of a string theory. This leads one to believe that they are part of a new kind of duality called $U$ duality.

In recent years the picture of string theory as a perturbative theory of strings has gone through a dramatic change. Higher dimensional objects called $p$-branes associated with non-perturbative stringy states were discovered (see e.g. [11, 12]). The $p$-branes are extended objects of $p + 1$ dimensions, which fit naturally to string theory as they define boundary conditions to the open string end points. This beautiful idea introduces a great deal of rich structure to the two previous ideas of compactifications and dualities since now the $p$-branes can curl around the compact dimensions and are mapped into different $p'$-branes under duality.

Out of this enormous corpus of knowledge of dualities, compactification and extended objects emerged a new idea of unification. The five superstring theories together with eleven dimensional $M$ theory $^2$ are seen as different limits of one unifying $U$-theory.

The plan of this thesis is as follows. In chapter 2 we briefly review the bosonic string in a flat background and fix our notations. Next we give an introduction to Witten’s cubic string field theory [13] and explain the renewed interest it has attracted, due to Sen’s conjecture on tachyon condensation [14].

In chapter 3 we find a new subalgebra of the star product in the matter sector. Its elements are squeezed states whose matrices commute with $(K_1)^2$. This subalgebra contains a large set of projectors. The states are represented by their eigenvalues and we find a mapping between the eigenvalues representation and other known representations. The silver is naturally in this subalgebra. Surprisingly, all the generalized butterfly states are also in this subalgebra, enabling us to analyze their spectrum.

In chapter 4 we derive two important tools for working in the $\kappa$ basis of string field theory. First we give an analytical expression for the finite part of the spectral density $\rho_{\text{fin}}$. This expression is relevant when both matter and ghost sectors are considered. Then we calculate the form of the matter part of
the Virasoro generators $L_n$ in the $\kappa$ basis, which construct string field theory’s derivation $Q_{BRST}$. We find that the Virasoro generators are given by one dimensional delta functions with complex arguments.

In chapter 5 we give the explicit form of the half-string representation in the continuous $\kappa$ basis. We show the comma structure of the three-vertex, when expanded around an arbitrary projector. We add the zero-mode and show that in the half-string representation it must be replaced by the mid-point degree of freedom. Adding the ghost sector to the three-vertex in the half-string representation is then straightforward. We demonstrate the simplicity of this formalism with some applications, such as gauge transformations and identification of subalgebras.

In the last chapter 6 we consider $\mathcal{N} = 1, 2$ superconformal mechanics in $0+1$ dimensions and show that if the Hamiltonian is invertible the superconformal generators can be used to construct half of the super Virasoro algebra. The full algebra can be derived if the special conformal generator is also invertible. The generators are quantized and a general prescription is given for the construction of the $\mathcal{N} = 1$ algebra independently of the specific details of the superconformal mechanics provided that in addition its quadratic Casimir operator vanishes.
Chapter 2

Review of bosonic string field theory

String field theory is a nonlocal theory of strings with an infinite number of fields in space-time. The theory is a nonperturbative off-shell formulation of string theory, it gives a systematic way of constructing perturbative string amplitudes, and is background independent. In recent years it attracted a new interest as a tool to confirm Sen’s conjectures regarding tachyon condensation. In this chapter we briefly review the bosonic string, Witten’s string field theory and we conclude with an outline of Sen’s conjectures.

2.1 The bosonic open string

We begin with a short description of the bosonic open string from the point of view of conformal field theory in two dimensions. Naturally, a fair cover of this subject is beyond the scope of this thesis. Further details on string theory and conformal field theory can be found in textbooks like [15, 16, 17] and in on-line comprehensive reviews like [18, 19, 20].

The classical string is a one dimensional object in space that sweep out a two dimensional world-sheet in space-time. It is parameterized by two coordinates \( \sigma \) and \( \tau \), and its motion in \( D \) dimensions is described by \( D \) functions \( X^\mu(\tau, \sigma) \) that are invariant under reparameterization

\[
X'^\mu(\tau', \sigma') = X^\mu(\tau, \sigma) \quad \mu = 1, 2, ..., D. \tag{2.1.1}
\]

The dynamics of the string is governed by the Polyakov action \( S_P \), which is a functional of the space-time embedding functions \( X^\mu(\tau, \sigma) \) and the world-sheet metric \( g_{ab}(\tau, \sigma) \), where \( a, b = 1, 2 \). Lorentz invariance fixes the space-time dimension to \( D = 26 \) for a flat background.
On a flat background, with a space-time metric \( \eta_{\mu\nu} \) and a constant \( \lambda \), the string world-sheet action is given by

\[
S_P[X, g] = \frac{1}{4\pi\alpha'} \int_{\Sigma} d\tau d\sigma \sqrt{g} g^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \frac{\lambda}{4\pi} \int_{\Sigma} d\tau d\sigma \sqrt{g} R^{(2)}(g) + \frac{\lambda}{2\pi} \int_{\partial\Sigma} ds \kappa ,
\]

(2.1.2)

where \( R^{(2)} \) is the two-dimensional world-sheet curvature, \( k \) is the geodesic curvature of the boundary and \( g \) is the determinant of the metric. The Regge slope parameter \( \alpha' \) is related to the string tension by

\[
T = \frac{1}{2\pi\alpha'} .
\]

(2.1.3)

The second and third terms in the action depends only on the topology of the world-sheet and compute \( \chi \), the Euler number \(^1\) of the world-sheet \( \Sigma \). Their significance is only when string interactions are considered.

The equations of motion for \( X^\mu \) and \( g_{ab} \) are

\[
\Delta X^\mu = \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} g^{ab} \partial_b X^\mu) = 0
\]

\[
T_{ab} \equiv \partial_a X^\mu \partial_b X_\mu - \frac{1}{2} g_{ab} \partial_c X^\mu \partial_c X_\mu = 0
\]

(2.1.4)

where \( T_{ab} \) is the energy-momentum tensor. For open strings, in a D25-brane background, we also specify the following boundary conditions

\[
\partial_n X^\mu |_{\partial\Sigma} = 0 \quad \mu = 0, 1, \ldots, 25
\]

(2.1.5)

where \( \partial_n \) is the normal derivative to the world-sheet boundary.

The action (2.1.2) is invariant under local reparameterization and Weyl transformations. For a flat Minkowski target-space background it also has a global Poincaré invariance. The gauge fixed action, in complex coordinates \( z = -e^{-i\sigma+\tau} \) and \( \bar{z} = -e^{i\sigma+\tau} \), takes the form

\[
S[X, b, c] = \frac{1}{2\pi\alpha'} \int d^2 z \partial_\sigma X^\mu \partial_{\bar{z}} X_\mu + \frac{1}{2\pi} \int d^2 z b_{zz} \partial_\sigma c^\bar{z} \partial_{\bar{z}} c^\bar{z} + \bar{b}_{zz} \partial_\sigma \bar{c} \partial_{\bar{z}} \bar{c}
\]

(2.1.6)

where \( c^\bar{z} \) and \( b_{zz} \) (\( c^\bar{z} \) and \( \bar{b}_{zz} \)) are the reparameterization ghosts (antighosts) fields. For the rest of this section we suppress space-time and conformal tensor
indices. In this condensed notation the matter and ghost energy-momentum
tensors are given by

\[ T^{(m)} = -\frac{1}{\alpha'} : \partial X \partial X : \]
\[ T^{(g)} = : \partial bc : -2\partial : bc : \]

where \( \partial \equiv \partial_z \) and \( : O : \) represent the normal ordering of an operator \( O \).

Beside a residual invariance under conformal transformations, this action is
also invariant under ghost and BRST transformations that are generated by

\[ N^g = \frac{1}{2\pi i} \oint dz : bc : \]
\[ Q_B = \frac{1}{2\pi i} \oint dzcT^{(m)} + \frac{1}{2} : cT^{(g)} : + \frac{3}{2} \partial^2 c \]

The equation of motions for the action (2.1.6) are

\[ \partial \bar{\partial} X = \bar{\partial} c = \bar{\partial} b = \partial c = \partial b = 0 . \]

and the Neumann boundary conditions for the matter and ghosts fields be-
comes

\[ \partial X(z) = \bar{\partial} X(\bar{z}) \]
\[ c(z) = c(\bar{z}) \]
\[ b(z) = \bar{b}(\bar{z}) \]

for \( \text{Im}(z) = 0 \). The matter and ghost fields in the action (2.1.6) can be expanded in modes

\[ X(z, \bar{z}) = x - i\alpha' p \log(z\bar{z}) + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n}{n} (z^{-n} + \bar{z}^{-n}) \]
\[ c(z) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+1}} \]
\[ b(z) = \sum_{n=-\infty}^{\infty} \frac{b_n}{\bar{z}^{n+2}} \]

and it is customary to denote \( \alpha_0 \equiv \sqrt{2\alpha' p} \). The modes obey the canonical
commutation relations

\[ [x, p] = i \]
\[ [\alpha_n, \alpha_m] = n\delta_{n+m,0} \]
\[ \{b_n, c_m\} = \delta_{m+n,0} \]
\[ \{b_n, b_m\} = \{c_n, c_m\} = 0, \]

and define the vacuum state. The matter \( SL(2, \mathbb{R}) \) invariant vacuum with momentum \( k \) is denoted by \( |0; k\rangle \) and is a ghost number zero state. There are several conventions to the ghost vacuum that differ by their ghost number \( N^g \)

\[ c_1 |0\rangle \quad N^g = -\frac{1}{2} \]
\[ c_0 c_1 |0\rangle \quad N^g = \frac{1}{2}, \]

and are motivated by the normalization

\[ \langle 0 | c_{-1} c_0 c_1 |0\rangle = 1. \] (2.1.14)

A general state in the Fock space take the form

\[ \sum_{\{n,m,l\}} \alpha_{-n_1} \cdots \alpha_{-n_i} c_{-m_1} \cdots c_{-m_j} b_{-l_1} \cdots b_{-l_j} |0; k\rangle, \]

where the vacuum \( |0; k\rangle \equiv |0; k\rangle_m \otimes |0\rangle_g \) is annihilated by

\[ \alpha_l |0; k\rangle = 0 \quad \text{for} \quad l \geq 1 \]
\[ c_m |0; k\rangle = 0 \quad \text{for} \quad m \geq 2 \]
\[ b_n |0; k\rangle = 0 \quad \text{for} \quad n \geq -1. \] (2.1.16)

The energy-momentum tensor of the gauge fixed action is a holomorphic operator in the matter and ghost sectors. Thus we can make the Laurent expansion

\[ T^{(m)}(z) = \sum_{n=-\infty}^{\infty} \frac{L_n^{(m)}}{z^{n+2}} \quad T^{(g)}(z) = \sum_{n=-\infty}^{\infty} \frac{L_n^{(g)}}{z^{n+2}}, \]

where \( L_n^{(m)} \) and \( L_n^{(g)} \) are known as the Virasoro generators. These generators obey the Virasoro algebra

\[ [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \] (2.1.18)
where $c$ is the central charge. For the matter sector $c^m = 1$, and for the ghost sector $c^g = -26$. The mode realization of the Virasoro generators is given by

$$
L_0^{(m)} = \alpha' p^2 + \sum_{m=0}^{\infty} \alpha_{-m} \alpha_m \\
L_n^{(m)} = \frac{1}{2} \sum_{m=-\infty}^{\infty} : \alpha_{n-m} \alpha_m : \\
L_n^{(g)} = \sum_{m=-\infty}^{\infty} (2n - m) : b_m c_{n-m} : -\delta_{n,0}.
$$

Evaluation of the integrals in eq. (2.1.8) gives the mode realization for the ghost number and BRST charges

$$
N^g = \sum_{n=1}^{\infty} (c_{-n} b_n - b_{-n} c_n) + c_0 b_0 - \frac{1}{2} \\
Q_B = \sum_{n=-\infty}^{\infty} c_n L_{-n}^{(m)} + \sum_{n,m=-\infty}^{\infty} \frac{n-m}{2} : c_n c_m c_{n-m} : -c_0.
$$

The BRST charge $Q_B$ will play a key role in the construction of Witten string field theory in the next chapter.

### 2.2 Witten’s theory

In 1986 Witten proposed a simple action that describes off shell open bosonic string field theory [13]. This action has the structure of a Chern Simons theory though in contrast to the later, the integration in the action is not interpreted as an integration on a three-manifold.

#### 2.2.1 Abstract setting of SFT

In order to ensure a large gauge symmetry, string field theory was first formulated in an axiomatic way. The action for a string field $\Psi$, which takes values in a graded algebra $\mathcal{A}$, is given by

$$
S[\Psi] = \frac{1}{2g^2} \int \Psi \star Q_B \Psi + \frac{2}{3} \Psi \star \Psi \star \Psi,
$$
where the derivation, the integration and the star product are linear and bilinear maps

\[ Q_B : \mathcal{A} \longrightarrow \mathcal{A} \]
\[ \int : \mathcal{A} \longrightarrow \mathbb{C} \]
\[ \ast : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A} . \] (2.2.2)

The set \( Q_B, \ast, \int \) together with a grading \( G \) of the string fields obeys

- **Associativity:** \( (\Phi \ast \Psi) \ast \chi = \Phi \ast (\Psi \ast \chi) \) \( \forall \Psi, \Phi, \chi \in \mathcal{A} \)
- **Derivation:** \( Q_B(\Psi \ast \Phi) = Q_B \Psi \ast \Phi + (-1)^{G \Psi} \Psi \ast Q_B \Phi \) \( \forall \Phi, \Psi \in \mathcal{A} \)
- **Cyclicity:** \( \int \Psi \ast \Phi = (-1)^{G \Psi G \Phi} \int \Phi \ast \Psi \) \( \forall \Psi, \Phi \in \mathcal{A} \)
- **Stokes:** \( \int Q_B \Psi = 0 \) \( \forall \Psi \in \mathcal{A} \)
- **Nilpotency:** \( Q_B^2 \Psi = 0 \) \( \forall \Psi \in \mathcal{A} \)

where the grading obeys

\[ G_{\Psi \ast \Phi} = G_\Psi + G_\Phi + G_\ast \]
\[ G_{Q_B \Psi} = 1 + G_\Psi . \] (2.2.4)

We require for consistency that the two terms in the action have the same grading so that \( 1 = G_\Psi + G_\ast \), and fix the arbitrariness of the grading by setting \( G_\ast = 0 \). This choice imply that the string field \( \Psi \) is a “1-form” with \( G_\psi = 1 \).

This Chern-Simons like setting is enough to grant gauge invariance under the infinitesimal transformation with the zero grading gauge parameter \( \Lambda \)

\[ \delta \Psi = Q_B \Lambda + \Psi \ast \Lambda - \Lambda \ast \Psi \quad \Lambda \in \mathcal{A} . \]

We define the “2-form” field strength \( F \)

\[ F \equiv Q_B \Psi + \Psi \ast \Psi \] (2.2.5)

which transforms homogeneously under the gauge transformation

\[ \delta F = F \ast \Lambda - \Lambda \ast F \] (2.2.6)

and obeys a Bianchi identity

\[ Q_B F + \Psi \ast F - F \ast \Psi = 0 . \] (2.2.7)
The equation of motion that is obtained from the action (2.2.1) is

\[ F = 0. \] (2.2.8)

We end this section by noting the following properties of the string field action that are important for the analysis of Sen’s conjectures.

1) A property of cubic actions that is satisfied by (2.2.1) is that around a classical solution \( \Psi = \Phi_0 + \Phi \)

\[ Q_B \Phi_0 + \Phi_0 \star \Phi_0 = 0 \] (2.2.9)

the action splits into two terms of identical functional form as the original action

\[ S[\Psi, Q_B \Psi] = S[\Phi_0, Q_B \Phi_0] + S[\Psi, \Phi \Phi], \] (2.2.10)

where the new kinetic operator \( Q \)

\[ Q \Phi = Q_B \Phi + \Phi_0 \star \Phi - (-1)^{G*} \Phi \star \Phi_0 \] (2.2.11)

obeys the same axioms as \( Q_B \).

2) Another property of the action is its transformation under field redefinition \( \Psi = e^K \Psi' \)

\[ S[\Psi, Q_B \Psi] = S[\Psi', Q' \Psi'] \] (2.2.12)

where \( K \) is a derivation operator that leave the integration invariant

\[ K(\Psi \star \Phi) = (K \Psi) \star \Phi + \Psi \star (K \Phi) \]

\[ \int K \Psi = 0 \quad \forall \Psi \in \mathcal{A}. \] (2.2.13)

The new action has the same functional form with a kinetic operator that is given by

\[ Q' = e^K Q_B e^{-K} \] (2.2.14)

If the operator \( K \) commute with \( Q_B \) this transformation is a symmetry of the action.
2.2.2 Open strings realization

Witten presented the formal structure of SFT in [13] and showed that the axioms in (2.2.3) are satisfied when the algebra

$$\mathcal{A} = \{\Psi[X(\sigma), c(\sigma), b(\sigma)]\}$$

is taken to be the space of string field functional $\Psi$ over the matter and ghosts fields of the bosonic open string in 26 dimensions, and the derivation operator $Q_B$ is the BRST operator in (2.1.20). The integration is given by sewing together the left and right halves of the string while the star product $\Psi \star \Phi$ is given by sewing the right halve of $\Psi$ to the left halve of $\Phi$. For simplicity we give the realization of these statements only for the matter sector.

The integration is given by the path integral

$$\int \Psi = \lim_{\delta \to 0} \int_{R_\delta} D X(\sigma, \tau) e^{-S[X(\sigma, \tau)]} \Psi[X(\sigma)]$$

$$= \int D X(\sigma) \Psi[X(\sigma)] \prod_{\sigma = \pi \over 2}^{\pi} \delta(X(\sigma) - X(\pi - \sigma)),$$

where the manifold $R_\delta$ is depicted in fig 2.1, and the boundary conditions are

Figure 2.1: The integration world-sheet $R_\delta$. (a) The world-sheet $R_\delta$ is flat except for a curvature delta-function singularity at $M$. (b) A parameterization of $R_\delta$, where the segments on the left and right sides of $M$ are glued.
Figure 2.2: The star multiplication is given by path integral on the region $S$ which is flat except at the point $N$, where the curvature has a delta function singularity.

given by

$$X(\sigma, \tau)|_{\tau=0} = X(\sigma)$$
$$\partial_\sigma X(\sigma, \tau)|_{\sigma=0} = \partial_\sigma X(\sigma, \tau)|_{\sigma=\pi} = 0$$

(2.2.17)

The $\star$-product is obtained by a similar path integral

$$\Psi \star \Phi[Z(\sigma)] = \lim_{\delta \to 0} \int_{S_\delta} D X(\sigma, \tau) e^{-S[X(\sigma, \tau)]} \Psi[X(\sigma)]\Phi[Y(\sigma)]$$

$$= \int \prod_{\sigma=\pi} \prod_{\sigma=\pi/2} d X(\sigma) \prod_{\sigma=0} d Y(\sigma) \Psi[X]\Phi[Y] \cdot \prod_{\sigma=\pi} \delta(X(\sigma) - Y(\pi - \sigma))$$

$$\equiv \int D X_R D Y_L \Psi[Z_L, X_R] \Phi[Y_L, Z_R] \delta(X_R - Y_L)$$

(2.2.18)

where the small world-sheet $S_\delta$ is depicted in fig 2.2 and

$$Z(\sigma) = \begin{cases} 
X(\sigma) & 0 \leq \sigma \leq \frac{\pi}{2} \\
Y(\sigma) & \frac{\pi}{2} \leq \sigma \leq \pi 
\end{cases}$$

(2.2.19)

The boundary conditions are given by

$$X(\sigma, t)|_{AB} = X(\sigma) \quad X(\sigma, t)|_{CD} = Y(\sigma) \quad X(\sigma, t)|_{EF} = Z(\sigma)$$

$$\partial_\sigma X(\sigma, t)|_{BC} = \partial_\sigma X(\sigma, t)|_{DE} = \partial_\sigma X(\sigma, t)|_{FA} = 0$$

(2.2.20)

This realization has also a reparameterization invariance that keep the midpoint fixed:

$$\sigma \longrightarrow f(\sigma)$$
$$f(\pi - \sigma) = \pi - f(\sigma)$$

(2.2.21)
The generators of this symmetry are derivations of the $\ast$-product, and are given by the combination of the Virasoro generators

$$K_n = L_n - (-1)^n L_{-n}.$$ 

Under a general field redefinition $\Psi = e^K \Psi'$, where $K = \sum v_n K_n$, the action transforms according to eq. (2.2.12).

### 2.2.3 Fock space representation

In this subsection we mainly follow the notation of [21, 22] and give a precise meaning to the delta function overlaps that represent integration and star multiplication in the open string realization of the previous section.

We expand $X(\sigma)$ in modes appropriate to open string boundary conditions

$$X(\sigma) = x_0 + \sqrt{2} \sum_{n=1}^{\infty} \frac{1}{n} x_n \cos n\sigma$$

and define creation and annihilation operators

$$a_n = \frac{1}{\sqrt{2n}} (p_n - i nx_n) \quad x_n = \frac{i}{\sqrt{2n}} (a_n - a_n^\dagger)$$

for $n \neq 0$ and

$$a_0 = \frac{1}{2} (p_0 - i 2x_0) \quad x_0 = \frac{i}{2} (a_0 - a_0^\dagger)$$

$$a_0^\dagger = \frac{1}{2} (p_0 + i 2x_0) \quad p_0 = (a_0 + a_0^\dagger)$$

where $p_n = -i \frac{\partial}{\partial x_n}$ is the conjugate momentum operator. The creation and annihilation operators obeys canonical commutation relations $[a_n, a_m^\dagger] = \delta_{nm}$ and define a vacuum $\langle 0 \rangle$. The state

$$|X(\sigma)\rangle = \prod_{n=1}^{\infty} |x_n\rangle = \prod_{n=1}^{\infty} e^{\frac{i}{2} a_n^\dagger a_n} e^{-i \sqrt{2n} a_n^\dagger x_n - \frac{n}{2} x_n x_n} \langle 0 |$$

is therefore an eigenstate of the operator $\hat{X}(\sigma)$ with an eigenvalue $X(\sigma)$. These notations allow us to form an equivalence between functionals and states

$$\Psi[X(\sigma)] \longleftrightarrow |\Psi\rangle = \int \mathcal{D}X \Psi[X(\sigma)] |X(\sigma)\rangle,$$
where the later can be expanded in $L_0$ eigen modes

$$|\Psi\rangle = (t(x) + A_\mu(x) a_\mu + ...) |0\rangle .$$  \hspace{1cm} (2.2.27)

Vertex states are defined in the dual space and corresponds to integration and star multiplication

$$\int \Psi \longrightarrow \langle I | \Psi \rangle$$

$$\int \Psi_1 \star \ldots \star \Psi_N \longrightarrow \langle V_N | \Psi_1 \rangle |\Psi_2\rangle \ldots |\Psi_N\rangle$$

\[\text{e.g}\]

$$|I\rangle = \exp \left[ -\frac{1}{2} (a^\dagger \mid C \mid a^\dagger) \right] |0\rangle$$  \hspace{1cm} (2.2.29)

$$|V_3\rangle = \exp \left[ -\frac{1}{2} (a^\dagger_r \mid V^{rs} \mid a^\dagger_s) \right] |0\rangle$$  \hspace{1cm} (2.2.30)

where $r, s = 1, 2, 3$ and $C$ is the twist matrix

$$C_{nm} = \delta_{nm}(-1)^n$$  \hspace{1cm} (2.2.31)

and the Neumann coefficients $V^{rs} \equiv C M^{rs}$ obeys

$$0 = [M^{rs}, M^{r's'}]$$

$$M^{rr'} = M^{r+1r+1}$$

$$M^{rr+1} = M^{r+1r+2},$$

i.e there are only three independent matrices out of the possible nine of the three-vertex. Hermitian conjugation is defined by BPZ [23] conjugation

$$\langle V_2 | \Psi \rangle \equiv \langle \Psi |_{BPZ} = |\Psi\rangle^\dagger$$

(2.2.33)

which is equivalent to the reality condition:

$$\Psi^*[X(\sigma)] = \Psi[X(\pi - \sigma)]$$

(2.2.34)

and the bilinear form is given by

$$\langle \Psi | \Phi \rangle = \int \Psi \star \Phi.$$  \hspace{1cm} (2.2.35)
2.3 Sen’s conjecture on Tachyon condensation

The new perspectives on the part of dualities and D-branes in string theory, that where obtain in the last decade, had lead Sen \cite{14} to remarkable conjectures on the roles of tachyons in string theory. These conjectures are best studied in the framework of string field theory.

Sen refer to the bosonic open string in 26 dimensions with the tachyonic mode as a theory in an unstable background of a $D_{25}$ brane. In this scenario the following physical processes are synonym: $D_{25}$ brane decay \ tachyon condensate \ tachyon obtain a vev (see e.g \cite{24, 25, 26}). More precisely the conjectures are summarized by the following statements:

- The tachyon potential $V(\phi)$, which is depicted in figure 2.3, has a local minimum which is equal to the $D_{25}$ mass

$$V(\phi_0) = -M_{D_{25}} \tag{2.3.1}$$

- Lower dimensional branes are solitonic solutions of the tachyon field.

- After the $D_{25}$ decay there are no branes on which the open string can end on, i.e we arrive to the close string vacuum.

A straightforward way to analyze the conjectures would be: to find a classical solution $\Phi_0$ for the vacuum, expand the action around the classical solution $\Psi = \Phi_0 + \Phi$, and analyze the spectrum of $\Phi$. The problem, however, is that there is no known closed form for $\Phi_0$. To overcome this problem Rastelli Sen and Zwiebach (RSZ) \cite{27} gave an ansatz for the form of the kinetic operator

$$Q = \sum_{n=0}^{\infty} a_n C_n, \tag{2.3.2}$$

where $C_n = c_n + (-1)^n c_{-n}$. By assumption this operator can be reached from $Q_B$ by the expansion around the vacuum and by field redefinition. This purely ghost $Q$ satisfies the following conditions

- It satisfies the axioms in eq. (2.2.3) which ensure gauge invariance.
Figure 2.3: According to Sen’s conjectures $A$ represent string theory with a tachyon mode in an unstable $D_{25}$ brane background, in contrast to the stable vacuum at $B$ where the tachyon obtain a vacuum expectation value (vev) and the $D$-brane decays.

- It has a trivial cohomology as is required by Sen’s conjecture.

- It is universal in the sense that it doesn’t depend on a specific brane background.

The next assumption of RSZ is that $D-p$ brane solutions of VSFT are given as product of matter and ghost wave-functions. The solutions and the equations of motions factorize according to

$$\Phi = \Phi_g \otimes \Phi_m$$

$$Q\Phi_g = -\Phi_g \ast \Phi_g$$

$$\Phi_m = \Phi_m \ast \Phi_m \quad (2.3.3)$$

We note that projectors are solutions to the equation of motion of the matter sector and that according to RSZ the ghost sector solution $\Phi_g$ is universal.

In the next chapter we will find a new family of projectors, i.e solutions to the matter sector equation, and analyze their spectrum. This family include known solutions as the sliver and the generalized butterfly.
Chapter 3

Squeezed state projectors in SFT

In this chapter we find a new subalgebra of the star product which consist of squeezed states whose matrices commute with \((K_1)^2\). This subalgebra contains a large set of projectors. The states are represented by their eigenvalues and we find a mapping between the eigenvalues representation and other known representations.

3.1 Introduction and summary

Cubic string field theory has attracted renewed interest, mainly due to Sen’s conjecture on tachyon condensation. For bosonic string theory the conjecture is that the open string tachyon potential has a minimum that the tachyon condenses to. The value of the potential at this minimum is exactly equal to the tension of a \(D\)-brane. The tachyon potential was calculated numerically in cubic open string field theory using the level truncation scheme [28, 29, 30, 31, 32]. These calculations agree very well with Sen’s conjecture.

Kostelecky and Potting attempted in [33] to find the vacuum of string field theory analytically. The basis for their solution in the matter sector was a squeezed state

\[
|\mathcal{S}\rangle = \exp \left( -\frac{1}{2} \sum_{n,m=1}^{\infty} a_{n}^{\dagger} S_{nm} a_{m}^{\dagger} \right) |0\rangle .
\]  

(3.1.1)

that solves the projection equation

\[
\mathcal{S} = \mathcal{S} \star \mathcal{S} .
\]

(3.1.2)

They made one more simplifying ansatz on the matrix \(T = CS\)

\[
[T, V^{rs}] = 0 , \quad r,s=1...3 ,
\]

(3.1.3)
where $C$ is the twist matrix in eq. (2.2.31) and $V^{rs}$ are the three-vertex matrices [21]. These assumptions allow for only two solutions. One trivial solution which is the identity state $S = C$ in eq. (2.2.29), and one non-trivial solution. A little earlier the subalgebra of wedge states was found [34]. Two of those states are projectors: the $360^\circ$ wedge, which corresponds to the identity state and the infinitely thin wedge called the sliver [35], which corresponds to the non-trivial solution of [33].

It seems surprising that the first two non-trivial projection states found, turned out to be the same state, especially due to the fact that in [34] CFT techniques were used while [33] used oscillators. The reason oscillator based calculations gave a wedge state is that the three-vertex matrices are related to the $120^\circ$ wedge state and all wedge state matrices commute. Therefore the ansatz (3.1.3) restricts the solutions to wedge state solutions.

Projectors are even more relevant in vacuum string field theory (VSFT), which is a formulation of SFT around the tachyon vacuum [27, 36, 37, 38]. In this formulation the kinetic operator $Q$ is purely ghost, meaning that the equation of motion for the matter sector is simply the projection equation (3.1.2). Therefore, projectors are solitonic solutions of VSFT associated with $D$-branes. The expectation that there will be only one type of $D25$-brane does not agree with the infinite number of spatially independent rank-one projectors. This suggests that all these projectors are related by a gauge transformation.

Rastelli Sen and Zwiebach found the spectrum of the three-vertex and wedge states matrices [39]. First the eigenvalues and eigenvectors of the matrix $K_1$ were calculated, where the matrix $K_1$ is defined as the action of the star algebra derivation $K_1 = L_1 + L_{-1}$ in the oscillator basis. The eigenvalues of $K_1$ are continuous in the range $-\infty < \kappa < \infty$ and are non-degenerate. $K_1$ also satisfies the commutation relations

\[
[M^{rs}, K_1] = [T_N, K_1] = 0, \quad r, s = 1, \ldots, 3, \quad N = 1, \ldots, \infty, \quad (3.1.4)
\]

where $V^{rs} = CM^{rs}$ are the three-vertex matrices and $V_N = CT_N$ are the wedge state matrices. The eigenvectors of $K_1$ are the eigenvectors of $M^{rs}, T_N$ because $K_1$ is non-degenerate and eq. (3.1.4). The spectroscopy results simplified many
elaborate computations. It was used in [40] to formulate the continuous Moyal representation of the star algebra, in [41] to study the gauge transformation of the vector state in VSFT, in [42] for an analytical calculation of tensions ratio, and in [43] for proving the equivalence of two definitions of $Q$.

The spectroscopy simplifies the calculations of [33] as well. We can work in the $K_1$ basis where the $M^{rs}$ matrices are diagonal, and require that $T = CS$ should also commute with $K_1$

$$[T, K_1] = 0. \tag{3.1.5}$$

Instead of equations involving infinite matrices, we get scalar equations for each eigenvalue $\kappa$. The condition (3.1.5) is exactly analog to the ansatz (3.1.3). Thus, repeating the calculations of [33], using the spectroscopy results, gives the same solutions.

The placement of the factors of $C$ in the above commutation relations is very rigid due to the fact that $C$ does not commute with $K_1$. The main idea of this chapter is to rely on the commutation relation

$$[C, (K_1)^2] = 0. \tag{3.1.6}$$

This is a result of the double degeneracy of $(K_1)^2$ where each eigenvalue $\kappa^2$ has two eigenvectors $v^{(\pm \kappa)}$. We solve the equations of [33] using a weaker ansatz

$$[S, (K_1)^2] = 0. \tag{3.1.7}$$

States that satisfy this ansatz form a subalgebra $\mathcal{H}_{\kappa^2}$ of the star product and are represented in the $K_1$ basis by a block diagonal matrix where each block is a two by two matrix. The advantage of the weaker ansatz is that now we get a much larger set of solutions. The calculations and the solutions are presented in section 3.2.

Identifying known projectors that satisfy the weaker ansatz (3.1.7) requires translating our solutions into more familiar representations. This is done in section 3.3. It turns out that the entire family of generalized butterfly states [38, 44, 45] is in the $\mathcal{H}_{\kappa^2}$ subalgebra and we find their spectra (3.3.36). The
fact that the butterfly state

\[ \exp\left(-\frac{1}{2}L_{-2}\right)|0\rangle = \exp\left(-\frac{1}{2}a_n^+V_{nm}B a_m^+\right)|0\rangle, \tag{3.1.8} \]

satisfies our ansatz, meaning \([V^B,(K_1)^2]=0\), was somewhat unexpected. It seems that other projectors with a simple Virasoro structure, do not satisfy our ansatz.

Not all our solutions correspond to surface states. One interesting such solution is the dual of the nothing state

\[ |S\rangle = \exp\left(-\frac{1}{2}a_n^+(-I_{nm})a_m^+\right). \tag{3.1.9} \]

It looks like the nothing state only with an opposite sign in the exponent. The nothing state describes a configuration of a string with an \(x_n\) independent wave function of the form \(\prod_{n=1}^{\infty} \delta(p_n)\), where \(p_n\) are the conjugate momenta. The dual of the nothing is \(\prod_{n=1}^{\infty} \delta(x_n)\).

In this chapter states are written up to normalization. The singular normalization of string field states is supposed to be corrected by the ghost sector, which is not treated here. We also ignored the zero modes, assuming that the string field is independent of them, i.e our solutions corresponds to a \(D25\)-brane. The space-time index \(\mu=0\ldots25\), is also suppressed. One can use the spectroscopy of the matter sector including the zero modes, and that of the ghost sector [46, 47, 48, 49], to generalize this work.

### 3.2 Projectors in the \(\mathcal{H}_{\kappa^2}\) subalgebra

Squeezed states whose matrices commute with \((K_1)^2\) form a subalgebra of the star product, which we denote \(\mathcal{H}_{\kappa^2}\). To prove this statement we write the expression of the star product of two squeezed states, \(|S_3\rangle = |S_1\rangle \star |S_2\rangle\) using [33]

\[
CS_3C = V_3^{11} + \left(V_3^{12}, V_3^{21}\right) \cdot \begin{pmatrix}
1 - S_1V_3^{11} & -S_1V_3^{12} \\
-S_2V_3^{21} & 1 - S_2V_3^{11}
\end{pmatrix}^{-1} \cdot \begin{pmatrix}
S_1V_3^{21} \\
S_2V_3^{12}
\end{pmatrix}, \tag{3.2.1}
\]
where $V_{3}^{rs}$ are the three-vertex matrices. In [39] it was shown that the matrices

$$M^{rs} = CV_{3}^{rs}$$

obey

$$[M^{rs}, K_1] = 0,$$  \hspace{1cm} (3.2.2)

The fact that $[C, (K_1)^2] = 0$ completes the proof, since $S_3$ is a function of matrices that commute with $(K_1)^2$, and therefore $[S_3, (K_1)^2] = 0$.

Eq. (3.2.2) and the nondegeneracy of $K_1$ implies that the $M^{rs}$ matrices are diagonal in the $K_1$ basis. In [39] their eigenvalues were found

$$\mu(k) \equiv \mu^{11}(k) = -\frac{1}{1 + 2 \cosh(k\pi/2)},$$

$$\mu^{12}(k) = \frac{1 + \exp(k\pi/2)}{1 + 2 \cosh(k\pi/2)},$$

$$\mu^{21}(k) = \mu^{12}(-k).$$  \hspace{1cm} (3.2.3)

To find squeezed state projectors one has to solve eq. (3.2.1), setting $S_1 = S_2 = S_3$. In the $H_{k^2}$ subalgebra the projector condition becomes a set of equations consisting of one scalar equation for $k = 0$ and one $2 \times 2$ matrix equation for each $k > 0$. We shall now solve this set of equations for all $k$ to find the condition for a state in the $H_{k^2}$ subalgebra to be a projector.

### 3.2.1 The $k = 0$ subspace

For the $k = 0$ eigenvalue there is a single normalizable eigenvector of $(K_1)^2$. We use the fact that it is twist odd to set $V_3^{rs} = -\mu^{rs}(0)$ in (3.2.1) and get

$$s_0 = \frac{1}{3} + \left(\frac{2}{3}, -\frac{2}{3}\right) \left(1 - \frac{1}{3}s_0 \begin{pmatrix} \frac{2}{3}s_0 & \frac{2}{3}s_0 \\ \frac{2}{3}s_0 & 1 - \frac{1}{3}s_0 \end{pmatrix}^{-1} \begin{pmatrix} s_0 & 0 \\ 0 & s_0 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} \right).$$  \hspace{1cm} (3.2.4)

The matrix that has to be inverted has an inverse for $s_0 \neq 1, -3$. The condition $s_0 \neq 1$ is related to the fact that $S$ defines a Bogoliubov transformation, and thus the eigenvalues of $S^\dagger S$ should obey

$$\lambda_{S^\dagger S} < 1.$$  \hspace{1cm} (3.2.5)

We know, however, that projectors of SFT are singular. When the matrix is inverted eq. (3.2.4) gives

$$s_0 = \pm 1,$$  \hspace{1cm} (3.2.6)
We see that although singular, \( s_0 = 1 \) is a solution to the equation. Indeed this is the solution for all surface state projectors with \( f(\pm i) = \infty \), where \( f(z) \) is the canonical transformation defining the state, as was shown in [45]. The \( s_0 = -1 \) solution can represent either a non-surface-state projector, or a surface state for which \( f(\pm i) \neq \infty \), as in the case of the star-algebra identity \( S = C \), for which

\[
f(z) = \frac{z}{1 - z^2}.
\]  

3.2.2 The \( \kappa \neq 0 \) subspace

The eigenvalue \( \kappa^2 \) of \((K_1)^2\) is double degenerate for \( \kappa \neq 0 \) with the eigenvectors \( v^{(\pm \kappa)} \). To solve the projection equation for the \( \pm \kappa \) pairs we need to work in the two dimensional subspace spanned by

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv v^{(-\kappa)} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv v^{(\kappa)}.
\]  

We avoid a double counting by taking only \( \kappa > 0 \). In this subspace the entities of eq. (3.2.1) are two by two matrices. The action of the \( C \) matrix in this subspace is

\[
C_\kappa = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},
\]  

since the vectors \( v_\pm^{(\kappa)} = \frac{1}{2} (v^{(-\kappa)} \mp v^{(\kappa)}) \) [39] are eigenvectors of \( C \) with eigenvalues \( \pm 1 \). Using eq. (3.2.3) we can now write the 3-vertex in this subspace

\[
V_{3}^{11} = \frac{1}{1 + 2 \cosh(\kappa \pi/2)} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},
\]

\[
V_{3}^{12} = \frac{-1}{1 + 2 \cosh(\kappa \pi/2)} \begin{pmatrix} 0 & 1 + \exp(\kappa \pi/2) \\ 1 + \exp(-\kappa \pi/2) & 0 \end{pmatrix},
\]

\[
V_{3}^{21} = (V_{3}^{12})^T.
\]

In this subspace a state in the \( \mathcal{H}_{\kappa^2} \) subalgebra is given by

\[
S_\kappa = \begin{pmatrix} s_1(\kappa) & s_2(\kappa) \\ s_2(\kappa) & s_3(\kappa) \end{pmatrix},
\]
which is symmetric since it represents a quadratic form. We impose the BPZ reality condition eq. (2.2.33) on the complex functions \( s_1, s_2, s_3 \). For squeezed states the reality condition translates into the condition

\[
CSC = S^\dagger,
\]

and for the \( \mathcal{H}_{\kappa^2} \) subalgebra we get the conditions

\[
\begin{align*}
CSC = S^\dagger, \\
(\begin{array}{cc}
  s_3 & s_2 \\
  s_2 & s_1 
\end{array}) = (\begin{array}{cc}
  s_1^* & s_3^* \\
  s_2^* & s_3^* 
\end{array}) \Rightarrow \begin{cases} 
  s_3 = s_1^* \\
  s_2 = s_2^* 
\end{cases}.
\]

Solving the projection equation we get two valid solutions (before imposing any reality condition)

\[
\begin{align*}
s_1 &= s_3 = 0, \quad s_2 = -1. \\
s_1s_3 &= s_2^2 - 2\cosh\left(\frac{\kappa\pi}{2}\right) s_2 + 1.
\end{align*}
\]

The first solution is \( S_\kappa = C_\kappa \). If we take this solution for all values of \( \kappa \neq 0 \) and combine it with the \( s_0 = -1 \) solution of \( \kappa = 0 \), we get \( S = C \), which is the star-algebra identity (3.2.7). The second solution is actually a two-parameter family of solutions. We parameterize the solutions using the two invariants

\[
\begin{align*}
u &\equiv -\det S_\kappa = 2\cosh\left(\frac{\kappa\pi}{2}\right) s_2 - 1, \\
u &\equiv \frac{\text{tr} S_\kappa}{2} = s_1 + s_3
\end{align*}
\]

with the inverse relations

\[
\begin{align*}
s_2 &= \frac{v + 1}{2\cosh\left(\frac{\kappa\pi}{2}\right)}, \\
s_{1,3} &= u \pm i\sqrt{\left(\frac{v + 1}{2\cosh\left(\frac{\kappa\pi}{2}\right)}\right)^2 - v - u^2},
\end{align*}
\]

where \( s_1 \) gets the plus sign and \( s_3 \) gets the minus sign, or vice versa. Every triplet \( s_{1,2,3} \) that comes from a choice of \( u, v \) is a solution of the projection
Figure 3.1: A projector is a curve in the $\kappa, u, v$ space. This curve can be described as a function from $\kappa > 0$ to a range in the $u, v$ plane which is defined by eq. (3.2.21). Notice that this range is $\kappa$ dependent, it is colored orange for $\kappa = 0.5$. In the limit $\kappa \to 0$, the allowed range is the whole triangle. But the requirement for continuity is that for $\kappa = 0$ the projector should end on the right edge or the left vertex of the triangle (colored green).

equation, though not all the solutions are legitimate. For BPZ states the restriction (3.2.5) becomes

$$\lambda_{S_\kappa S_\kappa} = (s_2 \pm |s_1|)^2 \leq 1,$$

(3.2.18)

where as in the case of $\kappa = 0$ we also consider singular transformations.

The BPZ reality condition implies that $u, v$ are real as well as the square root in (3.2.17). The condition for the solution to be twist invariant is

$$[S, C] = 0 \Rightarrow s_1 = s_3.$$  

(3.2.19)

When combined with the BPZ condition (3.2.13), twist invariance enforces the reality of $S_\kappa$. For the $u, v$ coordinates this condition reads

$$u^2 = \left(\frac{v + 1}{2 \cosh(\frac{4\kappa}{2})}\right)^2 - v.$$ 

(3.2.20)
Figure 3.2: Various projectors in the $\kappa, u, v$ space. The depth of the figure on the left parametrizes $\kappa$ with $\kappa = 0$ in the front and $\kappa = 3 \approx \infty$ in the back. The right figure is a projection of the left figure to the $u, v$ plane. The curves on the top surface are twist invariant projectors, they are generalized butterfly states to be described in section 3.3.3 (from top to bottom): (i) The sliver $\alpha = 0$. (ii) The butterfly $\alpha = 1$. (iii) The state $\alpha = 1.5$. (iv) The nothing $\alpha = 2$. The additional curve represents a generic non twist invariant projector.

The normalization requirement (3.2.18) and the BPZ reality condition on $S_\kappa$ in eq. (3.2.13) restricts $u, v$ to the region

$$v \geq -1,$$

$$\left( \frac{v + 1}{2 \cosh(\frac{3\pi}{2})} \right)^2 - v \geq u^2. \tag{3.2.21}$$

These inequalities determine the allowed range for the solutions, as a function of $\kappa$, as shown in figure 3.1. To build a projector one has to choose an allowed value for $(u, v)$ for every value of $\kappa > 0$ as illustrated in figure 3.2. This prescription allows for a large class of projectors that contains the generalized butterfly, including the sliver and the butterfly.

Since the continuous $\kappa$ basis is related to the oscillator basis by integration, $u, v$ should be integrable functions of $\kappa$. Moreover, we should identify functions
which differ on a zero measure set. Further restrictions on the functions \( u \) and \( v \) would be related to the identification of the class of legitimate string fields, which is still unknown. It should be large enough to contain D-branes \([22, 50]\), perturbative states around D-branes \([51]\), as well as closed strings \([52]\). Yet, it should not be too large \([53, 40]\). Continuity of \( u, v \) as a functions of \( \kappa \) restricts the class of string fields. We do not know if it has anything to do with the “correct” choice, but we shall henceforth mention some of its consequences. In fact without this restriction the analysis of \( \kappa = 0 \) in section 3.2.1 is meaningless.

Continuity of \( s_{1,2,3} \) at \( \kappa = 0 \) implies that one of the \( \kappa = 0 \) solutions (3.2.6) is reached in the \( \kappa \to 0 \) limit. The eigenvector \( u^{(\kappa=0)} \) is twist odd[39]. Therefore, to get the relevant solution of \( S_\kappa \) we use the projector on the twist odd eigenvalue

\[
P_\kappa^{(-)} = -\frac{1}{2}(C_\kappa - 1) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},
\]

(3.2.22)

In the limit \( \kappa \to 0 \) we get

\[
S_\kappa^{(-)} = P_\kappa^{(-)} S_\kappa P_\kappa^{(-)} \xrightarrow{\kappa \to 0} \left( u + \frac{v + 1}{2} \right) \cdot P_\kappa^{(-)} = s_0 \cdot P_\kappa^{(-)}
\]

\[
\Rightarrow \begin{cases} 
  v = 1 - 2u & s_0 = 1 \\
  v = -3 - 2u & s_0 = -1
\end{cases}
\]

(3.2.23)

We see that the SFT projectors (3.2.15) which are continuous with respect to \( \kappa \) will end either on the right segment of the figure and have \( s_0 = 1 \), or at the point \( u = v = -1 \), and have \( s_0 = -1 \). The identity (3.2.14) has \( s_0 = -1 \), since \( P_\kappa^{(-)} C_\kappa P_\kappa^{(-)} = -1 \cdot P_\kappa^{(-)} \). All the projectors in (3.2.15) are of rank one. This follows from the fact that a projector is given by a trajectory in the \( \kappa, u, v \) space. The rank can be given by a continuous function of this trajectory, all the trajectories are homotopic, and the previously known ones among them are of rank one. This observation is in accordance with the proof of [54] that all gaussian projectors apart of the identity are of rank one. Of course both our argument, and that of [54] may fail if the given projector is too singular. Another property of rank one projectors is their factorization to functions of the left and right part of the string. We show that the projectors in (3.2.15) factorize in 5.
3.2.3 The sliver, the butterfly and the nothing

In order to see how projectors are represented in the phase space that is depicted in figure 3.1 we look at the following examples. Among our solutions we can recognize known projectors as the sliver, the butterfly and the nothing. We shall see in section 3.3.3 that in fact all the generalized butterfly states are in $\mathcal{H}_{\kappa^2}$.

The spectrum of the sliver was given in [39]

$$\tau(\kappa) = -e^{-|\kappa|^{\pi}}, \quad (3.2.24)$$

or in our notation

$$T_\kappa = C_\kappa S_\kappa = -\exp\left(-\frac{\kappa \pi}{2}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow S_\kappa = \exp\left(-\frac{\kappa \pi}{2}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.2.25)$$

Using eq. (3.2.16) we get for the sliver $u = 0, v = \exp(-\kappa \pi)$. The butterfly’s matrix is given by

$$S_\kappa = \frac{1}{2 \cosh(\frac{\kappa \pi}{2})} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (3.2.26)$$

as we will show in eq. (3.3.36) and therefore, for the butterfly $u = \frac{1}{2 \cosh(\frac{\kappa \pi}{2})}, v = 0$. The nothing is the squeezed state defined by the identity matrix, $S = I$, therefore

$$S_\kappa = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.2.27)$$

that is $u = 1, v = -1$.

3.3 Relations to other representations

Now that we have the form of the $\mathcal{H}_{\kappa^2}$ projectors in the $K_1$ basis we would like to represent them in other forms. We start by transforming the states in $\mathcal{H}_{\kappa^2}$ to the oscillator basis. Then we derive a procedure to find the conformal map of squeezed states. Only surface state can pass this procedure, giving us a way to recognize which squeezed states are surface states. We demonstrate this procedure by some examples.
We also derive an inverse procedure for checking if a state is in $\mathcal{H}_{\kappa^2}$ and if so find the matrix $S_{\kappa}$ representing it. Finding the spectrum given $S_{\kappa}$ is straightforward. We use this procedure to show that the generalized butterfly states are in $\mathcal{H}_{\kappa^2}$ and to find their spectrum. Then we show that not all the $L_{-2m}$ projectors are in $\mathcal{H}_{\kappa^2}$.

We also compare the structure of the states in $\mathcal{H}_{\kappa^2}$ to their form in the half-string formalism. Transforming $S_{\kappa}$ to the continuous Moyal representation is straightforward and useful.

### 3.3.1 The oscillator basis

To calculate the matrices in the oscillator basis of states in $\mathcal{H}_{\kappa^2}$ we use the orthogonality and completeness of the $\kappa$ basis [43]. The basis vectors are defined by

$$|\kappa\rangle = \sum_{n=1}^{\infty} v^{(\kappa)}_n |n\rangle ,$$

(3.3.1)

where

$$(n| = |n\rangle_T = (0, 0, \ldots, 0, 1, 0, 0, \ldots) ,$$

(3.3.2)

with the 1 in the $n^{th}$ position of the vector. The coefficients

$$v^{(\kappa)}_n = (\kappa|n)$$

(3.3.3)

are given by the generating function

$$\sum_{n=1}^{\infty} \frac{v^{(\kappa)}_n}{\sqrt{n}} z^n = \frac{1}{\kappa}(1 - e^{-\kappa \tan^{-1}(z)}) \equiv f_{\kappa}(z) .$$

(3.3.4)

By inverting this relation we obtain

$$v^{(\kappa)}_n = \frac{\sqrt{n}}{2\pi i} \int \frac{f_{\kappa}(z)}{z^{n+1}} dz = \frac{1}{\sqrt{n}} \frac{1}{2\pi i} \int \frac{dz}{z^n} \partial_z f_{\kappa}(z)$$

$$= i^{n-1} \frac{\sqrt{n}}{2\pi} \mathcal{N}(\kappa) \int_{-\infty}^{\infty} \frac{e^{iku} \tanh^{n-1}(u)}{\cosh^2(u)} du .$$

(3.3.5)

In the first step the contour is around the origin, while in the second line the contour is deformed to the real axe, see [55, 56]. The orthogonality and
completeness relations in the $\kappa$ basis are

$$
(\kappa|\kappa') = \mathcal{N}(\kappa)\delta(\kappa - \kappa'),
$$

$$
1 = \int_{-\infty}^{\infty} d\kappa \frac{|\kappa\rangle \langle \kappa|}{\mathcal{N}(\kappa)},
$$

(3.3.6)

where

$$
\mathcal{N}(\kappa) = \frac{2}{\kappa} \sinh \left( \frac{\kappa\pi}{2} \right),
$$

(3.3.7)

is a normalization factor.

The matrix elements of a squeezed state can be written using a two-parameter generating function

$$
S_{nm} = \frac{1}{\sqrt{nm}} \oint dz dw \frac{1}{(2\pi i)^2} z^n w^m S(z, w),
$$

(3.3.8)

with the inverse relation

$$
S(z, w) = \sum_{m,n=1}^{\infty} \sqrt{nm} S_{nm} z^{n-1} w^{m-1}.
$$

(3.3.9)

The symmetry $S_{nm} = S_{mn}$ translates to the symmetry $S(z, w) = S(w, z)$.

We can now calculate the matrix elements of a state in the $\mathcal{H}_{\kappa^2}$ subalgebra. That is, given $S_\kappa$ for $\kappa > 0$ the matrix elements are

$$
S_{nm} = (n|S|m) = \int_{-\infty}^{\infty} \frac{dk dk'}{\mathcal{N}(\kappa)\mathcal{N}(\kappa')} (n|\kappa \rangle \langle \kappa|S|\kappa'\rangle \langle \kappa'|m\rangle =

\int_{0}^{\infty} \frac{dk dk'}{\mathcal{N}(\kappa)\mathcal{N}(\kappa')} \left( (n|\kappa\rangle \langle \kappa| \right) \left( (-\kappa|S|\kappa') \langle \kappa'| \right) \left( (\kappa|S|\kappa') \langle \kappa' \right) \left( (\kappa'|m) \right) =

\int_{0}^{\infty} \frac{dk}{\mathcal{N}(\kappa)} \left( (n|\kappa) \langle \kappa | \right) \left( s_1(\kappa) s_2(\kappa) \right) \left( (\kappa|m) \right) =

\frac{1}{\sqrt{nm}} \oint dz dw \frac{1}{(2\pi i)^2} z^n w^m \int_{0}^{\infty} \frac{dk}{\mathcal{N}(\kappa)} \left( \partial_z f_{-\kappa}(z) \partial_z f_{\kappa}(z) \right) S_\kappa \left( \partial_w f_{-\kappa}(w) \right) \left( \partial_w f_{\kappa}(w) \right).
$$

(3.3.10)

Notice that we limited ourselves to the $\mathcal{H}_{\kappa^2}$ subalgebra in passing from the second line to the third line. In the last equality we used (3.3.5). By defining

$$
s_{13}(\kappa) \equiv \begin{cases} 
s_3(\kappa) & \kappa > 0 \\
s_1(-\kappa) & \kappa < 0 
\end{cases},
$$

$$
s_{22}(\kappa) \equiv \begin{cases} 
s_2(\kappa) & \kappa > 0 \\
s_2(-\kappa) & \kappa < 0 
\end{cases},
$$

(3.3.11)
we get

\[ S(z, w) = \int_{-\infty}^{\infty} \frac{d\kappa}{\mathcal{N}(\kappa)} \left( \partial_z f_\kappa(z) \partial_w f_\kappa(w) s_{13}(\kappa) + \partial_z f_\kappa(z) \partial_w f_{-\kappa}(w) s_{22}(\kappa) \right) = \int_{-\infty}^{\infty} \frac{dk}{\mathcal{N}(\kappa)} e^{-\kappa(\tan^{-1}(z)+\tan^{-1}(w))} s_{13}(\kappa) + e^{\kappa(\tan^{-1}(w)-\tan^{-1}(z))} s_{22}(\kappa) \frac{(1 + z^2)(1 + w^2)}{(1 + z^2)(1 + w^2)}. \] (3.3.12)

This equation is what we were after: the oscillator matrix elements as a function of \( S_\kappa \).

### 3.3.2 Surface states

The matrix elements expression (3.3.8) for squeezed states has a similar structure to that of surface states [57, 58, 45]

\[ S_{nm} = -\frac{1}{\sqrt{nm}} \oint \frac{dzdw}{(2\pi i)^2 z^m w^n (f(z) - f(w))^2} f''(-z) f'(-w), \] (3.3.13)

where \( f(z) \) is the conformal transformation that defines the surface state. This raises the question which squeezed states are surface states, and what is their conformal transformation \( f(z) \). Squeezed states are surface states if and only if \( \exists f(z) \) such that

\[ S(z, w) \approx -\frac{f''(-z) f'(-w)}{(f(-z) - f(-w))^2}. \] (3.3.14)

By \( \approx \) we mean “equal up to poles”, since poles do not contribute to the contour integrals in eq. (3.3.13). However \( S(z, w) \) is regular near the origin, and

\[ \frac{f''(-z) f'(-w)}{(f(-z) - f(-w))^2} = \frac{1}{(z - w)^2} + \text{regular terms}. \] (3.3.15)

Therefore, the condition is that there exists \( f(z) \) such that

\[ S(z, w) = -\frac{f''(-z) f'(-w)}{(f(-z) - f(-w))^2} + \frac{1}{(z - w)^2}. \] (3.3.16)

If this is the case, then in particular

\[ S(z, 0) = -\frac{f''(-z)}{f'(-z)^2} + \frac{1}{z^2}, \] (3.3.17)
where we used $f(0) = 0, f'(0) = 1$ which is possible due to $SL(2, \mathbb{C})$ invariance. The solution of this equation gives us a candidate for $f(z)$

$$f^c(z) = \frac{z}{1 - z \int_0^{-z} S(\bar{z}, 0) d\bar{z}}. \quad (3.3.18)$$

Given $S(z, w)$, one can solve eq. (3.3.18) to get $f^c(z)$, then substitute the solution in eq. (3.3.16) and check if it reproduces $S(z, w)$. A squeezed state is a surface state if and only if $f^c(z)$ reproduces $S(z, w)$. We now turn to some examples.

**Example 1. Reconstructing the butterfly**

To evaluate the generating function (3.3.12) of the butterfly (3.2.26) we use the relation

$$\int_{-\infty}^{\infty} \frac{dk}{k} \frac{e^{ik}}{2 \sinh \left( \frac{\pi k}{2} \right) 2 \cosh \left( \frac{\pi k}{2} \right)} = \frac{1}{4 \cos(\frac{\pi}{2})^2}. \quad (3.3.19)$$

Replacing $c$ by $-\tan^{-1}(z) \pm \tan^{-1}(w)$ we obtain

$$S(z, w) = \frac{w^2 + z^2 - \frac{w^2 + z^2 + 2w^2z^2}{\sqrt{(1+w^2)(1+z^2)}}}{(w^2 - z^2)^2}. \quad (3.3.20)$$

By eq. (3.3.18), we get

$$f(z) = \frac{z}{\sqrt{1 + z^2}}, \quad (3.3.21)$$

which is the correct expression for the butterfly. Substituting this expression into (3.3.16) reproduces eq. (3.3.20), as it should.

**Example 2. The duals of the nothing and of the identity states**

We demonstrated in 3.2.3 that the nothing state has $u = 1, v = -1$, that is $S_\kappa = 1$. The dual of the nothing has $S_\kappa = -1$, and is the mirror image of the nothing in the $u, v$ plane, $u = v = -1$. Both projectors live at the boundary of figure 3.1, and saturate the inequality (3.3.16) for all $\kappa$, and are therefore very singular. It can be shown that the mirror images of the other butterfly states do not correspond to continuous projectors.
Equation (3.3.12) now gives
\[ S(z, w) = -\frac{1}{(1 - wz)^2}, \] (3.3.22)
Now we use eq. (3.3.18) to find the candidate \( f(z) \)
\[ f^c(z) = \frac{z}{1 - z^2}. \] (3.3.23)
We recognize \( f^c(z) \) as the conformal transformation of the identity state, which
is represented by \( C \), rather by \(-I\). Indeed, when we substitute (3.3.23) back
into (3.3.16) we get
\[ S^{Id}(z, w) = -\frac{1}{(1 + wz)^2}, \] (3.3.24)
instead of (3.3.22). This proves that the dual of the nothing state is not a
surface state.

The dual of the identity \( S_\kappa = -C_\kappa \) is not a projector. It has
\[ f^c(z) = \frac{z}{1 + z^2}, \] (3.3.25)
which is the conformal map of the nothing, meaning that it is also not a surface
state.

**Example 3. A non-orthogonal projector**

Here we want to give an example of another surface state projector. This
projector has \( v = 0, u = \frac{1}{2} \) for all \( \kappa \). Notice that it does not respect the reality
condition (3.2.13) meaning that it is a non-orthogonal projector as is discussed
in section 3.3.4. In the \( K_1 \) basis the projector is
\[ s_{13}(\kappa) = \frac{1}{2} \left( 1 \pm \tanh \left( \frac{\kappa \pi}{2} \right) \right), \] (3.3.26)
\[ s_{22}(\kappa) = \frac{1}{2 \cosh \left( \frac{\kappa \pi}{2} \right)}. \] (3.3.27)
Note that we are describing two projectors \((\pm)\). Non-orthogonal projectors
come in pairs, because they are asymmetric. Therefore, if we find one projector,
its twisted partner will give another. The conformal transformations of the
projectors are
\[ f(z) = \pm 1 + \frac{z \mp 1}{\sqrt{1 + z^2}}, \] (3.3.28)
We can always build two orthogonal projectors from a non-orthogonal pair by star multiplying them in different orders. The current states are a gluing of the generalized butterfly ($\alpha = \frac{2}{3}$) on one side with the nothing ($\alpha = 2$) on the other [59].

### 3.3.3 The inverse transformation

In the previous subsections we constructed the matrix representation and the conformal map representation for states in the $\mathcal{H}_{\kappa^2}$ subalgebra. It is natural to ask the opposite question. Does a given state belongs to $\mathcal{H}_{\kappa^2}$, and if so, what is its form in the $K_1$ basis. Given its form in the $K_1$ basis we can immediately check if it is a projector (3.2.15).

We can use eq. (3.3.16), or eq. (3.3.9) to define $S(z, w)$ for a surface state that is represented by $f(z)$ or for any other squeezed state that is given by the matrix $S_{nm}$. The question is, what are the conditions for the existence of the functions $s_{13}(\kappa)$ and $s_{22}(\kappa)$ which reproduce $S(z, w)$ via eq. (3.3.12), and what are these functions.

Inspecting eq. (3.3.12), we see that $S(z, w)(1 + z^2)(1 + w^2)$ cannot be an arbitrary function of $z, w$ for states in the $\mathcal{H}_{\kappa^2}$ subalgebra, but should rather be a sum of two terms

$$S(z, w)(1 + z^2)(1 + w^2) = F_1(\xi) + F_2(\zeta), \quad (3.3.29)$$

where we have defined

$$\xi = i(\tan^{-1}(z) + \tan^{-1}(w)),$$
$$\zeta = i(\tan^{-1}(z) - \tan^{-1}(w)). \quad (3.3.30)$$

From eq. (3.3.12) we see that $F_1(\xi)$ is the Fourier transform of $s_{13}(\kappa)/\mathcal{N}(\kappa)$, while $F_2(\zeta)$ is the Fourier transform of $s_{22}(\kappa)/\mathcal{N}(\kappa)$. We conclude that this split to a sum of functions is a necessary condition for the state to be in $\mathcal{H}_{\kappa^2}$ due to the form of eq. (3.3.12), and a sufficient condition since the Fourier transform is invertible.

Suppose now that $F_1$ and $F_2$ are given. The inverse Fourier Transform...
reads
\[ s_{13}(\kappa) = \frac{N(\kappa)}{2\pi} \int_{-\infty}^{\infty} e^{-i\kappa\xi} F_1(\xi) \, d\xi , \]
\[ s_{22}(\kappa) = \frac{N(\kappa)}{2\pi} \int_{-\infty}^{\infty} e^{-i\kappa\zeta} F_2(\zeta) \, d\zeta . \]  
(3.3.31)

These are the desired expressions for inverting the transformation. They apply only to states in \( \mathcal{H}_{\kappa^2} \), meaning, states of the form (3.3.29). We can use eq. (3.3.31,3.2.15) to check if a given state is a projector, as is illustrated below.

**Example 1. The spectrum of the generalized butterfly states**

The generalized butterfly states are a one-parameter family of projectors \([45]\). They are defined by the maps

\[ f(z) = \frac{1}{\alpha} \sin(\alpha \tan^{-1}(z)) , \]  
(3.3.32)

where \( 0 \leq \alpha \leq 2 \). Special cases include the sliver \( \alpha = 0 \), for which

\[ f(z) = \tan^{-1}(z) , \]  
(3.3.33)

the canonical butterfly \( \alpha = 1 \) (3.3.21), and the nothing state \( \alpha = 2 \) with

\[ f(z) = \frac{z}{1 + z^2} . \]  
(3.3.34)

We substitute the map (3.3.32) in eq.(3.3.16), and notice that indeed it splits according to eq. (3.3.29), with

\[ F_1(\xi) = \frac{\alpha^2}{4 \cosh \left( \frac{\alpha \xi}{2} \right)^2} , \]
\[ F_2(\zeta) = \frac{\alpha^2}{4 \sinh \left( \frac{\alpha \zeta}{2} \right)^2} - \frac{1}{\sinh(\zeta)^2} , \]  
(3.3.35)

which means that the generalized butterfly states are in \( \mathcal{H}_{\kappa^2} \). Using eq. (3.3.31) we get

\[ s_1 = s_3 = s_{13} = \frac{\sinh \left( \frac{\kappa \pi}{2} \right)}{\sinh \left( \frac{\kappa \pi}{\alpha} \right)} , \]
\[ s_2 = s_{22} = \cosh \left( \frac{\kappa \pi}{2} \right) - \coth \left( \frac{\kappa \pi}{\alpha} \right) \sinh \left( \frac{\kappa \pi}{2} \right) . \]  
(3.3.36)
To evaluate the integrals, we closed the integration contour with a semi-circle around the lower half plane, picking up an infinite number of residues along the lower half of the imaginary axis, which summed up to the given results. A simple consistency check shows that these matrix elements satisfy eq. (3.2.15), which verifies that the generalized butterfly states are indeed projectors.

**Example 2. The $L_{-2m}$ projectors**

Another class of projectors with a simple Virasoro representation was introduced in [45]. These projectors are surface states defined by the conformal maps

$$f_m(z) = \frac{z}{(1 - (-z^2)^m)^{1/2m}},$$  \hspace{1cm} (3.3.37)

with associated states

$$|P_{2m}\rangle = \exp\left(-\frac{1}{2m}L_{-2m}\right)|0\rangle.$$  \hspace{1cm} (3.3.38)

In order to check if a given surface state is in $\mathcal{H}_{\kappa^2}$, we found it useful to use a condition which is equivalent to eq. (3.3.29) for surface states

$$\Box \Box \log\left(\frac{f(z) - f(w)}{z - w}\right) = 0,$$  \hspace{1cm} (3.3.39)

where $\Box = \frac{d^2}{d\xi^2} - \frac{d^2}{d\zeta^2}$ is the two dimensional d’Alembertian and $\Diamond = \frac{d^2}{d\xi d\zeta}$ is the light-cone d’Alembertian, or vice versa. The relation between $z, w$ and $\xi, \zeta$ is given by eq. (3.3.30). This condition holds, as expected, for the case of the butterfly $m = 1$. A direct check shows that this equation does not hold for the next 100 cases and therefore that these projectors are not in the $\mathcal{H}_{\kappa^2}$ subalgebra. We expect that higher $m$ projectors will not fulfill this condition either.

### 3.3.4 Half-string representation

In this subsection we discuss how states in the $\mathcal{H}_{\kappa^2}$ subalgebra look in the half-string formalism [60, 22, 50, 61]. Finding rank one projectors in half-string formalism is extremely simple. Every normalized string wave functional
\( \Psi_P[x(\sigma)] \) that has a factorized form

\[
\Psi_P[x(\sigma)] = \chi_1[l(\sigma)]\chi_2[r(\sigma)],
\]

(3.3.40)

where \( l(\sigma) \sim x(\sigma) \) and \( r(\sigma) \sim x(\pi-\sigma) \) are the left and right sides of the string \(^1\), corresponds to a rank one projector. The string field reality condition (2.2.33) for a string functional reads

\[
\Psi_P^*[x(\sigma)] = \Psi_P[x(\pi-\sigma)] \Rightarrow \chi_2[r(\sigma)] = \chi_1^*[r(\sigma)].
\]

(3.3.41)

String fields can (almost) be regarded as matrices over the space of half-strings [13]. The BPZ reality condition implies that these matrices are Hermitian. A Hermitian projector is denoted “orthogonal projector” since its zero and one eigenvalue subspaces are orthogonal.

In [22, 50] projectors with real gaussian functionals were discussed. We consider also complex gaussians because they appear in \( \mathcal{H}_{\kappa^2} \)

\[
\Psi[X(\sigma)] = \exp\left(-\frac{1}{2}l_{2k-1}M_{2k-1,2j-1}l_{2j-1}\right) \exp\left(-\frac{1}{2}r_{2k-1}M_{2k-1,2j-1}^*r_{2j-1}\right)
\]

\[
= \exp\left(-\frac{1}{2}x_nL_{nm}x_m\right),
\]

(3.3.42)

where \( l_{2k+1}, r_{2j+1} \) and \( x_n \) are the Fourier modes of \( l(\sigma), r(\sigma) \) and \( x(\sigma) \). The form of the gaussians in the half-string formalism was restricted by the string field reality condition. For the full-string matrix \( L \) this condition reads

\[
L^* = CLC.
\]

(3.3.43)

Solving eq. (3.3.42) for \( L \) gives

\[
L_{2n-1,2m-1} = 2 \operatorname{Re} M_{2n-1,2m-1},
\]

\[
L_{2n,2m} = 2 (\operatorname{Re} M T^T)_{2n,2m},
\]

\[
L_{2n,2m-1} = L_{2m-1,2n} = 2i (T1m M)_{2n,2m-1},
\]

(3.3.44)

where the matrix \( T \) is defined by

\[
T_{2n,2m-1} = \frac{4}{\pi} \int_0^{\pi/2} \cos(2n\sigma) \cos((2m-1)\sigma) d\sigma = 2(-1)^{m+n+1} \pi \left( \frac{1}{2m - 1 + 2n} + \frac{1}{2m - 1 - 2n} \right).
\]

(3.3.45)
The relation between gaussian functionals and squeezed states is given by

\[ S = \frac{1 - ELE}{1 + ELE}, \tag{3.3.46} \]

where the diagonal matrix \( E \) is defined as in [22].

The case of a twist invariant projector is given by a real matrix \( M \), which corresponds to a real, block diagonal \( S_{nm} \). The form of \( S_{nm} \) restricts the Taylor expansion of \( S(z, w) \) to monomials \( z^k w^l \) with real coefficients, where \( k \) and \( l \) are both odd or both even. Inspection of the symmetry properties of the integrand in eq. (3.3.12) shows that this requirement is satisfied only if \( s_3(\kappa) = s_1(\kappa) \) which is the twist invariance condition (3.2.19). A general orthogonal projector eq. (3.3.41), is given by a complex matrix \( M \). In this case \( S_{nm} \) can have imaginary entries in the odd–even blocks. The analog condition for states in \( H_\kappa^2 \) is \( s_3(\kappa) = s_1^*(\kappa) \). Non-orthogonal, non-real projectors can have general \( s_1, s_2, s_3 \) and general \( S_{nm} \) matrices limited only by the projection condition.

### 3.4 Conclusions

In this chapter we set out to find squeezed state projectors whose matrix commute with \( (K_1)^2 \). The set \( H_\kappa^2 \) of squeezed states that commute with \( (K_1)^2 \) is a subalgebra of the star algebra. Analyzing this subalgebra is straightforward, using the explicit form of \( V_3 \) (3.2.10). This subalgebra obviously contains the wedge states, \( H_{\text{wedge}} \subset H_\kappa^2 \). The generalized butterfly states are also in \( H_\kappa^2 \), but some other surface states are not. There are also states in \( H_\kappa^2 \) which are not surface states.

All this makes \( H_\kappa^2 \) a convenient laboratory for the study of the star algebra. The BPZ reality condition is given by eq. (3.2.13). The condition for twist invariance is then that the matrix is real, eq. (3.2.19). The projection condition also has a simple form, eq. (3.2.15).

It would be interesting to try to address some of the open problems of string field theory in the \( H_\kappa^2 \) subalgebra. For example the allowed space of states in SFT is not well understood. In the \( H_\kappa^2 \) subalgebra this question...
should translate to conditions on the functions of $\kappa$. Those functions must be integrable to have any meaning. But we do not know what other constraints they have and specifically we do not know if we should impose continuity.

Addressing the above questions would require an analysis of the ghost sector and of the zero-modes. These two issues were not discussed in this chapter. It might be possible to incorporate them in the $\mathcal{H}_{\kappa^2}$ subalgebra using the known spectrum of the full vertex.
Chapter 4

Spectral density & the Virasoro operators in the continuous basis of SFT

In this chapter we derive two important tools for working in the $\kappa$ basis of string field theory. First we give an analytical expression for the finite part of the spectral density $\rho_{\text{fin}}$. This expression is relevant when both matter and ghost sectors are considered. Then we calculate the form of the matter part of the Virasoro generators $L_n$ in the $\kappa$ basis, which construct string field theory’s derivation $Q_B$.

4.1 Introduction and summary

In [39] Rastelli Sen and Zwiebach diagonalized the matrices of the star product and found their spectrum. The eigenvectors of these matrices form a continuous basis whose eigenvalues are in the range $-\infty < \kappa < \infty$. Working in this basis simplifies calculations involving the star product.

Calculating normalization of squeezed states in the continuous basis involves determinants of continuous matrices. If these matrices are diagonal $X = X_\kappa \delta(\kappa - \kappa')$, then the determinant has the form

$$\det X = \exp(\text{tr} \log X) = \exp \left( \int d\kappa \delta(\kappa - \kappa) \log X_\kappa \right). \quad (4.1.1)$$

A similar expression holds for matrices in the $H_\kappa$ subalgebra [55]. The delta function is the spectral density $\rho(\kappa)$ of the continuous basis. This spectral density diverges, and in the level-truncation regularization its divergence is $\kappa$ independent and behaves as $\rho^L(\kappa) = \frac{\log L}{2\pi}$. Because of this divergence, it seems that the determinant can only get the values 0, 1, $\infty$ depending on whether the integral $\int d\kappa \log X$ is negative, zero or positive. But $\rho(\kappa)$ has a finite
contribution $\rho_{\text{fin}}(\kappa)$ which is $\kappa$ dependent. When the ghost sector contributions cancel these infinities, $\rho_{\text{fin}}(\kappa)$ cannot be ignored, and the expression

$$\exp \left( \int d\kappa \rho_{\text{fin}}(\kappa) \log X \right),$$

should be taken into account. Belov and Konechny calculated $\rho_{\text{fin}}(\kappa)$ numerically in [62]. We find the analytic expression

$$\rho_{\text{fin}}(\kappa) = \frac{4 \log(2) - 2\gamma - \Psi\left(\frac{i\kappa}{2}\right) - \Psi\left(-\frac{i\kappa}{2}\right)}{4\pi},$$

(4.1.2)

where $\gamma$ is Euler’s constant, and $\Psi$ is the digamma (polygamma) function.

Another missing ingredient of the continuous basis is the form of the Virasoro generators. The Virasoro generators are used in string field theory to construct $Q_B$, from which the kinetic term around the perturbative vacuum is built. Other derivations built from the Virasoro generators can serve as kinetic terms as well [63]. The Virasoro generators are also useful in the construction of surface states [57], and in particular of surface state projectors [45]. In [40] it was noticed that the expression for $L_0$ in the continuous basis diverges. This is true for all the Virasoro generators. Nonetheless, we manage to find an analytic expression for them. For a single scalar field, $L_0$ is given by

$$L_0 = \alpha' p_0^2 + \sum_{n=1}^{\infty} na_n^\dagger a_n = \alpha' p_0^2 + \int_{-\infty}^{\infty} \frac{d\kappa d\kappa'}{\sqrt{N(\kappa)N(\kappa')}} g_{0,\kappa,\kappa'},$$

$$g_{0,\kappa,\kappa'} = \sum_{n=1}^{\infty} n\nu_n^\kappa \nu_n^\kappa' = \frac{\cosh(\frac{(\kappa+\kappa')\pi}{4})}{2} \left(\delta(\kappa-\kappa'+2i) + \delta(\kappa-\kappa'-2i)\right).$$

(4.1.3)

The use of complex arguments in the one dimensional delta function is somewhat unorthodox, but the definition of the delta function for complex arguments is essentially the same as for real arguments. We elaborate on the definition of the delta function and demonstrate its use in subsection 4.3.2.

The complex delta functions hide the high divergence of $L_0$ in the continuous basis. They appear in all the Virasoro generators. The midpoint preserving reparameterization generators $K_n = L_n - (-1)^n L_{-n}$ should have milder divergences [49], and indeed they do not contain complex delta functions.
This chapter is organized as follows. In section 4.2 we calculate the finite part of the spectral density. In section 4.3 we introduce a useful operator, which we denote $\mathcal{L}_\kappa$, and elaborate on the definition and use of the one dimensional delta function with a complex argument. We use these tools in section 4.4 to calculate the Virasoro generators. Sections 4.3 and 4.4 are independent of section 4.2.

### 4.2 The spectral density

The spectral density $\rho(\kappa)$ is used for calculating traces and determinants of operators that are diagonal in the continuous basis. The trace for example, of a diagonal operator $G$, is given by

$$
\text{tr} G = (n| G |n) = (n|\kappa) (\kappa| G |\kappa') (\kappa'|n)
$$

$$
= (n|\kappa) (\kappa|n) G_\kappa = \rho(\kappa) G_\kappa,
$$

where we define the spectral density

$$
\rho(\kappa) \equiv (\kappa|n) (n|\kappa) = \frac{1}{\mathcal{N}(\kappa)} \sum_{n=1}^{\infty} v_\kappa^n v_\kappa^n.
$$

From the invariance of the trace $\text{tr} G = (\kappa| G |\kappa)$, it is obvious that $\rho(\kappa)$ diverges like $\delta(0)$. In [39] it was shown that the leading term in level truncation regularization is $\rho^L(\kappa) = \frac{\log L}{2\pi}$, where $L$ denotes the level. This term is $\kappa$ independent. In [62] the finite, $\kappa$ dependent, contribution to the spectral density was defined as

$$
\rho^{\text{fin}}_L(\kappa) = \frac{1}{\mathcal{N}(\kappa)} \sum_{n=1}^{2L} v_\kappa^n v_\kappa^n = \frac{\log L/\pi}{2\pi} \sum_{n=1}^{L} \frac{1}{n},
$$

and evaluated numerically.

In this subsection we obtain an analytical expression for this term

$$
\rho^{\text{fin}}(\kappa) = \lim_{L \to \infty} \rho^{2L}_L(\kappa).
$$

We begin by regularizing the two diverging sums on the r.h.s of (4.2.3) with powers of the variable $z$ to get

$$
\rho^{\text{fin}}(z, \kappa) = \frac{1}{\mathcal{N}(\kappa)} \sum_{n=1}^{\infty} \left( v_{2n-1}^\kappa v_{2n-1}^\kappa z^n + v_{2n}^\kappa v_{2n}^\kappa z^n \right) - \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{z^n}{n},
$$

(4.2.5)
where \( |z| < 1 \). The last sum gives
\[
- \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{z^n}{n} = \frac{1}{2\pi} \log(1-z).
\] (4.2.6)

We evaluate the first sum by using eq. (3.3.5)
\[
\frac{1}{\mathcal{N}(\kappa)} \sum_{n=1}^{\infty} v_{2n-1}^\kappa v_{2n-1}^\kappa z^n = \frac{2z^{\frac{3}{2}}}{\mathcal{N}(\kappa)} \partial_z \sum_{n=1}^{\infty} \frac{v_{2n-1}^\kappa v_{2n-1}^\kappa}{2n-1} z^{n-\frac{1}{2}}
\]
\[
= \frac{2\mathcal{N}(\kappa) z^{\frac{3}{2}}}{\pi^2} \partial_z \int_0^\infty du \, dv \sqrt{z} \frac{\cos(\kappa u) \cos(\kappa v)}{\cosh^2(u) \cosh^2(v)} \sum_{n=1}^{\infty} (z \tanh^2(u) \tanh^2(v))^{n-1}
\]
\[
= \frac{2\mathcal{N}(\kappa) z^{\frac{3}{2}}}{\pi^2} \partial_z \int_0^\infty du \left( \sqrt{z} \frac{\cos(\kappa u) \cos(\kappa v)}{\cosh^2(u) \cosh^2(v) - z \sinh^2(u) \sinh^2(v)} \right) dv
\]
\[
= \frac{z}{\pi} \int_0^\infty \cos(\kappa v) \cos(\kappa \tanh^{-1}(\sqrt{z} \tanh(v))) \cosh^2(v) - z \sinh^2(v) dv.
\] (4.2.7)

In the last step we replaced \( \int_0^\infty du \cos(\kappa u) \) with \( \frac{1}{2} \int_{-\infty}^\infty du \exp(i \kappa u) \) and evaluated the \( u \) integral by closing the contour in the upper half plane for \( \kappa > 0 \) (with analytic continuation for \( \kappa \leq 0 \)) picking up the residues at \( u = \frac{(2n-1)\pi i}{2} \pm \tanh^{-1}(\sqrt{z} \tanh(v)) \). In a similar way we evaluate the second term on the r.h.s. of (4.2.5)
\[
\frac{1}{\mathcal{N}(\kappa)} \sum_{n=1}^{\infty} v_{2n}^\kappa v_{2n}^\kappa z^n = \frac{2z^{\frac{3}{2}}}{\mathcal{N}(\kappa)} \partial_z \sum_{n=1}^{\infty} \frac{v_{2n}^\kappa v_{2n}^\kappa}{2n} z^n
\]
\[
= \frac{2\mathcal{N}(\kappa) z^{\frac{3}{2}}}{\pi^2} \partial_z \int_0^\infty du \left( \sqrt{z} \frac{\sin(\kappa u) \sin(\kappa v) \tanh(u) \tanh(v)}{\cosh^2(u) \cosh^2(v) - z \sinh^2(u) \sinh^2(v)} \right) dv
\]
\[
= \frac{\sqrt{z}}{\pi} \int_0^\infty \sin(\kappa v) \sin(\kappa \tanh^{-1}(\sqrt{z} \tanh(v))) \cosh^2(v) - z \sinh^2(v) dv,
\] (4.2.8)

where now, in addition to the residues at \( u = \frac{(2n-1)\pi i}{2} \pm \tanh^{-1}(\sqrt{z} \tanh(v)) \), there are also residues at \( u = \frac{(2n-1)\pi i}{2} \). In the \( z \to 1 \) limit we add the integrals in equations (4.2.7) and (4.2.8) and neglect terms that behave like \((1 - z) \log(1 - z)\) to get
\[
\rho_{\text{fin}}(\kappa) = \lim_{z \to 1} \frac{1}{\pi} \int_0^\infty \cos \left( \kappa \left( v - \tanh^{-1}(\sqrt{z} \tanh(v)) \right) \right) \cosh^2(v) - z \sinh^2(v) dv + \frac{1}{2\pi} \log(1 - z).
\] (4.2.9)
To evaluate this integral we write $\sqrt{z} = 1 - \epsilon$, and drop terms, which become irrelevant in this limit

$$
\rho_{\text{fin}}(\kappa) = \lim_{\epsilon \to 0} \left( \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos \left( \frac{\kappa}{2} \log(1 + \frac{\epsilon}{2} e^{2v}) \right)}{1 + \frac{\epsilon}{2} e^{2v}} dv + \frac{1}{2\pi} \log(2\epsilon) \right)
$$

$$
= \lim_{\epsilon \to 0} \frac{1}{2\pi} \left( \int_{\frac{\epsilon}{2}}^{\infty} \frac{\cos \left( \frac{\kappa}{2} \right)}{e^{\epsilon} - 1} d\xi + \log(2\epsilon) \right),
$$

where $\xi = \log(1 + \frac{\epsilon}{2} e^{2v})$. To get rid of the infinities in the limit we calculate

$$
\partial_\kappa \rho_{\text{fin}}(\kappa) = - \frac{1}{4\pi} \int_{0}^{\infty} \frac{\xi \sin \left( \frac{\kappa}{2} \right)}{e^{\epsilon} - 1} d\xi,
$$

and since $\rho_{\text{fin}}(0) = \frac{\log(2)}{\pi}$ (as can be seen from eq. (4.2.10) or from a direct calculation) we get

$$
\rho_{\text{fin}}(\kappa) = \frac{\log(2)}{\pi} + \int_{0}^{\kappa} (\partial_\kappa \rho_{\text{fin}}(\kappa)) d\kappa = \frac{4\log(2) - 2\gamma - \Psi(\frac{\kappa}{2}) - \Psi(-\frac{\kappa}{2})}{4\pi},
$$

where $\gamma$ is Euler’s constant, and $\Psi$ is the digamma (polygamma) function. This result agrees with the numerical estimates of [62].

### 4.3 Mathematical preliminary

In the next section we shall see that in order to calculate the Virasoro generators in the $\kappa$ basis, we need to introduce the differential operator $L_\kappa$. This operator is a sum of two shift operators in the imaginary direction. We define and explain the use of this operator in 4.3.1.

We shall have to operate with $L_\kappa$ on delta functions, generating delta functions with complex arguments, creatures that we usually do not encounter. Thus, in 4.3.2 we elaborate on the definition of these delta functions and demonstrate their use.

#### 4.3.1 The operator $L_\kappa$

Our task in this section is to find an operator $L_\kappa$ which obeys

$$
L_\kappa(\kappa v_n^\kappa) = n v_n^\kappa.
$$
Equivalently this operator should obey the relation
\[ \mathcal{L}_\kappa \frac{\kappa v}{\sqrt{n}} z^n = z \partial_z \frac{v}{\sqrt{n}} z^n. \] (4.3.2)

Summing over \( n \), and using the generating function (3.3.4), we get
\[ \mathcal{L}_\kappa (1 - e^{-\kappa u}) = \frac{1}{2} \sin(2u) e^{-\kappa u}, \] (4.3.3)
where \( u = \tan(z) \). We see that
\[ \mathcal{L}_\kappa = \frac{1}{2} \sin(2\partial_\kappa) \] (4.3.4)
satisfies the above equation.

The operator \( \mathcal{L}_\kappa \) is a sum of two shift operators
\[ \mathcal{L}_\kappa f(\kappa) = \frac{\exp(2i\partial_\kappa) - \exp(-2i\partial_\kappa)}{4i} f(\kappa) \] (4.3.5)
and as such is defined only for functions which can be Taylor expanded with convergence radius \( r > 2 \). Since \( \mathcal{L}_\kappa \) contains only odd powers of \( \partial_\kappa \) we get for functions with convergence radius \( r > 2 \) on the real axis an integration by parts formula
\[ \int_{-\infty}^{\infty} f(\kappa) \mathcal{L}_\kappa g(\kappa) d\kappa = -\int_{-\infty}^{\infty} g(\kappa) \mathcal{L}_\kappa f(\kappa) d\kappa. \] (4.3.6)

The case \( r = 2 \) should be handled with care. We illustrate it with an example that will be used in section 4.4
\[ \exp(2i\partial_\kappa) \frac{1}{\cosh\left(\frac{(\kappa-\kappa')\pi}{4}\right)} = \frac{1}{\cosh\left(\frac{(\kappa-\kappa')\pi}{4}\right)} = \frac{-i}{\sinh\left(\frac{(\kappa-\kappa')\pi}{4}\right)}, \] (4.3.7)
where \( \kappa' \neq \kappa \). For \( \kappa' = \kappa \) the radius of convergence is \( r = 2 \), and the above expression is undefined. To extract the singular part of it we write
\[ \exp(2i\partial_\kappa) \frac{1}{\cosh\left(\frac{(\kappa-\kappa')\pi}{4}\right)} = \lim_{\epsilon \to 0^-} \left( \exp((2 - \epsilon)i\partial_\kappa) \frac{1}{\cosh\left(\frac{(\kappa-\kappa')\pi}{4}\right)} \right) 
= \lim_{\epsilon \to 0^-} \frac{-i}{\sinh\left(\frac{(\kappa-\kappa')\pi}{4}\right) - i\epsilon} = -i \mathcal{P} \frac{1}{\sinh\left(\frac{(\kappa-\kappa')\pi}{4}\right)} + \pi \delta\left(\sinh\left(\frac{(\kappa-\kappa')\pi}{4}\right)\right) \] (4.3.8)
where $\mathcal{P}$ represents the principal value of the function. In a similar fashion

$$\exp(-2i\partial_\kappa) \frac{1}{\cosh\left(\frac{(\kappa-\kappa')\pi}{4}\right)} = i\mathcal{P} \frac{1}{\sinh\left(\frac{(\kappa-\kappa')\pi}{4}\right)} + 4\delta(\kappa-\kappa') \quad (4.3.9)$$

We can take the symmetric and antisymmetric parts of these shift operators and write

$$\cos(2\partial_\kappa) \left( \frac{1}{\cosh\left(\frac{(\kappa-\kappa')\pi}{4}\right)} \right) = 4\delta(\kappa-\kappa') , \quad (4.3.10)$$

$$\sin(2\partial_\kappa) \left( \frac{1}{\cosh\left(\frac{(\kappa-\kappa')\pi}{4}\right)} \right) = -\mathcal{P} \frac{1}{\sinh\left(\frac{(\kappa-\kappa')\pi}{4}\right)} .$$

The antisymmetric part is of importance because of (4.3.4). The importance of the symmetric part lies in the “trigonometric” identity

$$\sin(2\partial_\kappa)(f(\kappa)g(\kappa)) = (\sin(2\partial_\kappa)f(\kappa))(\cos(2\partial_\kappa)g(\kappa)) + (\cos(2\partial_\kappa)f(\kappa))(\sin(2\partial_\kappa)g(\kappa)) . \quad (4.3.11)$$

We shall expand the definition of $L_\kappa$ to functions whose radius of convergence is $r < 2$ at the end of the next subsection.

4.3.2 The one dimensional delta function with complex argument

In this subsection we present the properties of the complex delta function, as it will emerge when we shall operate with $L_\kappa$ on $\delta(\kappa-\kappa')$ in the calculations of the Virasoro generators. Formally, $L_\kappa \delta(\kappa-\kappa')$ is an infinite sum of delta function derivatives, but does this sum converge to a distribution? We demonstrate that this sum is an object very similar to a regular distribution.

So far, we defined the operation of $L_\kappa$ only on functions with convergence radius $r > 2$. However, for the delta function, there is no notion of convergence radius because it is a distribution. We consider the definition of the delta function as a “limit” of functions,

$$\delta(\kappa) = \lim_{\epsilon \to 0} \delta_\epsilon(\kappa) . \quad (4.3.12)$$

Different $\delta_\epsilon$ sequences have different radii of convergence. While the sequence used in (4.3.8) has a zero radius, other sequences, such as the limit of gaussians,

$$\delta_\epsilon(\kappa) = \frac{1}{\sqrt{\pi\epsilon}} e^{\frac{-\kappa^2}{4\epsilon}} , \quad (4.3.13)$$

$$\quad$$

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are analytic in the whole complex plane. Henceforth, we define the delta function using a sequence of this type. We can now define

$$\mathcal{L}_\epsilon \delta (\kappa - \kappa') = \lim_{\epsilon \to 0} \mathcal{L}_\epsilon \delta_\epsilon (\kappa - \kappa') \equiv \frac{1}{4i}(\delta (\kappa - \kappa' + 2i) - \delta (\kappa - \kappa' + 2i)). \quad (4.3.14)$$

The complex arguments of the delta functions may seem strange, but in fact integrations involving these delta functions are similar to the familiar case of a real argument. This can be seen using the behaviors at (real directed) infinity of $\delta_\epsilon$ and contour arguments. These contour arguments only apply when we convolute the delta function with an analytic function $f(\kappa)$ that has no poles on the way to the new contour of integration. Thus, the complex delta function acts as a distribution, when confined to this class of functions.

Suppose that $f(\kappa)$ has simple poles $\kappa_n$ in the range $0 \leq \Im(\kappa) \leq 2$. To evaluate the integral

$$\int_{-\infty}^{\infty} \delta (\kappa - 2i) f(\kappa) d\kappa , \quad (4.3.15)$$

we displace the contour to the line $\Im(\kappa) = 2$, and pick up the residues along the way. Poles for which $0 < \Im(\kappa_n) < 2$, contribute $2\pi i \delta (\kappa_n - 2i) \text{res}_{\kappa_n} (f(\kappa))$, while poles for which $\Im(\kappa_n) = 2$ contribute $\pi i \delta (\kappa_n - 2i) \text{res}_{\kappa_n} (f(\kappa))$. The last case to consider is the case of integrating the principle part of poles located on the real line. In this case we again get a contribution of $\pi i \delta (\kappa_n - 2i) \text{res}_{\kappa_n} (f(\kappa))$ to the integral. All in all

$$\int_{-\infty}^{\infty} \delta (\kappa - 2i) P f(\kappa) d\kappa = f(2i) + 2\pi i \sum_{0 < \Im(\kappa_n) < 2} \text{res}_{\kappa_n} (f(\kappa)) \delta (\kappa_n - 2i) + \pi i \sum_{\Im(\kappa_n) = 0, 2} \text{res}_{\kappa_n} (f(\kappa)) \delta (\kappa_n - 2i). \quad (4.3.16)$$

Due to a change in the orientation of integration

$$\int_{-\infty}^{\infty} \delta (\kappa + 2i) P f(\kappa) d\kappa = f(-2i) - 2\pi i \sum_{-2 < \Im(\kappa_n) < 0} \text{res}_{\kappa_n} (f(\kappa)) \delta (\kappa_n + 2i) - \pi i \sum_{\Im(\kappa_n) = -2, 0} \text{res}_{\kappa_n} (f(\kappa)) \delta (\kappa_n + 2i). \quad (4.3.17)$$
We see that unlike regular distributions, the convolution of these generalized distributions with analytic functions can result in generalized distributions, rather than functions. The incorporation of multiple poles is straightforward and adds terms with derivatives of the delta function.

Finally we introduce a recipe for handling expressions such as $\mathcal{L}_\kappa f(\kappa)$, where $f(\kappa)$ has a radius of convergence $r < 2$

$$\mathcal{L}_\kappa f(\kappa) = \mathcal{L}_\kappa \int \delta(\kappa - \tilde{\kappa}) f(\tilde{\kappa}) d\tilde{\kappa} \equiv \int \mathcal{L}_\kappa \delta(\kappa - \tilde{\kappa}) f(\tilde{\kappa}) d\tilde{\kappa}.$$ (4.3.18)

This definition involves changing the order of integration with the limit in the definition of the delta function (4.3.12). As we naturally think of this limit as being taken after all integrations were performed, we shall refer to the r.h.s of this equation as the definition of $\mathcal{L}_\kappa f(\kappa)$. For the case $r > 2$ this definition coincides with (4.3.5).

### 4.4 The Virasoro generators in the $\kappa$ basis

In this section we obtain the form of the Virasoro generators in the $\kappa$ basis for a single scalar field. Since the expressions are cumbersome, we start in 4.4.1 by finding $L_0$, which is the simplest one. It is also the most useful one, in particular when working in the Siegel gauge. In subsection 4.4.2 we calculate $L_{\pm 1}$ and confirm the closure of the sl(2) algebra directly. Next, we get the general expression for all the generators and obtain identities that follow from the Virasoro algebra. All of the above is done in the zero momentum sector for clarity. We conclude by giving the general expressions for the generators including the zero mode.

#### 4.4.1 Calculating $L_0$

Formally, writing the Virasoro generator $L_0$ in the $\kappa$ basis involves a simple change of basis

$$L_0 = \sum_{n=1}^{\infty} n a_\kappa^\dagger a_n = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\kappa' a_\kappa'^\dagger a_\kappa n n^{\kappa'} \nu^{\kappa'} n = \int_{-\infty}^{\infty} d\kappa' \frac{a_\kappa'^\dagger a_\kappa}{\sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}} g^{\kappa,\kappa'}_{0}, \quad (4.4.1)$$
where we define
\[ g_{0}^{\kappa,\kappa'} \equiv \sum_{n=1}^{\infty} n v_{n}^{\kappa} v_{n}^{\kappa'} \quad (4.4.2) \]

However, it was shown in [40] that this sum is highly divergent. Thus, \( g_{0}^{\kappa,\kappa'} \) cannot be given by a regular function. We derive an analytic expression for it as a generalized distribution by acting on both sides of the completeness relation (3.3.6) with the \( \mathcal{L}_{\kappa} \) operator

\[ g_{0}^{\kappa,\kappa'} = \mathcal{L}_{\kappa} \left( \kappa \sum_{n=1}^{\infty} n v_{n}^{\kappa} v_{n}^{\kappa'} \right) = \mathcal{L}_{\kappa} (\kappa \mathcal{N}(\kappa) \delta(\kappa-\kappa')) \]
\[ = \frac{\sinh \left( \frac{\kappa \pi}{2} \right)}{2i} (\delta(\kappa-\kappa'+2i) - \delta(\kappa-\kappa'-2i)) \]
\[ = \frac{\cosh \left( \frac{(\kappa' + \kappa) \pi}{4} \right)}{2} (\delta(\kappa-\kappa'+2i) + \delta(\kappa-\kappa'-2i)) , \quad (4.4.3) \]

where in the last step we used the properties of the delta function in order to give a manifestly symmetric expression for \( g_{0}^{\kappa,\kappa'} \).

We conclude this subsection with a consistency check. Define the covariant metric by the following convergent sum

\[ g_{0}^{\kappa,\kappa'} \equiv \sum_{n=1}^{\infty} \frac{n v_{n}^{\kappa} v_{n}^{\kappa'}}{n} . \quad (4.4.4) \]

By the completeness relation (3.3.6) and the definition of \( \mathcal{L}_{\kappa} \) (4.3.1), \( g_{0}^{\kappa,\kappa'} \) obeys the identity

\[ \mathcal{L}_{\kappa} (\kappa g_{0}^{\kappa,\kappa'}) = \mathcal{N}(\kappa) \delta(\kappa-\kappa') . \quad (4.4.5) \]

We evaluate this metric by the methods of [43] and obtain

\[ g_{0}^{\kappa,\kappa'} = \frac{2 \sinh \left( \frac{\kappa \pi}{4} \right) \sinh \left( \frac{\kappa' \pi}{4} \right)}{\kappa \kappa' \cosh \left( \frac{(\kappa-\kappa') \pi}{4} \right)} . \quad (4.4.6) \]

This expression can be used to verify eq. (4.4.5) directly

\[ \mathcal{L}_{\kappa} (\kappa g_{0}^{\kappa,\kappa'}) = \frac{\sinh \left( \frac{\kappa \pi}{4} \right)}{\kappa'} \left( \sin(2\partial_{\kappa}) \sinh \left( \frac{\kappa \pi}{4} \right) \right) \left( \cos(2\partial_{\kappa}) \frac{1}{\cosh \left( \frac{(\kappa-\kappa') \pi}{4} \right)} \right) \]
\[ = \frac{4 \sinh \left( \frac{\kappa' \pi}{4} \right) \cosh \left( \frac{\kappa \pi}{4} \right) \delta(\kappa-\kappa')}{\kappa'} = \mathcal{N}(\kappa) \delta(\kappa-\kappa') , \quad (4.4.7) \]
where we used eq. (4.3.10),(4.3.11), and the fact that
\[
\cos(2\partial_\kappa) \sinh \left( \frac{\kappa \pi}{4} \right) = 0, \quad \sin(2\partial_\kappa) \sinh \left( \frac{\kappa \pi}{4} \right) = \cosh \left( \frac{\kappa \pi}{4} \right).
\] (4.4.8)

Using the complex delta function integration rules (4.3.16),(4.3.17), one can verify that the metrics obey
\[
\int_{-\infty}^{\infty} \frac{d\tilde{\kappa}}{N(\tilde{\kappa})} g_{\kappa,\tilde{\kappa}}^0 g_{\tilde{\kappa},\kappa'}^0 = N(\kappa) \delta(\kappa - \kappa'),
\] (4.4.9)
as expected.

### 4.4.2 Calculating $L_{\pm 1}$

We use similar methods to obtain the remaining sl(2) generators. Actually, it is enough to find $L_{-1}$, since $L_1$ can be obtained by hermitian conjugation. We write $L_{-1}$ in the $\kappa$ basis
\[
L_{-1} = \sum_{n=1}^{\infty} \sqrt{n(n+1)} a_{n+1}^\dagger a_n
\]
\[
= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{d\kappa d\kappa'}{\sqrt{N(\kappa)N(\kappa')}} \sqrt{n(n+1)} v_n^\kappa v_{n+1}^{\kappa'} = \int_{-\infty}^{\infty} \frac{d\kappa d\kappa'}{\sqrt{N(\kappa)N(\kappa')}} g_{1,\kappa}^\kappa g_{1,\kappa'}^\kappa',
\] (4.4.10)
where
\[
g_{1,\kappa}^\kappa \equiv \sum_{n=1}^{\infty} \sqrt{n(n+1)} v_n^\kappa v_{n+1}^{\kappa'}.
\] (4.4.11)

Since the sum does not converge, we define the well behaved expression
\[
g_{1,\kappa}^\kappa \equiv \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}} v_n^\kappa v_{n+1}^{\kappa'}.
\] (4.4.12)

We will get $g_{1,\kappa}^\kappa$ using
\[
g_{1,\kappa}^\kappa = L_{\kappa} L_{\kappa'}(\kappa\kappa' g_{1,\kappa}^\kappa).
\] (4.4.13)

Substituting the integral expression for $v_n^\kappa$ (3.3.5) we obtain
\[
g_{1,\kappa}^\kappa = i \frac{N(\kappa)N(\kappa')}{4\pi^2} \int_{-\infty}^{\infty} \frac{du dv e^{i(\kappa u + \kappa' v)}}{\cosh^2(u) \cosh^2(v)} \sum_{n=1}^{\infty} (-1)^{n-1} \tanh^{n-1}(u) \tanh^n(v)
\]
\[
= i \frac{N(\kappa)N(\kappa')}{4\pi^2} \int_{-\infty}^{\infty} \frac{du dv e^{i(\kappa u + \kappa' v)} \tanh(v)}{\cosh(u) \cosh(v) \cosh(u + v)}
\]
\[
= \frac{1}{\kappa} - \frac{(\kappa - \kappa') \sinh \left( \frac{\kappa \pi}{4} \right)}{2\kappa \kappa' \sinh \left( \frac{(\kappa - \kappa') \pi}{4} \right) \cosh \left( \frac{(\kappa - \kappa') \pi}{4} \right)}.
\] (4.4.14)
A direct calculation using (4.3.10),(4.3.11) gives
\[
\mathcal{L}_\kappa(\kappa g^1_{\kappa,\kappa'}) = \mathcal{P} \frac{\sinh\left(\frac{\kappa'\pi}{2}\right)}{\kappa' \sinh\left(\frac{\pi}{2}\right)}. \tag{4.4.15}
\]

As this expression has zero radius of convergence for \(\kappa = \kappa'\), we use (4.3.18) and get
\[
g^1_{\kappa,\kappa'} = \mathcal{L}_{\kappa'} \int_{-\infty}^{\infty} d\kappa' \delta(\kappa - \kappa') \mathcal{P} \frac{\sinh\left(\frac{\kappa'\pi}{2}\right)}{\sinh\left(\frac{\kappa - \kappa'}{2}\right)} \sinh\left(\frac{\pi}{2}\right) \sinh\left(\frac{\kappa - \kappa'}{2}\right)
\]
\[
= \int_{-\infty}^{\infty} d\kappa' \mathcal{P} \frac{(\delta(\kappa - \kappa' - 2i) - \delta(\kappa - \kappa' + 2i)) \sinh\left(\frac{\kappa'\pi}{2}\right)}{4i \sinh\left(\frac{\kappa - \kappa'}{2}\right)} \tag{4.4.16}
\]
\[
= -\frac{\sinh\left(\frac{\kappa'\pi}{2}\right)}{2} (\delta(\kappa - \kappa' + 2i) + \delta(\kappa - \kappa' - 2i) + 2\delta(\kappa - \kappa')),
\]
where in the last step we used the integration rules (4.3.16),(4.3.17).

To verify that the \textit{sl}(2) algebra indeed holds, we write
\[
[L_1, L_{-1}] = \int_{-\infty}^{\infty} \frac{d\kappa d\kappa_2 d\kappa_3 d\kappa_4}{4\sqrt{\mathcal{N}(\kappa_1)\mathcal{N}(\kappa_2)\mathcal{N}(\kappa_3)\mathcal{N}(\kappa_4)}} \sinh\left(\frac{\kappa_1\pi}{2}\right) \sinh\left(\frac{\kappa_3\pi}{2}\right) \left[a^\dagger_{\kappa_1} a_{\kappa_2}, a^\dagger_{\kappa_4} a_{\kappa_3}\right]
\]
\[
\times (\delta(\kappa_1 - \kappa_2 + 2i) + \delta(\kappa_1 - \kappa_2 - 2i) + 2\delta(\kappa_1 - \kappa_2))
\]
\[
\times (\delta(\kappa_3 - \kappa_4 + 2i) + \delta(\kappa_3 - \kappa_4 - 2i) + 2\delta(\kappa_3 - \kappa_4)) \tag{4.4.17}
\]
\[
= \int_{-\infty}^{\infty} \frac{dr d\kappa d\kappa' d\kappa_2 d\kappa_3 d\kappa_4}{4\sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')\mathcal{N}(\kappa_2)\mathcal{N}(\kappa_3)\mathcal{N}(\kappa_4)}} \sinh\left(\frac{\kappa\pi}{2}\right) \sinh\left(\frac{\kappa'\pi}{2}\right) \sinh\left(\frac{\kappa_2\pi}{2}\right) \sinh\left(\frac{\kappa_3\pi}{2}\right)
\]
\[
\times (\delta(\kappa - \kappa_2 + 2i) + \delta(\kappa - \kappa_2 - 2i) + 2\delta(\kappa - \kappa_2))
\]
\[
\times (\delta(\kappa' - \kappa_2 + 2i) + \delta(\kappa' - \kappa_2 - 2i) + 2\delta(\kappa' - \kappa_2)) = 2L_0.
\]

The derivation of the other commutation relations is similar.
4.4.3 General $L_n$

Our strategy should be familiar by now, with a minor change, the appearance of creation (annihilation) operators bilinears. We write

$$L_{-m} = \frac{1}{2} \sum_{n=1}^{m-1} \sqrt{n(m - n)} a^{\dagger}_{m/n-a^{\dagger}_{m-n}} + \sum_{n=1}^{\infty} \sqrt{n(n + m)} a^{\dagger}_{n+m} a_n$$

$$L_m = L_{-m} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk^2 dk'^2}{\sqrt{N(k)N(k')}} h^{k,k'}_m + \int_{-\infty}^{\infty} \frac{dk^2 dk'^2}{\sqrt{N(k)N(k')}} g^{k,k'}_m,$$  \hspace{1cm} (4.4.18)

where

$$g^{k,k'}_m \equiv \sum_{n=1}^{\infty} \sqrt{n(n + m)} v_n^{k} v_{n+m}^{k'},$$

$$h^{k,k'}_m \equiv \sum_{n=1}^{m-1} \sqrt{n(m - n)} v_n^{k'} v_{m-n}^{k}.$$  \hspace{1cm} (4.4.19)

We start by calculating

$$g^{m}_{k,k'} \equiv \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n + m)}} v_n^{k} v_{n+m}^{k'}$$

$$= i^m \frac{N(k)N(k')}{4\pi^2} \int_{-\infty}^{\infty} du dv e^{i(ku + ku')} \sum_{n=0}^{\infty} (-1)^n \tanh^n(u) \tanh^{n+m}(v)$$

$$= i^m \frac{N(k)N(k')}{4\pi^2} \int_{-\infty}^{\infty} du dv e^{i(ku + ku')} \tanh^m(v)$$

$$= i^{m+1} \frac{N(k')}{2\pi k} \int_{-\infty}^{\infty} dv \frac{\tanh^{m-1}(v)}{\cosh^2(v)} \left( e^{i(k' - v)} - e^{-i(k')} \right).$$  \hspace{1cm} (4.4.20)

We can now obtain

$$g^{k,k'}_m = \mathcal{L}_k \mathcal{L}_k(k,k') g^{m}_{k,k'} = \mathcal{L}_k \mathcal{L}_k \frac{i^{m+1} k' N(k')}{2\pi} \int_{-\infty}^{\infty} dv \frac{\tanh^{m-1}(v) e^{i(k' - v)}}{\cosh^2(v)}$$

$$= k' N(k') \mathcal{L}_k \mathcal{L}_k \frac{v_{m-k}^{k'}}{\sqrt{m N(k' - k)}},$$  \hspace{1cm} (4.4.21)

where in the second equality we noticed that the $k$ independent term drops under the action of the derivative. In the last equality we recognized the integral expression for $v^{k'}_n$ (3.3.5), and used the anticommutation relation

$$\{\mathcal{L}_{k'}, k' N(k')\} = 0.$$  \hspace{1cm} (4.4.22)
In order to continue we recall that \( v_m^{\kappa - \kappa'} \) are polynomials with respect to \( \kappa, \kappa' \), and thus have infinite radius of convergence. Next, we notice that \( \mathcal{L}_\kappa \mathcal{L}_\kappa \) acts on a function of \( \kappa_- \equiv \kappa' - \kappa \). We define \( \kappa_+ \equiv \kappa + \kappa' \), and get

\[
g_m^{\kappa, \kappa'} = \frac{\sinh(\frac{\kappa' \pi}{2})}{8 \sqrt{m}} (\cos(4\partial_-) - \cos(4\partial_+)) \frac{\kappa_- v_m^{\kappa - \kappa'}}{\sinh(\frac{\kappa_- \pi}{2})} = \frac{\sinh(\frac{\kappa' \pi}{2})}{16 \sqrt{m}} \int_{-\infty}^{\infty} d\tilde{k} \tilde{k} v_m^{\kappa - \kappa'} \delta(\kappa_- - \tilde{k} + 4i) + \delta(\kappa_- - \tilde{k} - 4i) - 2\delta(\kappa_- - \tilde{k})
\]

\[
= \sinh \left( \frac{\kappa' \pi}{2} \right) \left( \frac{q_m(\kappa_-)}{\sinh \left( \frac{\kappa_- \pi}{2} \right)} - m \sin \left( \frac{m\pi}{2} \right) \delta(\kappa_-) \right) + \frac{1}{2} \cosh \left( \frac{\kappa_+ \pi}{4} \right) (i^m \delta(\kappa_- - 2i) - (-i)^m \delta(\kappa_- + 2i)) ,
\]

where we used

\[
v_m^{\pm2i} = (\mp i)^{m-1} , \quad v_m^{\pm4i} = (\mp i)^{m-1} m ,
\]

and defined the polynomial

\[
q_m(\kappa) \equiv \frac{(\kappa + 4i)v_m^{\kappa + 4i} + (\kappa - 4i)v_m^{\kappa - 4i} - 2\kappa v_m^{\kappa}}{16 \sqrt{m}} = \oint \frac{e^{-\kappa \tan^{-1}(z)}dz}{z^{m-1}(1 + z^2)^2}.
\]

Finally, we find \( h_m^{\kappa, \kappa'} \)

\[
h_m^{\kappa, \kappa'} = \mathcal{L}_\kappa \mathcal{L}_\kappa \sum_{n=1}^{m-1} \frac{1}{(2\pi i)^2} \oint \frac{dz dw}{z^{n+1}w^{m-n+1}} e^{-\kappa \tan^{-1}(z) - \kappa' \tan^{-1}(w)}
\]

\[
= \mathcal{L}_\kappa \mathcal{L}_\kappa \oint \frac{dz dw}{(2\pi i)^2} \frac{1}{z^{m+1}} \left( \frac{1}{w - z} - \frac{1}{w} \right) e^{-\kappa \tan^{-1}(z) - \kappa' \tan^{-1}(w)}
\]

\[
= -\mathcal{L}_\kappa \mathcal{L}_\kappa \frac{\kappa_+ v_m^{\kappa_+}}{\sqrt{m}} = q_m(\kappa_+) .
\]

Substituting the Virasoro generators in the continuous basis representation
eq. (4.4.18) in the algebra (2.1.18), with \( c = 1 \), gives the identities

\[
(n - m) g_{n+m} = [g_n, g_m], \quad (4.4.27a)
\]

\[
(n - m) h_{n+m} = h_n g_m + (h_n g_m)^T - h_m g_n - (h_m g_n)^T, \quad (4.4.27b)
\]

\[
\text{Tr}(h_m h_n) = \frac{1}{6} \delta_{n,m}(n^3 - n), \quad (4.4.27c)
\]

\[
(n + m) g_{n-m} = g_n g_m^T - g_m g_n + h_m h_n \quad n \geq m, \quad (4.4.27d)
\]

\[
(n + m) h_{n-m} = g_m h_n + (g_m h_n)^T \quad n \geq m, \quad (4.4.27e)
\]

\[
0 = g_n h_m + (g_n h_m)^T \quad n \geq m. \quad (4.4.27f)
\]

where matrix multiplication is implicit. Summation of a continuous index \( \tilde{\kappa} \), in
the multiplication as well as in the trace, should be understood as integration
\[
\int \frac{d\tilde{\kappa}}{\mathcal{N}(\kappa)}.
\]

To demonstrate the formalism we give a direct proof of two of the above
identities. The commutators of the \( g_m \) in eq. (4.4.27a) are obtained by using
its form in eq. (4.4.21) and the expression for \( v_\kappa^\kappa \) in (3.3.5)

\[
[g_n, g_m]^{\kappa\kappa'} = \kappa' \mathcal{N}(\kappa') \mathcal{L}_\kappa \mathcal{L}_{\kappa'} \int \frac{d\tilde{\kappa}}{\sqrt{mn}} \tilde{\kappa} \mathcal{L}_{\tilde{\kappa}} \left( \frac{v_\kappa^{\tilde{\kappa}}}{\mathcal{N}(\kappa - \tilde{\kappa})} \right) \mathcal{L}_{\tilde{\kappa}} \left( \frac{v_m^{\kappa'-\tilde{\kappa}}}{\mathcal{N}(\tilde{\kappa} - \kappa')} \right) - (n \leftrightarrow m)
\]

\[
= \kappa' \mathcal{N}(\kappa') \mathcal{L}_\kappa \mathcal{L}_{\kappa'} \mathcal{L}_\kappa \mathcal{L}_{\kappa'} \int \frac{d\tilde{\kappa}}{\sqrt{mn}} \tilde{\kappa} \mathcal{N}(\kappa - \tilde{\kappa}) \mathcal{N}(\tilde{\kappa} - \kappa') - (n \leftrightarrow m)
\]

\[
= \kappa' \mathcal{N}(\kappa') \mathcal{L}_\kappa^2 \mathcal{L}_{\kappa'}^2 \frac{i^{n+m-1} \sqrt{mn}}{2\pi} \int du \partial_u \left( \frac{e^{-i\kappa u} \tanh^{n-1} u}{\cos^2 u} \right) \frac{e^{i\kappa' u} \tanh^{m-1} u}{\cos^2 u} - (n \leftrightarrow m)
\]

\[
= (n - m) \kappa' \mathcal{N}(\kappa') \mathcal{L}_\kappa \mathcal{L}_{\kappa'} \frac{v_\kappa^{\kappa'-n}}{\sqrt{n+m \mathcal{N}(\kappa - \kappa')}} = (n - m) g_{n+m}^{\kappa\kappa'},
\]

(4.4.28)

where in the second step we used the symmetry \( \mathcal{L}_{\tilde{\kappa}} f(\kappa - \tilde{\kappa}) = -\mathcal{L}_\kappa f(\kappa - \tilde{\kappa}) \)
and in the third step we performed the integration with respect to \( \tilde{\kappa} \). In the
last step we acted with one of the \( \mathcal{L}_\kappa \) operators and with one \( \mathcal{L}_{\kappa'} \).

For the identity (4.4.27c) we use generating function techniques. The r.h.s
of the identity gives

\[
\sum_{n,m \geq 1} z^m w^n \frac{1}{6} \delta_{n,m}(n^3 - n) = \frac{z^2 w^2}{(1 - z w)^2},
\]

(4.4.29)
while for the l.h.s
\begin{align*}
\sum_{n,m \geq 1} z^m w^n & \int \frac{h_n^{\kappa \kappa'} h_m^{\kappa' \kappa}}{\mathcal{N}(\kappa) \mathcal{N}(\kappa')} d\kappa d\kappa' \\
& = \frac{z^2 w^2}{(1 + z^2)^2} \int \frac{d\kappa d\kappa'}{\mathcal{N}(\kappa) \mathcal{N}(\kappa')} e^{-(\kappa + \kappa')(\tan^{-1} z + \tan^{-1} w)} \\
& = \frac{z^2 w^2}{(1 + z^2)^2} \left( \frac{1}{\cos^2(\tan^{-1} z + \tan^{-1} w)} \right)^2 = \frac{z^2 w^2}{(1 - zw)^2},
\end{align*}

where we used (4.4.26) to sum over \(n, m\).

### 4.4.4 The zero mode

Adding the zero mode to the Virasoro generators is simple, as all we have to do is to substitute
\begin{align*}
L_{-n} & \to L_{-n} + \sqrt{n} a_n^\dagger \sqrt{2\alpha' p_0} = L_{-n} + \sqrt{2\alpha'} \int d\kappa \sqrt{n} \nu_n^{\kappa} a_n^\dagger \sqrt{\mathcal{N}(\kappa)} p_0, \\
L_n & \to L_n + \sqrt{n} a_n \sqrt{2\alpha'} p_0 = L_n + \sqrt{2\alpha'} \int d\kappa \sqrt{n} \nu_n^{\kappa} a_n \sqrt{\mathcal{N}(\kappa)} p_0, \\
L_0 & \to L_0 + \alpha' p_0^2. \tag{4.4.31}
\end{align*}

There are new identities that follow from the Virasoro algebra (2.1.18)
\begin{align*}
(n - m)(\sqrt{n + m} \nu_{n+m}) & = g_m^T(\sqrt{n} \nu_n) - g_n^T(\sqrt{m} \nu_m), \tag{4.4.32a} \\
0 & = g_n(\sqrt{m} \nu_m) + h_m(\sqrt{n} \nu_n) \quad n \geq m, \tag{4.4.32b} \\
(n + m)(\sqrt{n - m} \nu_{n-m}) & = g_m(\sqrt{n} \nu_n) + h_n(\sqrt{m} \nu_m) \quad n \geq m. \tag{4.4.32c}
\end{align*}

This completes the derivation of the matter sector Virasoro generators.

### 4.5 Conclusions

In this chapter we found an analytic expression for the finite part of the spectral density \(\rho_{fin}\) and the form of the Virasoro generators in the continuous basis. We hope that it would help solve some of the withstanding problems of string field theory.
The most pretentious goal is to find an analytic solution to string field theory’s equation of motion $Q_B \Psi + \Psi \ast \Psi = 0$. One way to achieve this might be to follow the analytical methods of [33] in the continuous basis.

For the total Virasoro generators, the form of the ghost sector is needed. The expressions in the ghost sector resemble those of the matter sector [49]. Thus, they can be calculated using the methods of this chapter.
Chapter 5

Continuous half-string representation of SFT

In this chapter we give the explicit form of the half-string representation in the continuous $\kappa$ basis. We show the comma structure of the three-vertex, when expanded around an arbitrary projector. We add the zero-mode and show how in the half-string representation it is replaced by the mid-point degree of freedom. Adding the ghost sector to the three-vertex in the half-string representation is then straightforward. We demonstrate the simplicity of this formalism with some applications, such as gauge transformations and identification of subalgebras.

5.1 Introduction

As a byproduct of the new interest in string field theory, further ways to simplify the star product were developed, most notably the half-string formulation [60, 22, 50, 61], spectroscopy of the three-vertex [39, 43], and the Moyal formulations [64, 54, 65, 40, 49]. Different subalgebras of the star product were recognized, usually belonging to the subalgebra of (shifted) squeezed states. Among these are the surface states, wedge states and $H_{\kappa^2}$ [34, 55].

The simplification of the star product usually complicates the form of the BRST operator $Q_B$. It may seem that this price is worth paying since $Q_B$ is limited to the quadratic term. However, in order to find the analytic solution, one should deal with both the star product and $Q_B$. In the continuous basis the situation is more complicated, as it was shown in [40] that $Q_B$ cannot be described by a function in this basis. This problem was partially solved when expressions for $Q_B$ as a generalized distribution and as a difference operator were provided in [56, 66].

In this chapter we present the continuous half-string formalism. We use the
fact that Bogoliubov transformations, which are defined by squeezed states projectors, transform the full-string into a half-string basis [33]. Starting from the continuous basis, which is reviewed in section 5.2, we can choose any projector in $\mathcal{H}_{\alpha z}$ to get a continuous half-string basis. The star product and integration then become

$$
(\Psi \star \Phi)(l^\kappa, r^\kappa, x_m) = \int D y^\kappa \Psi(l^\kappa, y^\kappa, x_m) \Phi(-y^\kappa, r^\kappa, x_m),
$$

$$
\int \Psi = \int D y^\kappa dx_m \Psi(y^\kappa, -y^\kappa, x_m).
$$

(5.1.1)

This result is independent of the projector we choose to work with, and it greatly simplifies calculations in the star-algebra. The detailed derivation of the continuous half-string basis can be found in section 5.3, where we also calculate explicitly the transformation to the sliver basis and to the butterfly basis. The half-string formalism in the sliver basis was studied in [61], while the half-string formalism of [60] is actually the half-string in the butterfly basis [45]. We end the section by incorporating the zero mode and the ghost sector into the formalism, enabling us to calculate the normalization of the vertices. The infinite factors in the normalization turn out correctly at the critical dimension $D = 26$.

In section 5.4 we demonstrate some applications of the formalism. In 5.4.1 we identify a subalgebra which is a doubling of the wedge state subalgebra. In 5.4.2 we find the explicit form of the gauge transformation among different butterflies and section 5.5 is devoted to conclusions. The properties of the Bogoliubov transformation and its relation to squeezed states are summarized in appendix A.

### 5.2 Preliminaries

#### 5.2.1 The continuous basis

String field theory has an infinite number of degrees of freedom. These degrees of freedom describe the string configuration and are usually represented by the
mode expansion of the string configuration and its conjugate momenta

\[ X(\sigma) = \sqrt{2\alpha'} \sum_{n=0}^{\infty} x_n u_n(\sigma), \quad P(\sigma) = \frac{1}{\sqrt{2\alpha'}} \sum_{n=0}^{\infty} p_n u_n(\sigma). \]  

(5.2.1)

In this notation all the modes are dimensionless, and the zero modes \( x_0, p_0 \) are equal to the physical zero modes \( x = \sqrt{2\alpha'} x_0, p = p_0/\sqrt{2\alpha'} \) for \( 2\alpha' = 1 \). It is conventional to use \( a_0 \) instead of what we mark as \( p_0 \), we use this notation to emphasize that the zero mode is not an oscillatory one.

The canonical basis

\[ u_0(\sigma) = 1, \quad u_n(\sigma) = \sqrt{2} \cos(n\sigma), \]  

(5.2.2)

is complete and orthogonal under the inner product

\[ \int_0^\pi \frac{d\sigma}{\pi} u_n(\sigma) u_m(\sigma) = \delta_{nm}, \quad \frac{1}{\pi} \sum_{n=0}^{\infty} u_n(\sigma) u_n(\sigma') = \delta(\sigma - \sigma'). \]  

(5.2.3)

The creation and annihilation operators \( a_n = 1/\sqrt{2} \left( \frac{p_n}{\sqrt{\omega_n}} - i\sqrt{\omega_n} x_n \right), \quad a_n^\dagger = 1/\sqrt{2} \left( \frac{p_n}{\sqrt{\omega_n}} + i\sqrt{\omega_n} x_n \right), \) obey the commutation relations \([a_n, a_m^\dagger] = \delta_{nm}\), and the inverse relations are

\[ x_n = \frac{i}{\sqrt{2} \sqrt{\omega_n}} (a_n - a_n^\dagger), \quad p_n = \frac{1}{\sqrt{2} \sqrt{\omega_n}} (a_n + a_n^\dagger). \]  

(5.2.4)

(5.2.5)

By taking \( \omega_n = n \) the Hamiltonian gets the canonical form

\[ H = \frac{1}{2} \int \frac{d\sigma}{\pi} \left( 2\alpha' P^2(\sigma) + \frac{X^2(\sigma)}{2\alpha'} \right) = \frac{1}{2} \sum_{n=0}^{\infty} (p_n^2 + n^2 x_n^2) = \frac{p_0^2}{2} + \sum_{n=1}^{\infty} n(a_n^\dagger a_n + 1/2). \]  

(5.2.6)

The normal ordering constant will henceforth be omitted.

The transformation to the continuous \( \kappa \) basis [39] is given by

\[ a_\kappa = \sum_{n=1}^{\infty} v_\kappa^n a_n, \]  

(5.2.7)

where in contrast to our conventions in section 3 and 4 (see eq. (3.3.4)) we includes the normalization factor \( N(\kappa) \) of eq. (3.3.7) in the definition of the generating function and define \( v_\kappa^n \) accordingly

\[ f_\kappa(z) = \sum_{n=1}^{\infty} v_\kappa^n z^n = \frac{1}{\sqrt{N(\kappa)}} \frac{1 - e^{-\kappa \tan^{-1} z}}{\kappa}. \]  

(5.2.8)
For this definition the completeness and orthogonality relations of eq. (3.3.6) take the canonical form

$$\sum_{n=1}^{\infty} v_n^\kappa v_n^{\kappa'} = \delta(\kappa - \kappa'), \quad \int d\kappa v_n^\kappa v_m^{\kappa} = \delta_{nm} .$$  \tag{5.2.9}

In this basis $K_1$ is diagonal and the three-vertex is simplified. It is an orthogonal change of basis for the creation and annihilation operators that does not alter the vacuum $|\Omega\rangle$ which is annihilated by all $a_n$ and therefore by all $a_\kappa$. The change to the continuous basis does not involve the zero mode, which remains a discrete degree of freedom. (A different approach is to diagonalize the three-vertex with the zero-mode [46, 47, 67].)

From the above transformation law we deduce the transformation of the coordinates

$$x^\kappa = i \frac{1}{\sqrt{2}} \frac{1}{\omega_\kappa} (a_\kappa - a_\kappa^\dagger) = i \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} v_n^\kappa (a_n - a_n^\dagger) = \frac{1}{\sqrt{2\omega_\kappa}} \sum_{n=1}^{\infty} v_n^\kappa \sqrt{n} x_n ,$$

$$x_n = \frac{1}{\sqrt{n}} \int d\kappa v_n^\kappa \sqrt{\omega_\kappa} x_\kappa . \tag{5.2.10}$$

The eigenstate of the coordinate $x^\kappa$ is

$$|x^\kappa\rangle = \left(\frac{\omega_\kappa}{\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{1}{2} \omega_\kappa (x^\kappa)^2 - i\sqrt{2\omega_\kappa} x^\kappa a_\kappa^\dagger + \frac{1}{2} a_\kappa^{1\dagger}\right) |\Omega\rangle . \tag{5.2.11}$$

For $\omega_n$ there is a canonical choice $\omega_n = n$, for which the states generated by acting with creation operators on the vacuum are eigenstates of the Hamiltonian. For $\omega_\kappa$ there is no such canonical choice. Therefore, we keep this factor explicit in this section, and in the following sections we set $\omega_\kappa = 1$. The basis for $X(\sigma)$ is found from the auxiliary calculation

$$\frac{X(\sigma)}{\sqrt{2\alpha'}} = \sum_{n=0}^{\infty} x_n u_n(\sigma) = x_0 + \int d\kappa x^\kappa \sqrt{\omega_\kappa} \sum_{n=1}^{\infty} v_n^\kappa u_n(\sigma) \equiv x_0 + \int d\kappa x^\kappa u_\kappa(\sigma) . \tag{5.2.12}$$

The generating function can be used to obtain an explicit expression for the new basis,

$$u_\kappa(\sigma) = \sqrt{\omega_\kappa} \sum_{n=1}^{\infty} \frac{v_n^\kappa}{\sqrt{n}} u_n(\sigma) = \sqrt{\omega_\kappa} \sum_{n=1}^{\infty} \frac{v_n^\kappa}{\sqrt{n}} \sqrt{2} \Re e^{in\sigma} = \sqrt{2\omega_\kappa} \Re f_\kappa(e^{i\sigma}) , \tag{5.2.13}$$
These basis functions are not orthogonal and in (4.4.6) their inner product was found to give the metric

\[
g_{\kappa\kappa'} = \int \frac{d\sigma}{\pi} u_{\kappa}(\sigma) u_{\kappa'}(\sigma) = \sqrt{\omega_{\kappa}\omega_{\kappa'}} \sum_{n=1}^{\infty} \frac{1}{n} v^\kappa_n v^\kappa' =
\]

\[
\sqrt{\omega_{\kappa}\omega_{\kappa'}} \frac{2 \sinh(\frac{\kappa \pi}{4}) \sinh(\frac{\kappa' \pi}{4})}{\sqrt{N(\kappa)N(\kappa')\kappa\kappa' \cosh(\frac{(\kappa-\kappa')\pi}{4})}} = \sqrt{\omega_{\kappa}\omega_{\kappa'}} \frac{\tanh(\frac{\kappa \pi}{4}) \tanh(\frac{\kappa' \pi}{4})}{2 \sqrt{\kappa\kappa'} \cosh(\frac{(\kappa-\kappa')\pi}{4})}.
\]  

(5.2.14)

Repeating the calculation for the conjugate momenta gives

\[
p_{\kappa} = \sqrt{\omega_{\kappa}} \sum_{n=1}^{\infty} \frac{v^\kappa_n}{\sqrt{n}} p_n,
\]

(5.2.15)

\[
u^\kappa(\sigma) = \frac{1}{\sqrt{\omega_{\kappa}}} \sum_{n=1}^{\infty} \sqrt{n} v^\kappa_n(\sigma) = \frac{1}{\sqrt{\omega_{\kappa}}} \sqrt{2\pi} z \frac{d}{dz} f_n(z) \bigg|_{z=e^{i\sigma}}.
\]

(5.2.16)

The (very singular) inverse metric was calculated in eq. (4.4.3)

\[
g^{\kappa\kappa'} = \int \frac{d\sigma}{\pi} u^\kappa(\sigma) u^{\kappa'}(\sigma) = \frac{1}{\sqrt{\omega_{\kappa}\omega_{\kappa'}}} \sum_{n=1}^{\infty} n v^\kappa_n v^{\kappa'}
\]

\[
= \frac{\sqrt{\kappa\kappa'}}{4\sqrt{\omega_{\kappa}\omega_{\kappa'}}} (\delta(\kappa - \kappa' + 2i) + \delta(\kappa - \kappa' - 2i)).
\]

(5.2.17)

This expression is very useful since it gives the expression for the Hamiltonian in the \( \kappa \) basis

\[
H = \sum_{n=1}^{\infty} n a^\dagger_n a_n = \int d\kappa d\kappa' \sum_{n=1}^{\infty} n v^\kappa_n v^{\kappa'} a^\dagger_n a_n
\]

\[
= \int d\kappa d\kappa' \sqrt{\omega_{\kappa}\omega_{\kappa'}} g^{\kappa\kappa'} a^\dagger_n a_n,
\]

(5.2.18)

which is \( L_0 \) and is also the matter part of \( Q_B \) in the Siegel gauge.

Finally, we write the three-vertex in the continuous basis

\[
|V_3(P_0)\rangle = \delta(p_0^1 + p_0^2 + p_0^3) \exp \left( -\frac{1}{2} A^\dagger V_3 A^\dagger + P_0 V_0 A^\dagger - \frac{1}{2} V_0 V_0^\dagger \right) |\Omega\rangle_{123},
\]

(5.2.19)

where \( A^\dagger, V_3, V_0, P_0 \) are tensors on the three Fock spaces

\[
A^\dagger = \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \\ a_3^\dagger \end{pmatrix}, \quad V_3 = \begin{pmatrix} V_{11}^3 \\ V_{12}^3 \\ V_{21}^3 \\ V_{21}^3 \\ V_{12}^3 \\ V_{11}^3 \\ V_{21}^3 \\ V_{12}^3 \end{pmatrix}, \quad V_0 = \begin{pmatrix} V_{11}^0 \\ V_{12}^0 \\ V_{21}^0 \\ V_{21}^0 \\ V_{12}^0 \\ V_{11}^0 \\ V_{21}^0 \end{pmatrix}, \quad P_0 = \begin{pmatrix} p_0^1 \\ p_0^2 \\ p_0^3 \end{pmatrix},
\]

(5.2.20)
and an integration over $\kappa$ is assumed. It is convenient to pair up creation operators which have the same $(K_1)^2$ eigenvalue
\[
a^\dagger = \begin{pmatrix} a^\dagger_{-\kappa} \\ a^\dagger_{\kappa} \end{pmatrix}.
\] (5.2.21)

In this notation we choose $\kappa > 0$, while $\kappa = 0$ has to be treated separately.

The three-vertex coefficients of $V_3$ are given by eq. (3.2.10) while those of $V_0$ are
\[
V_0^{11} = \sqrt{2\alpha'} \frac{1}{3} \sqrt{N(\kappa)} \tanh \frac{\kappa \pi}{4} \begin{pmatrix} 1 \\ -1 \end{pmatrix},
\]
\[
V_0^{12} = \sqrt{2\alpha'} \frac{1}{2} \sqrt{N(\kappa)} \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \frac{1}{2} V_0^{11},
\] (5.2.22)
\[
V_0^{21} = \sqrt{2\alpha'} \frac{1}{2} \sqrt{N(\kappa)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} V_0^{11},
\]
\[
V_{00} = (2\alpha') \frac{1}{2} \log \left( \frac{27}{16} \right).
\]

From the form of the three-vertex, it can be seen that squeezed states, whose defining matrix is block diagonal with two-by-two blocks, which mix $a_\kappa$ with $a_{-\kappa}$, form a subalgebra. This is the $H_{\kappa^2}$ subalgebra of [55].

For $\kappa = 0$ the only contribution to the three-vertex comes from
\[
|V_3^{\kappa=0}\rangle = \exp \left( -\frac{1}{2} A^\dagger_{\kappa=0} V_3^{\kappa=0} A^\dagger_{\kappa=0} \right) |\Omega\rangle, \quad V_3^{\kappa=0} = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}.
\] (5.2.23)

In the functional basis, using (5.2.11) this gives a factor of
\[
V_3^{\kappa=0}(x_{1,2,3}) = \langle x_1^{\kappa=0} | \langle x_2^{\kappa=0} | \langle x_3^{\kappa=0} | V_3^{\kappa=0} \rangle = \frac{\sqrt{3}}{2} \left( \frac{\omega_0}{\pi} \right)^{1/4} \delta(x_1^{\kappa=0} + x_2^{\kappa=0} + x_3^{\kappa=0}).
\] (5.2.24)

The $\kappa = 0$ eigenvalue may seem to be negligible, since it is a single point in a continuous set of states. However, there are many calculations in the literature in which the opposite is true. In [68] this effect was called twist anomaly. A careful treatment of this twist anomaly gave the correct ratio of D-brane tensions [42] and the correct tachyon mass around VSFT’s D-brane [69].
The importance of $\kappa = 0$ can also be seen in the Moyal formalism. Without it, $L_0$ has the wrong spectrum. Instead of having eigenstates for all integer eigenvalues, one gets eigenstates with even integer eigenvalues only, which are doubly degenerate. This is the spectrum of two strings with half the length of the original string. Only when regularizing $L_0$ to include $\kappa = 0$, the correct spectrum is obtained [70]. Also, when disregarding $\kappa = 0$, it turns out that the butterfly solves the equation of motion of string field theory in the Siegel gauge. This is due to the fact that the butterfly state is the vacuum state of the two halves of the string and without $\kappa = 0$, string field theory essentially becomes a free theory for the two halves of the string. Ignoring $\kappa = 0$ in the Lagrangian allows for an arbitrary value for $x_{\kappa=0}$, giving string configurations in which the string splits in the middle. However, for most of our calculations in this chapter the $\kappa = 0$ mode will be irrelevant.

### 5.2.2 Half-string representation

The half-string representation was suggested already in [13] as a convenient representation for string field theory. We split the string into left and right degrees of freedom

$$X(\sigma) = l(\sigma) + r(\sigma) + x_m, \quad (5.2.25)$$

where $r(\sigma) \equiv r(\pi - \sigma)$, so that both $l(\sigma)$ and $r(\sigma)$ are defined between 0 and $\pi/2$ and vanish between $\pi/2$ and $\pi$. The mid-point degree of freedom $x_m$, appears because we chose Dirichlet boundary condition for $l(\sigma), r(\sigma)$ at the mid-point.

The integration and star product can now be written as

$$\int \Psi = \int Dy(\sigma) dx_m \Psi(y, y, x_m), \quad (5.2.26)$$

$$(\Psi_1 \star \Psi_2)(l, r, x_m) = \int Dy(\sigma) \Psi_1(l, y, x_m) \Psi_2(y, r, x_m). \quad (5.2.27)$$

In this representation the string field can be treated as a matrix over the space of half-strings, up to mid-point issues.

In [60] string field theory was formulated by expanding the half-string in
modes

\[
l(\sigma) = \sqrt{2} \sum_{n=1}^{\infty} l_n \cos(2n - 1) \sigma, \tag{5.2.28}
\]

\[
r(\sigma) = \sqrt{2} \sum_{n=1}^{\infty} r_n \cos(2n - 1) \sigma. \tag{5.2.29}
\]

We relate creation and annihilation operators to these modes as in the case of the full-string modes (5.2.4). Note that the vacuum state \(|\Omega_h⟩\), which is annihilated by all the half-string annihilation operators \(a_{l,r}^{n}\), is not the same as the vacuum state \(|\Omega⟩\) annihilated by the full-string oscillators \(a_n\).

The three-vertex has a simple functional form in this basis,

\[
V_h^3(l_1^{1,2,3}, r_1^{1,2,3}, x_m^{1,2,3}) = \delta(x_1^1 - x_m^2)\delta(x_2^2 - x_m^3) \prod_n \delta(r_n^1 - l_n^2)\delta(r_n^2 - l_n^3)\delta(r_n^3 - l_n^1),
\]

and we can use the state formalism (5.2.11) to calculate the three-vertex state

\[
|V_h^3(p_1^{1,2,3})⟩ = 3 \prod_{s=1}^{3} \int dx_m^s e^{-i \sum_{s=1}^{3} p_m^s x_m^s} \prod_{n=1}^{\infty} dl_n^s dr_n^s V_h^3(l_1^{1,2,3}, r_1^{1,2,3}, x_m^{1,2,3}) |l_n^s, r_n^s⟩\]

\[
= \delta(p_1^1 + p_2^2 + p_3^3) \exp \left( -\frac{1}{2} \sum_{s,t=1}^{3} \sum_{n,m=1}^{\infty} (a_{h}^{s})_n^m (V_h^3)^{st}_{nm} (a_{h}^{t})_m^n \right) |Ω_h⟩_{123},
\]

where \(a_{hn} = (a_{ln}, a_{rn})\), and \(V_h^3\) is composed of the matrices

\[
(V_h^3)^{11}_{nm} = (V_h^3)^{22}_{nm} = (V_h^3)^{33}_{nm} = 0, \quad (V_h^3)^{00}_{nm} = 0,
\]

\[
(V_h^3)^{12}_{nm} = (V_h^3)^{23}_{nm} = (V_h^3)^{31}_{nm} = 0, \quad (V_h^3)^{0\delta}_{nm} = \delta_{nm}, \quad (V_h^3)^{\delta0}_{nm} = 0,
\]

\[
(V_h^3)^{21}_{nm} = (V_h^3)^{32}_{nm} = (V_h^3)^{13}_{nm} = 0, \quad (V_h^3)^{\delta0}_{nm} = \delta_{nm}.
\]

Notice that the vertex in (5.2.30) multiplies states of the form \(Ψ[l, r] = ⟨l, r|Ψ⟩\).

Therefore, to calculate \(|V_h^3⟩\), we used the BPZ conjugate of the vertex

\[
bpz (V_h^3(l_1^{1,2,3}, r_1^{1,2,3}, x_m^{1,2,3})) = V_h^3(l_1^{1,2,3}, r_1^{1,2,3}, x_m^{1,2,3}). \tag{5.2.33}
\]
5.3 Continuous half-string representation

In this section we derive the form of the vertex in the continuous half-string representation. We start with the \( p = 0 \) matter sector. We then add the zero mode, which is replaced by the mid-point degree of freedom. Finally we discuss the ghost sector, which enables us to calculate the normalization of the vertices.

5.3.1 Zero momentum

In [60] the half-string representation was described using discrete half-string oscillators. We suggest using a continuous set of oscillators. We use the fact that the form of the three-vertex for the half-string should always be as in eq. (5.2.30). Eq. (5.2.31),(5.2.32) then imply that the half-string vacuum state always obeys

\[
|\Omega_h \star \Omega_h\rangle_3 = i\langle \Omega_h | 2\langle \Omega_h | |V_3^h\rangle_{123} = \exp\left(-\frac{1}{2}a_3^{\dagger}V^{33}a_3\right) |\Omega_h\rangle_3 = |\Omega_h\rangle_3,
\]

meaning that this vacuum state is a projector. Here we assumed that the vacuum state is BPZ real, meaning \(|\Omega_h\rangle = \langle \Omega_h |\). We can now find half-string representations using the fact that every squeezed state projector describes a Bogoliubov transformation into a half-string basis (see appendix A for details on the Bogoliubov transformation). For a given squeezed state matrix \( S \), we can find a matrix \( W \) that satisfies

\[
W^{\dagger}W = (1 - SS^*)^{-1}
\]

If \( S \) describes a projector, then the set of operators

\[
a^h_n = W_{nm}a_m + U_{nm}a^\dagger_m, \quad a^{h\dagger}_n = W^*_{nm}a^\dagger_m + U^*_{nm}a_m
\]

where \( U = WS \), describe oscillators of the half-string for which the squeezed state is the new vacuum. Remember that \( S \) defines \( W \) only up to a unitary transformation of the modes. Therefore, this half-string basis could contain a mixture of left and right modes, meaning that we still do not have left-right factorization.

The representation of any squeezed state in the half-string basis can be obtained using the expressions in A.2, where we think of the squeezed state as
defining another Bogoliubov transformation. The three-vertex is a (singular) squeezed state in a tripled tensor space of oscillators. Therefore, the three-vertex in the new half-string basis is

\[
V^h_3 = \begin{pmatrix}
V^h_{11} & V^h_{12} & V^h_{21} \\
V^h_{21} & V^h_{11} & V^h_{12} \\
V^h_{12} & V^h_{21} & V^h_{11}
\end{pmatrix}
= (W^\dagger_3 - V_3 U^\dagger_3)^{-1}(V_3 W^T_3 - U^T_3),
\]

(5.3.3)

where \( W_3 = 1_3 \otimes W \) and \( U_3 = 1_3 \otimes U \).

For states in \( \mathcal{H}_{k^2} \) [55], we can calculate \( V^h_3 \) explicitly. A general state in \( \mathcal{H}_{k^2} \) is a squeezed state of the form \( \exp(-\frac{1}{2}a^\dagger S a^\dagger) |\Omega\rangle \), where \( a^\dagger \) is defined in (5.2.21). If the state is BPZ real then \( S \) can be represented by the matrix

\[
S = \begin{pmatrix}
s_1 + is_3 & s_2 \\
s_2 & s_1 - is_3
\end{pmatrix} = s_1 \mathbf{1} + s_2 \sigma_1 + is_3 \sigma_3,
\]

(5.3.4)

where \( \sigma_i \) are the Pauli matrices and \( s_i \) are real.

\( W \) is defined through (A.1.4) up to a multiplication by unitary matrix. We choose to work with a Hermitian \( W \). There are still four such choices related to the choice of the sign of the two eigenvalues. We choose \( W \) with positive eigenvalues, as in (A.1.6). To calculate \( W \) we write

\[
1 - SS^* = A(1 \cosh \alpha + \hat{\sigma} \sinh \alpha) = A \exp(\alpha \hat{\sigma}),
\]

(5.3.5)

where

\[
A^2 = (1 - s_1^2 - s_2^2 - s_3^2)^2 - 4s_2^2(s_1^2 + s_3^2),
\]

\[
\hat{\sigma} = \frac{s_1 \sigma_1 - s_3 \sigma_2}{\sqrt{s_1^2 + s_3^2}} \quad (\sigma^2 = 1),
\]

(5.3.6)

\[
\alpha = \tanh^{-1}\left(\frac{-2s_2 \sqrt{s_1^2 + s_3^2}}{1 - s_1^2 - s_2^2 - s_3^2}\right),
\]

and we get

\[
W = (1 - SS^*)^{-1/2} = A^{-1/2}(1 \cosh \frac{\alpha}{2} - \hat{\sigma} \sinh \frac{\alpha}{2}).
\]

(5.3.7)

Taking a state \( S \), which is a projector, we get for the transformed three-vertex

\[
V^h_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad V^h_{12} = - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad V^h_{21} = - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

(5.3.8)
From the form of $V_3^h$ we deduce that our choice of Hermitian $W$ gives the correct half-string factorization,

$$a^h_\kappa = \begin{pmatrix} a^\dagger_\kappa \\ a^\dagger_\kappa \end{pmatrix} = W \begin{pmatrix} a_\kappa \\ a_\kappa \end{pmatrix} + U \begin{pmatrix} a^\dagger_{-\kappa} \\ a^\dagger_{-\kappa} \end{pmatrix}. \quad (5.3.9)$$

Moreover, the condition on $S$ that the three-vertex will be of the form (5.3.8) is exactly the condition found in [55] for $S$ to be a projector. We do not get the identity solution $S = C$ this way, because it is of infinite rank, while only rank one projectors have left–right factorization.

The sign of the $V_3^h$ matrices is opposite to that of the usual half-string vertex (5.2.32). This sign can be reversed by taking $W$ with one positive and one negative eigenvalue, but we chose to work with the positively defined $W$. In this convention we get that the twist matrix transforms to itself for any real $S$,

$$C_h = (W - CU)^{-1}(CW - U) = C = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (5.3.10)$$

We can get the three-vertex in the functional form using the expression for eigenstates of the coordinate $x^\kappa$ (5.2.11), where our coordinates are $l^\kappa, r^\kappa$. The three-vertex becomes

$$V_3^h(l^\kappa_{1,2,3}, r^\kappa_{1,2,3}) = \delta(r^\kappa_1 + l^\kappa_2)\delta(r^\kappa_2 + l^\kappa_3)\delta(r^\kappa_3 + l^\kappa_1), \quad (5.3.11)$$

where a product over all continuous values of $\kappa > 0$ is assumed. The signs in this three-vertex functional are not the standard signs used as a result of the different sign in the three-vertex state. We will see that, nevertheless, they give the correct gluing of the strings.

Our results are true for all projectors, but for the sake of simplicity, we focus on the sliver and the butterfly

$$S_s = e^{-\frac{\kappa\pi}{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_b = \frac{1}{2\cosh(\frac{\kappa\pi}{2})} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$W_s = \frac{1}{\sqrt{1 - e^{-\kappa\pi}}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W_b = \frac{1}{2\sinh(\frac{\kappa\pi}{2})} \begin{pmatrix} e^{\frac{\kappa\pi}{2}} & e^{-\frac{\kappa\pi}{2}} \\ e^{-\frac{\kappa\pi}{2}} & e^{\frac{\kappa\pi}{2}} \end{pmatrix}, \quad (5.3.12)$$

$$U_s = \frac{e^{-\frac{\kappa\pi}{2}}}{\sqrt{1 - e^{-\kappa\pi}}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U_b = \frac{1}{2\sinh(\frac{\kappa\pi}{2})} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (5.3.12)$$
We can use the relation between the Bogoliubov transformation and coordinate transformation \((A.3.2)\) to get the shape of the half-string modes.

\[
\tilde{u}_h^h(\sigma) = \begin{pmatrix} \tilde{u}_l^l(\sigma) \\ \tilde{u}_r^r(\sigma) \end{pmatrix} = (W + U) \begin{pmatrix} u_{-\kappa}(\sigma) \\ u_\kappa(\sigma) \end{pmatrix},  \tag{5.3.13}
\]

\[
\tilde{u}_h^c(\sigma) = \begin{pmatrix} \tilde{u}_l^c(\sigma) \\ \tilde{u}_r^c(\sigma) \end{pmatrix} = (W - U) \begin{pmatrix} u^{-\kappa}(\sigma) \\ u^\kappa(\sigma) \end{pmatrix}.  \tag{5.3.14}
\]

The tilde over \(u_h^\kappa(\sigma)\) marks that this result does not include the zero mode, which will be treated in the next section. A direct analysis shows that \(\tilde{u}_l^l(\sigma)\), which is related to the left half-string shape in position space is constant for \(\sigma > \pi/2\).

### 5.3.2 The zero-mode

In all the calculations so far, the zero-mode was ignored by setting \(p = 0\). In this section we reinstate it in the half-string formalism. The method of calculating the three-vertex is the same, we treat it as a Bogoliubov transformation. The difference is that now, in addition to the term quadratic in creation operators, we have a linear term which depends on the zero-mode momenta. We treat the momenta as parameters of a shifted Bogoliubov transformation. The calculation results in a diverging coefficient for the \(p_0^2\) term.

We can avoid this divergence by a change of variables from the zero-mode to the mid-point. The mid-point degree of freedom is defined as

\[
x_m = \frac{X(\pi/2)}{\sqrt{2\alpha'}} = x_0 + \sum_{n=1}^\infty J_n x_n = x_0 + \int d\kappa J_\kappa x_\kappa  \tag{5.3.15}
\]

\[
J_{2n} \equiv \sqrt{2}(-1)^n \quad J_{2n+1} \equiv 0  \tag{5.3.16}
\]

\[
J_\kappa = \sum_{n=1}^\infty \frac{\nu_\kappa^2}{\sqrt{n}} J_n = \sqrt{2}\Re f_\kappa(i) = \mathcal{P} \frac{\sqrt{2}}{\kappa \sqrt{N(\kappa)}}, \tag{5.3.17}
\]

where \(J_\kappa\) should be regarded as a distribution. We transform \(J_\kappa\) to the half-string basis using \((5.3.13)\)

\[
J^h_\kappa = \begin{pmatrix} J_l^h \\ J_r^h \end{pmatrix} = (W + U) \begin{pmatrix} J_{-\kappa} \\ J_\kappa \end{pmatrix}.  \tag{5.3.18}
\]
In the case of the butterfly basis we get that $J^h_κ = J_κ$. $J^h_κ$ is exactly the constant value of $u^h_κ(σ)$

$$
\tilde{u}^l_κ(π/2 < σ < π) = J^l_κ \quad \tilde{u}^r_κ(0 < σ < π/2) = J^r_κ. \quad (5.3.19)
$$

The expansions of the string configuration around the zero-mode and around the mid-point are

$$
\frac{X(σ)}{\sqrt{2α'}} = x_0 + \int dκ (l^κ \tilde{u}^l_κ(σ) + r^κ \tilde{u}^r_κ(σ)) = x_m + \int dκ (l^κ u^l_κ(σ) + r^κ u^r_κ(σ)), \quad (5.3.20)
$$

where $u^h_κ(σ) = \tilde{u}^h_κ(σ) - J^h_κ$. We find the related Bogoliubov transformation from the transformation of the coordinates

$$
\left( \begin{array}{c}
  x_m \\
  h^κ 
\end{array} \right) = \left( \begin{array}{cc}
  1 & J^T_κ \\
  0 & W - U
\end{array} \right) \left( \begin{array}{c}
  x_0 \\
  x^κ 
\end{array} \right) \equiv M \left( \begin{array}{c}
  x_0 \\
  x^κ 
\end{array} \right). \quad (5.3.21)
$$

We use a vector notation where the first entry represents a single degree of freedom (the mid-point or the zero mode) and the second entry represents a continuous infinite set of degrees of freedom (left and right of the half-string or $±κ$ of the full-string). $M$ is an infinite matrix, but it can be easily inverted since we have $(W - U)^{-1} = W + U$ and $J^h_κ = (W + U)J_κ$, so that

$$
M^{-1} = \left( \begin{array}{cc}
  1 & -J^T_κ \\
  0 & W + U
\end{array} \right). \quad (5.3.22)
$$

The transpose of this matrix transforms the momenta and specifically, for the mid-point momenta, we get $p_m = p_0$. The transformation of the creation operators is then

$$
a^\dagger_h = Wa^\dagger_κ + Ua_κ - \frac{1}{\sqrt{2}} J^h_κ p_0. \quad (5.3.23)
$$

We can now use the results of A.2 to transform the three-vertex (5.2.19). Setting $V = V_3$, $μ = P_0 V_0$ we find the three-vertex of the continuous half-string including the mid-point,

$$
|V^h_3(P_m)⟩ = \delta(p^l_m + p^2_m + p^3_m) \exp \left( -\frac{1}{2} V_0 P_m^2 + \frac{1}{2} μ^T (1 + V_3)^{-1} μ \right) \cdot \exp \left( -\frac{1}{2} μ^T (1 + V^h_3)^{-1} μ + μ A^\dagger_h - \frac{1}{2} A^\dagger_h V^h_3 A^\dagger_h \right)|Ω_h⟩_{123}. \quad (5.3.24)
$$
where \( A_h = (a^1_h, a^2_h, a^3_h) \), \( P_m = (p^1_m, p^2_m, p^3_m) \). A direct calculation shows that after applying the \( \delta \)-function over the momenta, we get \( \hat{\mu} = 0 \) as expected. In the derivation of (5.3.24) we have replaced

\[
(1 - V_3)(1 - V_3^2)^{-1} = (1 + V_3)^{-1}.
\]

This is supposedly forbidden since \( V_3 \) has the eigenvalue 1 in its spectrum. The result also seems meaningless because the other eigenvalue of \( V_3 \) is \(-1\). However, \( \mu \) turns out to be an eigenvector of \( V_3 \) with the eigenvalue 1. (Notice that \( \mu \) is \( P_0 \)-dependent, but it is an eigenvector of \( V_3 \) for all values of \( P_0 \).) We are left with the term \( \frac{1}{4} \mu^T \mu \), which can be easily calculated in the \( \kappa \) basis, and after applying the \( \delta \) function over the momenta, this term exactly cancels \( V_{00} \) so that we are left with the expected three-vertex

\[
|V^h_3(P_m)\rangle = \delta(p^1_m + p^2_m + p^3_m) \exp \left(-\frac{1}{2}A^\dagger_h V^h_3 A_h \right) |\Omega_h\rangle_{123}.
\]

5.3.3 The ghost sector

The ghost sector of string field theory is more complicated than the matter sector, due to the mid-point insertion required in the star product and integration. The origin of this insertion is the singularity in the world-sheet metric. It is easier to write the star-product in the bosonized ghost sector as was done originally in [13]. The insertion is naturally defined in the half-string formulation, where the mid-point is one of the basic degrees of freedom.

The mode expansion of the bosonized ghost \( \phi(\sigma) \) is the same as the matter coordinates with the exception that the zero point momentum is discretized to half-integer values. This momentum gives the ghost number of the state by \( N_G = \hat{p}_0 + 3/2 \), where \( \hat{p}_0 \) is anti-Hermitian and it satisfies the following relations with its eigenstate

\[
\hat{p}_0 |p_0\rangle = p_0 |p_0\rangle, \quad \langle p_0 | \hat{p}_0 = -p_0 \langle p_0 |,
\]

\[
\langle p_0 | p'_0 \rangle = \delta_{p_0 + p'_0}, \quad \sum_{p_0} |p_0\rangle \langle -p_0| = 1, \quad (|p_0\rangle)^\dagger = |p_0\rangle.
\]

The star product and the integration in the ghost sector with the insertion
at the mid-point \( \phi_m = \phi(\pi/2) \) can be written as

\[
(\Psi_1 * \Psi_2)(l^\kappa, r^\kappa, \phi_m) = e^{\frac{3i}{2} \phi_m} \int D y^\kappa \Psi_1(l^\kappa, y^\kappa, \phi_m) \Psi_2(-y^\kappa, r^\kappa, \phi_m),
\]

\[
\int \Psi = \int D y^\kappa d\phi_m e^{-\frac{3i}{2} \phi_m} \Psi(y^\kappa, -y^\kappa, \phi_m),
\]

\[
(5.3.28)
\]

To see that the integral vanishes unless \( \Psi \) has ghost number 3, we write a state with momentum \( p_0 \) as \( \langle \phi_0 | \Psi, p_0 \rangle = \exp(i p_0 \phi_0) | \Psi \rangle \). The relation between \( \phi_0 \) and \( \phi_m \) is as in (5.3.15), implying that the integral vanishes unless \( p_0 = \frac{3}{2} \), i.e. \( N_G = 3 \). For the same reason the zero mode momentum obeys \( p_{03} = p_{01} + p_{02} + \frac{3}{2} \) under the star product and therefore the ghost number is additive. This leads to the known facts that solutions of the projection equation must have ghost number zero, and solutions of string field theory must have ghost number one.

The vertices states in the continuous half-string basis are of the form

\[
|V_N\rangle = \gamma_N \exp \left( -\frac{1}{2} a_h^s \dagger V_{Nst} a_h^s \right) \sum_{p_m^s} \delta_{p_m^s - \frac{3}{2} (N-2)} \prod_{r=s}^{N} |\Omega, p_m^s\rangle.
\]

\[
(5.3.29)
\]

The mid-point momenta \( p_m^s \) are the relevant degrees of freedom for the half-string, but they are equal to the zero-mode momenta \( p_m = p_0 \). The normalization factors \( \gamma_N \) can be calculated using the fact that the overlap of two surface states should be 1 [58]. We use the vacuum state, which in the continuous half-string basis is

\[
|\Omega, p_0\rangle = N_\Omega e^{\mu_h a_h^s \dagger S a_h^s} |\Omega_h, p_m\rangle,
\]

\[
\mu_h = -\frac{1}{\sqrt{2}} (-1 + S) \sqrt{\kappa} p_m, \quad N_\Omega = \det(1 - S^2) \frac{(D+1)}{4} e^{-\frac{1}{2} \mu_h T (1-S)^{-1} \mu_h},
\]

\[
(5.3.30)
\]

where we used the transformation (5.3.23). The vacuum state has a ghost number zero, meaning \( p_0 = -\frac{3}{2} \), but to calculate overlaps we also need the shifted vacuum state with \( p_0 = \frac{3}{2} \). The overlap of the vacuum with the identity (the one-vertex) is

\[
\langle \Omega, \frac{3}{2} | I \rangle = \gamma_1 N_\Omega \det(1 + CS)^{-\frac{D+1}{2}} e^{-\frac{1}{2} \mu_h T (1-CS)^{-1} C \mu_h}.
\]

\[
(5.3.31)
\]

There are two types of divergences in this expression. One comes from the determinants and the other comes from the exponents. For the determinants,
we use the relation det $M = \exp \text{tr} \log M$. If the matrix $M$ is diagonal in the $\kappa$ basis with eigenvalues $m_\kappa$, we have $\log M = \int d\kappa \rho(\kappa) \log m_\kappa$, where the spectral density $\rho(\kappa)$ diverges. In level truncation it diverges as $\frac{\log L}{2\pi}$. In 4.2 the finite contribution was calculated

$$\rho^L(\kappa) = \frac{1}{2\pi} \sum_{n=1}^{L/2} \frac{1}{n} + \rho^L_{\text{fin}}(\kappa),$$

and

$$\rho_{\text{fin}}(\kappa) = \lim_{L \to \infty} \rho^L_{\text{fin}}(\kappa) = \frac{4 \log(2) - 2\gamma - \Psi\left(\frac{\kappa}{2}\right) - \Psi\left(-\frac{\kappa}{2}\right)}{4\pi}. \quad (5.3.32)$$

The divergence in the exponents comes from the scalar product in the $\kappa$ basis, which results in integrals of $\frac{1}{\kappa^2}$. This divergence can also be regulated in level truncation by noticing that

$$\lim_{L \to \infty} \sum_{n=1}^{L/2} \frac{1}{n} = \sum_{n,m=1}^{\infty} \frac{J_n J_m}{\sqrt{nm}} \delta_{nm},$$

and

$$e^{-\int_{-\infty}^{\infty} d\kappa \sum_{n=1}^{\infty} \frac{J_n J_n^\kappa}{n} \sum_{m=1}^{\infty} \frac{J_m J_m^\kappa}{m}} = \int_{-\infty}^{\infty} d\kappa J_\kappa J_\kappa = \int_{0}^{\infty} \frac{4d\kappa}{\kappa^2 N(\kappa)}. \quad (5.3.33)$$

Therefore, the two types of divergences can be compared.

We now have everything we need to calculate $\gamma_1$ from the requirement that the overlap (5.3.31) equals one

$$\gamma_1 = e^{-\left(D+1\right)\left(\frac{1}{8} \sum \frac{1}{d_i} + d_1\right)} e^{\frac{1}{8} \sum \frac{1}{d_1}}, \quad d_1 = \frac{1}{4} \int_{0}^{\infty} d\kappa \rho_{\text{fin}}(\kappa) \log \left(\coth^2 \frac{\kappa\pi}{2}\right) = 0.15758., \quad (5.3.34)$$

where the first exponent comes from the determinant and the second is the exponent from the ghost sector. We can calculate the normalizations of all the vertices using the relations

$$|V_N\rangle = \langle I|V_{N+1}\rangle \Rightarrow \gamma_N = \gamma_1 \gamma_{N+1} \Rightarrow \gamma_N = \gamma_1^{2-N}. \quad (5.3.35)$$

Next, we do a couple of calculations, to check the validity of our results. Repeating the calculation for the reflector (the two-vertex) gives

$$1 \langle \Omega, \frac{3}{2} | 2 \langle \Omega, -\frac{3}{2} | R \rangle_{12} = \gamma_2 N_1^2 \text{det}(1 + V_2 S_2)^{-\frac{D+1}{2}} e^{-\frac{1}{2} \frac{\mu}{\nu}^T (1-S_2 V_2)^{-1} V_2 \mu/2} = 1, \quad (5.3.36)$$
where $S_2 = 1 \otimes S, \mu_{2h} = (\mu_h^1, \mu_h^2)$. Here we get $\gamma_2 = 1$ independently of the number of dimensions $D$, we are working in. This result is analogous to the calculation of the overlap of the vacuum state with itself

$$|\Omega, p_0\rangle = \langle \Omega, p_0| \mathcal{R}\rangle \Rightarrow \langle \Omega, \frac{3}{2} | \Omega, -\frac{3}{2}\rangle = \langle \Omega, \frac{3}{2} | \langle \Omega, -\frac{3}{2} | \mathcal{R}\rangle = 1.$$  \hspace{1cm} (5.3.37)

Next we calculate explicitly the normalization of the three-vertex

$$1\langle \Omega, \frac{3}{2} | 2 \langle \Omega, -\frac{3}{2} | 3 \langle \Omega, -\frac{3}{2} | V_3\rangle_{123} = \gamma_3 N_3^\Omega \text{det}(1 + V_3 S_3)^{-\frac{D+3}{2}} e^{-\frac{1}{2} \mu_{2h}^3 (1 - S_3 V_3)^{-1} V_3 \mu_{2h}} = 1$$

$$\Rightarrow \gamma_3 = e^{(D+1)(\frac{3}{14} \sum \frac{1}{\pi} + d_\delta) e^\frac{3}{8} \sum \frac{1}{\pi} + e_3}, \quad e_3 = \frac{3}{2} \log \frac{27}{16},$$

$$d_3 = \frac{1}{4} \int_0^\infty d\kappa \rho_{\text{fin}}(\kappa) \log \left(4 \text{csch} \left(\frac{3\kappa \pi}{4}\right)^4 \sinh \left(\frac{\kappa \pi}{2}\right)^6 \right) = 0.09296 \ldots$$  \hspace{1cm} (5.3.38)

Comparing the diverging terms, we get that the relation $\gamma_3 = \gamma_1^{-1}$ holds only for the critical dimension $D = 26$. Unfortunately, the finite parts do not match. Such a discrepancy also appeared in the calculation of the overlap of the two wedge states $\langle 3 | 3 \rangle$ in [62]. This means that the level truncation regularization we are using does not preserve the Virasoro algebra. This anomaly can come from the $\kappa = 0$ state which we ignored or from the calculation of the determinants. It can also come from using the bosonized ghost. Truncating the fermionic ghost $c =: e^{i\phi}$ : to level $L$ is not the same as truncating the bosonized ghost $\phi$ to same level, which can be seen from their relation $(c)_L = (e^{i\phi}_L :)_L \neq e^{i\phi}_L :$.

We summarize this section by calculating the full-string three-vertex from its definition in the half-string basis (5.3.29) using the bosonized ghost sector. For this we need to use the transformation

$$a_{\kappa}^\dagger = W a_h^\dagger - U a_h + \frac{1}{\sqrt{2}} J_{\kappa} p_m .$$  \hspace{1cm} (5.3.39)

Using again the results of A.2 we get

$$|V_3(P_0)\rangle = \gamma_3 \text{det} \left(1 - \frac{V_3^2}{1 - V_3^2}\right)^{\frac{1}{4}} \delta^{p_0 + p_0^3 + p_0^3 - \frac{3}{2}} \cdot \exp \left(-\frac{1}{2} A_{\kappa}^\dagger V_3 A_{\kappa} + P_0 V_0 A_{\kappa}^\dagger + \frac{1}{2} P_0 V_0 P_0\right) |\Omega\rangle_{123} .$$  \hspace{1cm} (5.3.40)
The vertex is different from the three-vertex of the matter sector, requiring the following redefinition of the three-vertex matrices

\[ V_{3}^{st} \rightarrow V_{3}^{st}, \]
\[ V_{0}^{st} \rightarrow V_{0}^{st} - \frac{\sqrt{2}}{3} J_{\kappa}, \]
\[ V_{00}^{st} \rightarrow V_{00}\delta_{st} + \frac{1}{3} \sum_{n} \frac{1}{n} - \frac{1}{6} \log \frac{27}{16}. \]

(5.3.41)

The vertex of the matter sector (5.2.19) is recovered if we set the momenta conservation relation \( p_{0}^{1} + p_{0}^{2} + p_{0}^{3} = 0 \). The determinant factor cannot be calculated directly, since it is zero divided by zero. We calculate it using the relation between \( V_{3} \) and \( V_{3}^{h} \)

\[ V_{3} = (W_{3} + V_{3}^{h}U_{3})^{-1}(V_{3}^{h}W_{3} + U_{3}) \Rightarrow \text{det} \left( \frac{1 - V_{3}^{2}}{1 - V_{3}^{h2}} \right) = \text{det} \left( \frac{1 - S_{3}^{2}}{(1 + V_{3}^{h}S_{3})^{2}} \right). \]

(5.3.42)

As a check of the transformation, we repeat the calculation (5.3.38) in the full-string basis

\[ 1\langle \Omega, \frac{3}{2} | 2\langle \Omega, -\frac{3}{2} | 3\langle \Omega, -\frac{3}{2} | V_{3}\rangle_{123} = \gamma_{3} \text{det} \left( \frac{1 - S_{3}^{2}}{(1 + V_{3}^{h}S_{3})^{2}} \right)^{\frac{D+1}{4}} e^{-\frac{1}{2} P_{b}V_{00}P_{b}} = 1. \]

(5.3.43)

This demonstrates that the anomaly we got is not related to the transformation. It is related either to the \( \kappa \) basis or to the bosonized ghost or to both.

### 5.4 Applications

Now that we have the continuous half-string formalism, we can use it in various calculations. The multiplication of two \( \mathcal{H}_{\kappa} \) states becomes very simple in this representation. Parameterizing two states \( S_{i} \) by

\[ S_{i}^{h} = \begin{pmatrix} a_{i} & b_{i} \\ b_{i} & c_{i} \end{pmatrix}, \]

(5.4.1)

where \( a_{i}, b_{i}, c_{i} \) are functions of \( \kappa > 0 \), we get for \( S^{h} \equiv S_{1}^{h} \ast S_{2}^{h} \)

\[ S^{h} = \frac{1}{1 - a_{2}c_{1}} \begin{pmatrix} a_{1}(1 - a_{2}c_{1}) + a_{2}b_{1}^{2} & -b_{1}b_{2} \\ -b_{1}b_{2} & c_{2}(1 - a_{2}c_{1}) + c_{1}b_{2}^{2} \end{pmatrix}. \]

(5.4.2)
This expression is equivalent to a gaussian integration in the functional language (up to normalization)

$$
\langle l^\kappa, r^\kappa | S^h \rangle = \langle l^\kappa, r^\kappa | S_1^h * S_2^h \rangle = \int dy_\kappa e^{-\frac{1}{2}(l^\kappa \ y^\kappa) L^h_1 (l^\kappa \ y^\kappa)} e^{-\frac{1}{2}(-y^\kappa \ r^\kappa) L^h_2 (-y^\kappa \ r^\kappa)},
$$

(5.4.3)

where

$$
L^h = \frac{1 - S^h}{1 + S^h}.
$$

(5.4.4)

### 5.4.1 Doubling of the wedge subalgebra

The wedge states $|n\rangle$ are squeezed states, given by the matrix [61]

$$
\Sigma_n = C T_n,
$$

(5.4.5)

where

$$
T_n = \frac{T + (-T)^n}{1 - (-T)^n}, \quad T = -e^{-\frac{\alpha \pi}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

(5.4.6)

The case $n = 1$ gives the identity state $|I\rangle$, $n = 2$ gives the vacuum state $|\Omega\rangle$ and $n \to \infty$ is the sliver.

Using (A.2.4), we get the expression for the wedge matrices $\Sigma_n$ in the sliver basis

$$
\Sigma^h_n = -e^{-\frac{\alpha \pi}{2}(n-1)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

(5.4.7)

The case $n \to \infty$ gives

$$
\Sigma^h_\infty = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
$$

(5.4.8)

which is the sliver matrix in the sliver basis. The case $n = 2$ gives

$$
\Sigma^h_2 = -\Sigma_\infty,
$$

(5.4.9)

meaning that the matrix defining the vacuum state in the sliver basis is minus the sliver matrix in the regular basis, as expected (A.1.11).
Using the expression for the star product of two $\mathcal{H}_{\kappa^2}$ states in the sliver basis (5.4.2) we verify the known algebra

$$|n\rangle \star |m\rangle = |n + m - 1\rangle . \quad (5.4.10)$$

By inspecting eq. (5.4.2),(5.4.7), we recognize that this algebra can be doubled by defining $\Sigma_n^+ = \Sigma_n^h$ and $\Sigma_n^- = -\Sigma_n^h$. The corresponding states $|n, \pm\rangle$ form a subalgebra of the star product with the product given by

$$|n, r\rangle \star |m, s\rangle = |n + m - 1, rs\rangle , \quad (5.4.11)$$

where $r, s = \pm 1$. This subalgebra is a direct product of $\mathbb{Z}_2$ with the wedge subalgebra, up to the identification of the sliver $|\infty, +\rangle = |\infty, -\rangle$.

### 5.4.2 The gauge transformation of the sliver to other butterflies

The spectroscopy of the general butterflies performed in [55] supported the natural assumption that in vacuum string field theory all these states are related by gauge transformations. It is possible to define states with different matter and ghost content. In vacuum string field theory it is common to assume a universal ghost part. Moreover, it was shown in [55] that states with the same ghost factor, whose matter factor consists of different matter butterflies, are orthogonal. In this subsection we explicitly present the gauge transformations among such states using the gauge factorization ansatz of [41], and, therefore, from now on we treat only the matter sector. It should be possible to perform similar gauge transformation in the ghost sector due to the enlarged gauge freedom of vacuum string field theory [38].

The generalized butterflies can be described by a parameter $0 \leq a \leq \infty$, which is related to $\alpha$ of [45] by $a = \frac{2-\alpha}{\alpha}$. According to the butterfly spectroscopy they are in $\mathcal{H}_{\kappa^2}$ and their defining matrices were found to be

$$S_a = \frac{1}{\sinh(\frac{\kappa \pi}{2} (1 + a))} \begin{pmatrix} \sinh(\frac{\kappa \pi}{2}) & \sinh(\frac{\kappa \pi}{2}a) \\ \sinh(\frac{\kappa \pi}{2}a) & \sinh(\frac{\kappa \pi}{2}) \end{pmatrix} , \quad (5.4.12)$$

where $a = 0$ is the nothing state, $a = 1$ is the canonical butterfly, and the limit $a \to \infty$ is the sliver. Using these results and eq. (A.2.4) we find that in the
sliver basis they transform to

\[ S^h_a = e^{-\frac{2\pi a}{\kappa}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.4.13} \]

The left–right factorization of the projectors is evident from the structure of the matrix. The entries off the diagonal which mix left and right modes vanish. If we take a general two-dimensional matrix, the condition that its off-diagonal entries vanish in the sliver basis is (once again) the projection condition of [55].

An infinitesimal gauge transformation is given by

\[ \delta \Phi = \Phi \star \Lambda - \Lambda \star \Phi. \tag{5.4.14} \]

Exponentiating this transformation, we get

\[ \Phi \to U^{-1} \star \Phi \star U, \tag{5.4.15} \]

where

\[ U = \exp_s(\Lambda) = \mathcal{I} + \Lambda + \frac{1}{2} \Lambda \star \Lambda + \ldots. \tag{5.4.16} \]

Here \( \mathcal{I} \) is the star-algebra identity element and \( U^{-1} \) is the inverse with respect to the star product of \( U \), i.e.

\[ U \star U^{-1} = U^{-1} \star U = \mathcal{I}. \tag{5.4.17} \]

For simplicity, we limit the search to squeezed states in \( \mathcal{H}_s^2 \) whose defining matrix is real. Given a squeezed state \( U \) in \( \mathcal{H}_s^2 \), defined by the matrix \( U^h \) we find using (5.4.2) that \( U^{-1} \) is also a squeezed state given by \( CU^{-1}C \). It can be inferred from [54] that this result is not limited to \( \mathcal{H}_s^2 \). We shall not restrict ourselves to BPZ-real states, since \( U \) is not a physical states, but a gauge transformation.

For a real matrix to define a normalizable squeezed state all its eigenvalues \( \lambda \) should be less than unity in absolute value. As we allow for singular squeezed states, such as the identity, we relax the condition to \( |\lambda| \leq 1 \). Define \( \lambda_{1,2} \) to be the eigenvalues of \( U^h \). The eigenvalues of \( U^{-1} \) are \( \frac{1}{\lambda_{1,2}} \). The conditions

\[ |\lambda_{1,2}| \leq 1, \quad \left| \frac{1}{\lambda_{1,2}} \right| \leq 1, \tag{5.4.18} \]
imply
\[ |\lambda_{1,2}| = 1. \]  
(5.4.19)

As we assumed that \( U^h \) is real and, being a quadratic form, it is necessarily symmetric, it follows that both eigenvalues are real, and thus
\[ \lambda_{1,2} = \pm 1. \]  
(5.4.20)

We would like to find a one-parameter family \( U_a \) which relates the sliver to any other generalized butterfly. Continuity with respect to \( a \), and \( U_\infty = C \), imply that one eigenvalue equals +1, and the other equals −1. To summarize, \( U^h \) is real, symmetric, traceless and \( \det(U^h) = -1 \).

A natural parameterization is
\[ U(\theta) = \begin{pmatrix} \sin(\theta) & -\cos(\theta) \\ -\cos(\theta) & -\sin(\theta) \end{pmatrix}, \]  
(5.4.21)

where \( \theta = 0 \) is the identity state. With this parameterization the multiplication rule (5.4.2) translates into
\[ U(\theta) \ast U(\phi) = U(\psi) \quad \sin(\psi) = \frac{\sin(\theta) + \sin(\phi)}{1 + \sin(\theta)\sin(\phi)}, \]  
(5.4.22)

so indeed we get a one parameter group. This multiplication rule is singular around \( \sin(\theta)\sin(\phi) = -1 \). Although \( \sin(\theta)\sin(\phi) \neq -1 \) implies that
\[ \left| \frac{\sin(\theta) + \sin(\phi)}{1 + \sin(\theta)\sin(\phi)} \right| \leq 1 \]  
and so the multiplication rule is well defined in a neighborhood of the singular points, the singularity is real, as the limit of the expression at the singular points is direction dependent. The multiplication \( U(\frac{\pi}{2}) \ast U\left(-\frac{\pi}{2}\right) \) can be regularized in different ways so as to produce any \( U(\theta) \) as a result. In string field theory we got used to anomalies of the star product at \( \kappa = 0 \). However, here the anomaly can appear at any \( \kappa \).

The group of gauge transformations (5.4.21) with \( \theta \in (-\frac{\pi}{2}, \frac{3\pi}{2}) \setminus \{\frac{\pi}{2}\} \) is a direct product of \( \mathbb{Z}_2 \) with the subgroup parameterized by \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\). The elements of the \( \mathbb{Z}_2 \) are the identity \( \mathcal{I} = U(0) \) and the identity dual \( \mathcal{J} = U(\pi) \). The two topologically disconnected sections of the gauge group are related by \( U(\pi - \theta) = \mathcal{J} \ast U(\theta) \). \( \mathcal{J} \) behaves as a root of the identity \( \mathcal{J} \ast \mathcal{J} = \mathcal{I} \), and it
transfers all squeezed states to themselves

\[ S = J^{-1} \star S \star J, \]

(5.4.23)

or in other words, it star-commutes with all squeezed states. This can be seen most easily in the continuous half-string functional form

\[ \langle l^K, r^K | J \rangle = \delta(l^K - r^K) \Rightarrow \left\{ \begin{array}{ll}
\Psi[l^K, r^K] \star J &= \Psi[l^K, -r^K] \\
J \star \Psi[l^K, r^K] &= \Psi[-l^K, r^K]
\end{array} \right. . \]

(5.4.24)

Therefore, \( J \) star-commutes with all states in which \( l^K, r^K \) appear in even powers, and particularly with all squeezed states, meaning that these states are not charged under the \( Z_2 \).

The subgroup parameterized by \( -\frac{\pi}{2} < \theta < \frac{\pi}{2} \), can be parameterized as

\[ U(t) = \begin{pmatrix} \tanh(t) & -\text{sech}(t) \\
-\text{sech}(t) & -\tanh(t) \end{pmatrix}, \]

(5.4.25)

where \( -\infty < t < \infty \). Now the star-multiplication rule simplifies to a linear relation

\[ U(t_1) \star U(t_2) = U(t_1 + t_2), \]

(5.4.26)

and the anomaly is hidden at \( t = \pm \infty \). In this parameterization all gauge transformations are generated by infinitesimal gauge transformations \( \Lambda \).

We can calculate \( \Lambda \) (5.4.14) as was done in [41]. Temporarily restoring the \( \kappa \) dependence we have

\[ |U[t_\kappa]\rangle = \exp \left( -\frac{1}{2} \int_0^\infty d\kappa \left( a_l^{\dagger \kappa} a_l^{\dagger \kappa} \right) \left( \begin{array}{cc} \tanh(t_\kappa) & -\text{sech}(t_\kappa) \\
-\text{sech}(t_\kappa) & -\tanh(t_\kappa) \end{array} \right) \left( \begin{array}{cc} a_l^{\dagger \kappa} \\
a_r^{\dagger \kappa} \end{array} \right) \right) |\Omega\rangle_h, \]

(5.4.27)

from which we get

\[ |\Lambda[t_\kappa]\rangle = \frac{d}{ds} |U[st_\kappa]\rangle \bigg|_{s=0} = -\frac{1}{2} \int_0^\infty d\kappa t_\kappa (a_l^{\dagger \kappa} a_l^{\dagger \kappa} - a_r^{\dagger \kappa} a_r^{\dagger \kappa}) |\Omega\rangle_h. \]

(5.4.28)

For finding a gauge transformation \( U_a \) from the sliver to a generalized butterfly (5.4.13) we write

\[ U_a^{-1} \star S_h^a \star U_a = S_h^a, \]

(5.4.29)
and use the expression (5.4.25) to get
\[
\tanh(t) = -e^{-\frac{\kappa \pi}{2} a}, \quad U_a = \begin{pmatrix}
-e^{-\frac{\kappa \pi}{2} a} & -\sqrt{1 - e^{-\kappa \pi a}} \\
-\sqrt{1 - e^{-\kappa \pi a}} & e^{-\frac{\kappa \pi}{2} a}
\end{pmatrix}.
\] (5.4.30)

We see that the gauge transformation approaches the singular limit in all cases when \( \kappa \to 0 \). All the butterflies share the eigenvalue 1 for \( \kappa = 0 \), and thus should have a trivial gauge transformation for this value. The discontinuity of the transformation near \( \kappa = 0 \) is related to the discontinuity of the transformation to the sliver basis. For the nothing state \( a = 0 \), the transformation is singular for all \( \kappa \), due to the singular eigenvalues of the nothing. Thus, the nothing should not be thought of as equivalent to the other states. The singularity of the nothing can also be seen if we try to find a Bogoliubov transformation to the nothing basis. This transformation is singular and has to be regularized, and the final result of the transformation is regularization dependent.

We can define the transformation \( U_{ab} \) between two butterflies
\[
U_{ab}^{-1} \star S_b^h \star U_{ab} = S_a^h,
\] (5.4.31)
by composing two gauge transformations using
\[
\tanh(t) = \frac{\sinh\left(\frac{\kappa \pi}{4} (a - b)\right)}{\sinh\left(\frac{\kappa \pi}{4} (a + b)\right)},
\]
\[
U_{ab} = \frac{1}{\sinh\left(\frac{\kappa \pi}{4} (a + b)\right)} \begin{pmatrix}
\sinh\left(\frac{\kappa \pi}{4} (a - b)\right) & -\sqrt{\sinh\left(\frac{\kappa \pi}{2} a\right) \sinh\left(\frac{\kappa \pi}{2} b\right)} \\
-\sqrt{\sinh\left(\frac{\kappa \pi}{2} a\right) \sinh\left(\frac{\kappa \pi}{2} b\right)} & -\sinh\left(\frac{\kappa \pi}{4} (a - b)\right)
\end{pmatrix}.
\] (5.4.32)

It is instructive to imply the gauge transformation \( U_1 \) to the wedge states. This transformation sends the sliver to the butterfly, and so should transform the wedge states to “butterfly wedge states”. These states form a subalgebra isomorphic to that of wedge states, and can be used as a regularization of the butterfly. Using \( U_1 \) and the expression (5.4.7) for the wedge states, we get that the butterfly wedge states in the sliver basis are defined by the matrix
\[
\Sigma_n^h = \frac{1}{\sinh\left(\frac{\kappa \pi}{2} n\right)} \begin{pmatrix}
\sinh\left(\frac{\kappa \pi}{2} (n - 1)\right) & -\sinh\left(\frac{\kappa \pi}{2}\right) \\
-\sinh\left(\frac{\kappa \pi}{2}\right) & \sinh\left(\frac{\kappa \pi}{2} (n - 1)\right)
\end{pmatrix}.
\] (5.4.33)
In particular we see that the identity state \( n = 1 \) is invariant, while the sliver \( n = \infty \) transforms into the butterfly. This construction could have been especially useful if the butterfly wedge states were surface states. This could have been the case if \( U_1, U_1^{-1} \) were surface states. We can see that this is not the case. The vacuum state \( n = 2 \) transforms to \( \Sigma_2^h \), which in the full-string \( \kappa \) basis becomes

\[
\Sigma_2^h = \begin{pmatrix}
e^{-\frac{\pi \kappa}{2}} & 0 \\
0 & e^{-\frac{\pi \kappa}{2}}
\end{pmatrix}.
\] (5.4.34)

Using the methods of section 3 of [55] we find that the candidate conformal transformation of this state is \( \tan^{-1}(z) \), which describes the sliver. Thus \( \Sigma_2 \) is not a surface state.

### 5.5 Conclusions

In this chapter we demonstrated how the transformation to the half-string basis works in the continuous basis. The star-product in this formalism is as simple as it gets. This allows us to show that all the butterfly projectors, which are all suppose to describe the same D-brane, are indeed related by a gauge transformation. In the half-string formalism we were able to find a parameterization of the gauge transformation (5.4.25) which is linear under the star-multiplication (5.4.26) and to write explicitly the gauge transformation between the butterflies.

Using the known form of the vertex in the half-string basis we were able to reconstruct the zero-mode dependence of the vertex in the full-string basis. Our results can be generalized to the fermionic ghost. To transform the fermionic ghost to the continuous half-string basis, we need to repeat the analysis of the Bogoliubov transformation that we presented in the appendix for anticommuting operators.
Chapter 6

Superconformal mechanics and super-Virasoro algebra

In this chapter we consider $\mathcal{N} = 1, 2$ superconformal mechanics in $0 + 1$ dimensions and derive the conditions to enlarge their symmetry group to the full super-Virasoro. The generators are quantized and a general prescription is given for the construction of the $\mathcal{N} = 1$ algebra independently of the specific details of the superconformal mechanics provided that in addition its quadratic Casimir operator vanishes.

6.1 Introduction

There is an ongoing interest in conformal mechanics, since the early work of [71], and in the supersymmetric generalization [72, 73, 74]. These one dimensional conformal field theories admit exact solutions to problems that can be accessed only perturbatively in higher dimensions, due to the existence of the powerful conformal symmetry that constrains the dynamics. Although these $d = 1$ conformal mechanics are relatively simple they are still not trivial. A geometrical picture that relates the one dimensional field equations to geodesics in the group space of $SO(1, 2)$ and $SU(1, 1|1)/U(1)$ was constructed in [75, 76].

Recently, the $AdS_{p+2}/CFT_{p+1}$ conjecture, see e.g [77, 78], has raised a new interest in conformal mechanics, i.e the case of $p = 0$. This conjecture proposes that in an appropriate limit certain conformal field theories in $p + 1$ dimensions are dual to superstring theory on an $AdS$ space in $p + 2$ dimensions times some compact manifold. The case of conformal quantum mechanics and the corresponding $AdS_2$ [79] may offer more insights to this conjecture due to
Although the basis to this chapter is mainly algebraic, it is instructive to bear in mind concrete realizations of physical systems which are governed by (super) conformal mechanics. An illustrative example is that of the physics of a test particle in the near horizon region of a $d = 4 \, \mathcal{N} = 2$ extreme Reissner-Nordström black hole [80, 81]. This particle is described by conformal mechanics where the canonical coordinates are the radial coordinate and radial momentum in $d = 4$. The main interest in such systems is due to the fact that black holes provide the arena where gravity and quantum mechanics match [82]. Understanding these systems shed light on the intriguing problems of quantum gravity. Another example is that of a non-relativistic spinning particle coupled to a magnetic field and a scalar potential [83] 1.

In this chapter we follow the analysis in [84, 85] on conformal symmetry and the Virasoro algebra and extend it to the case of $\mathcal{N} = 1, 2$ super conformal symmetry. In section 6.2 we identify the superconformal algebra as the subalgebra of the super-Virasoro algebra. In section 6.3 we obtain classical recursion equations for the generators and find a representation of the full algebra for a free particle and an interacting one. At the level of Poisson brackets this algebra close to $\mathcal{N} = 1, 2$ Neveu-Schwarz super-Virasoro algebra, with a $U(1)$ Kac-Moody algebra [86] for the latter case. In section 6.4 we describe the quantization of the generators and give a general construction of $\mathcal{N} = 1$ super-Virasoro generators out of the superconformal ones. The results of this chapter were published in [87].

### 6.2 Generators of Superconformal Quantum Mechanics

The generators of $0 + 1$ dimensional superconformal quantum mechanics obey the super algebra of $osp(2|2) \cong su(1, 1|1)$ [81]. Their non-trivial commutation
and anti-commutation relations can be written in the following way:

\[
[H, D]_\pm = i H \pm i K \pm 2i D ,
\]

\[
[Q_i, Q_j]_\pm = 2\delta_{ij} H \pm 2\delta_{ij} K \pm 2\delta_{ij} D \pm \epsilon_{ij} B ,
\]

\[
[D, Q_i]_\pm = -\frac{i}{2} Q_i \pm i S_i ,
\]

\[
[K, Q_i]_\pm = -i S_i \pm i Q_i ,
\]

\[
[B, Q_i]_\pm = -i \epsilon_{ij} Q_j \pm i \epsilon_{ij} S_j , \quad i, j = 1, 2 ,
\]

where \([, ]_\pm\) stands for commutators and anticommutators. The following identification of the generators

\[
L_{-1} = H , \quad L_0 = -D , \quad L_1 = K ,
\]

\[
G^i_{-1/2} = Q_i , \quad G^i_{1/2} = -S_i , \quad B_0 = B
\]

can be used to recast the algebra \((6.2.1)\) as a subalgebra of the super-Virasoro algebra with a \(U(1)\) charge

\[
[L_m, L_n]_\pm = i(m - n)L_{m+n}
\]

\[
[L_m, G^i_r]_\pm = i\frac{1}{2}(m - r)G^i_{m+r}
\]

\[
[G^i_r, G^j_s]_\pm = 2\delta_{ij} L_{r+s} + \epsilon_{ij}(r - s)B_{r+s}
\]

\[
[B_k, L_m]_\pm = ikB_{k+m}
\]

\[
[B_k, G^i_r]_\pm = -i\epsilon_{ij} G^j_{k+r}
\]

for \(m, n = -1, 0, 1, r, s = -\frac{1}{2}, \frac{1}{2}\) and \(k = 0\). A realization of this algebra is given for the theory of a free particle by

\[
K = \frac{1}{2}x^2 , \quad D = -\frac{1}{4}(xp + px) , \quad H = \frac{1}{2}p^2
\]

\[
Q^i = \psi^i p , \quad S^i = -\psi^i x , \quad B = \frac{1}{2}[\psi^1, \psi^2]_-
\]

where \(\psi^i, i = 1, 2\) are Grassmann coordinates \([\psi^i, \psi^j]_+ = \delta^{ij}\).

For completion we give an explicit form of the conserved charges associated with this algebra written in a way that is compatible with the whole
super-Virasoro algebra. Time translation is generated by $H = L_{-1}$ so for any quantum generators $G$ the Heisenberg representation gives
\[ G(t) = e^{iL_{-1}t}Ge^{-iL_{-1}t} = \sum_{l=0}^{\infty} \frac{(-it)^l}{l!}[[G, L_{-1}], L_{-1}, ..., L_{-1}], \quad (6.2.4) \]
where the first term corresponding to $l = 0$ is $G$. The nested commutators are easily calculated for any super-Virasoro generator
\[ G_r^i(t) = \sum_{l=0}^{\infty} \left( \frac{r+1/2}{l} \right) G_{r-l}^i t^l \]
\[ L_n(t) = \sum_{l=0}^{\infty} \left( \frac{n+1}{l} \right) L_{n-l} t^l \]
\[ B_k(t) = \sum_{l=0}^{\infty} \left( \frac{k}{l} \right) B_{k-l} t^l, \]
which are finite sums for $r \geq -1/2$, $n \geq -1$ and $k \geq 0$. If, for example, $k = -m < 0$ we may use analytic continuation
\[ l!(\frac{k}{l}) \to \lim_{\epsilon \to 0} \frac{\Gamma(-m + 1 + \epsilon)}{\Gamma(-m - l + 1 + \epsilon)} = (-1)^l \frac{\Gamma(m + l)}{\Gamma(m)}. \]
The generators in (6.2.5) are conserved by construction
\[ \frac{dG}{dt} = i[G, L_{-1}] + \frac{\partial G}{\partial t} = 0. \quad (6.2.6) \]

### 6.3 Classical algebra

In order to find the complete super-Virasoro algebra we use the classical algebra
\[ \{L_m, L_n\} = (m - n)L_{m+n} \]
\[ \{L_m, G_r^i\} = \left( \frac{1}{2} m - r \right) G_{m+r}^i \]
\[ \{G_r^i, G_s^j\} = 2 \delta_{ij} L_{r+s} + \epsilon_{ij} (r - s) B_{r+s} \]
\[ \{B_k, L_m\} = k B_{k+m} \]
\[ \{B_k, G_r^i\} = -\epsilon_{ij} G_{k+r}^j \]
\[ \{B_k, B_m\} = 0, \]
\[ \{B_k, L_m\} = \frac{k}{m} B_{k+m} \]
where \{ , \} denote the even Poisson Brackets [88]

\[
\{ G, K \} = \frac{\partial G}{\partial x} \frac{\partial K}{\partial p} - \frac{\partial G}{\partial p} \frac{\partial K}{\partial x} - \sum_{i=1}^{2} (-1)^{p(Q)} \frac{\partial G}{\partial \theta^i} \frac{\partial K}{\partial \theta^l},
\]

(6.3.2)

with \( p(Q) = 1 \) for an odd generator and zero otherwise. This definition gives

for even coordinates \( xp - px = 0 \)

\[
\{ x, p \} = 1
\]

(6.3.3)

and for odd coordinates \( \theta^i \theta^j + \theta^j \theta^i = 0 \)

\[
\{ \theta^i, \theta^j \} = \delta^{ij}.
\]

(6.3.4)

In what will follow we use the explicit representation of the global generators, e.g the generators that are given in equation (6.2.3), and the classical algebra to obtain a set of solvable differential equations. This is done by calculating the Poisson Brackets of a general generator of the algebra with one of the global generators and demanding that it will be equal to the r.h.s of the algebra (6.3.1), as was done in [84] for the \( L_n \) generators.

**6.3.1 Free Particle**

We begin with the super conformal generators of the free particle that are given in equation (6.2.3). The Poisson Brackets of the \( G^i_r \) generators with the Hamiltonian and the special conformal generator gives

\[
-p \frac{\partial G^i_r}{\partial x} = \{ H, G^i_r \} = \{ L_{-1}, G^i_r \} = \left( -\frac{1}{2} - r \right) G^i_{r-1}
\]

\[
\frac{x}{2} \frac{\partial G^i_r}{\partial p} = \{ K, G^i_r \} = \{ L_1, G^i_r \} = \left( \frac{1}{2} - r \right) G^i_{r+1}.
\]

(6.3.5)

The integrability condition \( \frac{\partial^2 G^i_r}{\partial p^2 x} = \frac{\partial^2 G^i_r}{\partial x p^2} \) gives the recursion formula for \( G^i_r \)

\[
\frac{1}{x^2}(r - \frac{1}{2})G^i_{r+1} = -\frac{1}{p^2}(r + \frac{1}{2})G^i_{r-1} + \frac{2r}{xp} G^i_r.
\]

(6.3.6)
Similar equations for $L_n$ and $B_k$ exist and are easily solved. The classical Virasoro generators are then given by

$$
G_r^i = \theta^i x^{1/2 + r} p^{1/2 - r} \\
L_n = \frac{1}{2} x^{1+n} p^{1-n} \\
B_k = B x^k p^{-k}
$$

for $n, k \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$, $B = \frac{1}{2}[\theta^1, \theta^2]$. For $n \geq -1$, $r \geq -1/2$ and $k \geq 0$ we can recast these generators as

$$
L_n = L_{-1}(L_0 L_{-1})^{n+1} \\
G_r^i = G_{-1/2}(L_0 L_{-1})^{r+1/2} \\
B_k = B_0(L_0 L_{-1})^k.
$$

This is an extension to [85]. The representation, however, is not “unitary” $L_n^* \neq L_{-n}$. To get a unitary representation we consider the linear combination of DFF [71]

$$
L_1 = \frac{1}{2}(aH - \frac{K}{a} - 2iD) = \frac{1}{2} z^2 \\
L_0 = \frac{1}{2}(aH + \frac{K}{a}) = \frac{1}{2} z z^* \\
L_{-1} = \frac{1}{2}(aH - \frac{K}{a} + 2iD) = \frac{1}{2} z^* z,
$$

where we define

$$
z = \sqrt{ap - \frac{ix}{\sqrt{a}}} \sqrt{2} \quad z^* = \sqrt{ap + \frac{ix}{\sqrt{a}}} \sqrt{2}.
$$

To obtain the entire algebra we can solve the differential equations again or observe that the new generators have the same form as in equation (6.3.7) with $x \rightarrow z$ and $p \rightarrow z^*$. Therefore the solution

$$
L_n = \frac{1}{2} z^{1+n} z^{*1-n} \\
B_k = B z^k z^{*-k} \\
G_r^i = \theta^i z^{1/2 + r} z^{*1/2 - r}
$$

(6.3.11)
obey the algebra up to new factors of $i$ due to the fact that the coordinate transformation (6.3.10) is canonical up to a scale, i.e $\{z, z^*\} = -i$. Another difference between the two representations, besides unitarity, is that for the generators in (6.3.11) time translation is generated by $L_0$ and not by $L_{-1}$.

### 6.3.2 An Interacting Particle

To obtain the generators of an interacting particle we note that the classical Virasoro algebra is invariant under $L_n \to f^n L_n$, $G^i_r \to f^r G^i_r$ and $B_k \to f^k B_k$ where $f = f(L_0)$. Since for (6.3.7) $L_0 = 2xp$ we get

$$
G^i_r = \theta^i x^{\frac{r}{2} + \frac{1}{2}} p^k f^k, \\
L_n = \frac{1}{2} x^{1+n} p^{1-n} f^n, \\
B_k = B x^k p^{-k} f^k.
$$

(6.3.12)

The Hamiltonian of a particle in a conformal potential is obtained for $f^{-1} = 1 + \frac{g}{x^2 p^2}$

$$
L_{-1} = H = \frac{1}{2} (p^2 + \frac{g}{x^2}).
$$

(6.3.13)

For a superconformal potential we first recall the general method [89] for supersymmetrizing a classical and quantum mechanical Hamiltonian. Given a Hamiltonian $H = \frac{1}{2} p^2 + V$ we can define the odd generators

$$
Q^i = (p1 + W, x) \psi^i,
$$

(6.3.14)

where $W, x = \frac{dW}{dx}$, which obey the following relation

$$
\{Q^i, Q^j\} = \delta^{ij} (p^2 + W^2, x^2) - 2B W, xx.
$$

(6.3.15)

By solving $\sqrt{2V} = W, x$ we get

$$
\{Q^i, Q^j\} = 2\delta^{ij} H_{\text{susy}},
$$

(6.3.16)

$$
H_{\text{susy}} = \frac{1}{2} (p^2 + W, x^2 - 2B W, xx).
$$

(6.3.17)
The potential for conformal mechanics is \( V = \frac{g^2}{2x^2} \), which is solved for \( W = \frac{1}{2}g \log x^2 \). These expressions for the Hamiltonian and the supersymmetric generators motivate the following ansatz

\[
G^i_r = x^{\frac{1}{2}+r} p^{\frac{1}{2}-r} f^r (1 + \frac{g^2}{u^2})^{-r-1/2}(1 + \frac{g}{u})^{ij} \psi^j \\
L_n = \frac{1}{2} x^{1+n} p^{1-n} f^n (1 + \frac{g^2 + 2gB}{u^2})^{-n} \\
B_k = x^k p^{-k} f^k (1 + \frac{g^2}{u^2})^{-k} B ,
\]

which obey (6.3.1) and can be recast in the form of (6.3.8).

One can use the classical analog of (6.2.4) to obtain an explicit representation of the conserved charges associated with the classical generators, e.g. the use of the generators in equation (6.3.12) will give

\[
G^i_r (t) = \theta^i p/\sqrt{f}(x/p f + t)^{r+1/2} \\
L_n (t) = \frac{1}{2} f p^2 (x/p f + t)^{n+1} \\
B_k (t) = B(x/p f + t)^k ,
\]

which is the subalgebra of \( w_\infty \) obtained in [90].

**6.4 Quantization of the generators**

For the quantization of the generators we first rescale the algebra and absorb all factors of \( i \). For clearness we give the \( \mathcal{N} = 0, 1, 2 \) algebras:

\[
\mathcal{N} = 0 : \quad [L_m, L_n]_\pm = (m - n)L_{m+n} \\
\mathcal{N} = 1 : \quad [L_m, L_n]_\pm = (m - n)L_{m+n} \\
[L_m, G_r]_\pm = (\frac{1}{2}m - r)G_{m+r} \\
[G_r, G_s]_+ = 2L_{r+s}
\]
\[ N = 2 : \quad [L_m, L_n]_- = (m - n)L_{m+n} \]
\[ [L_m, G^i_r]_- = (\frac{1}{2}m - r)G^i_{m+r} \]
\[ [G^i_r, G^j_s]_+ = 2\delta^{ij}L_{r+s} + (r - s)\epsilon^{ij}B_{r+s} \] (6.4.3)
\[ [B_k, B_n]_- = 0 \]
\[ [B_k, L_n]_- = kB_{k+n} \]
\[ [B_k, G^i_r]_- = -\epsilon^{ij}G^j_{k+r} \]

and also the Casimir operator of each of the (super) \( sl(2, \mathbb{R}) \) subalgebras:

\[ N = 0 : \quad L^2 = L_0^2 + L_0 - L_1L_{-1} \]
\[ N = 1 : \quad 1C^2 = L^2 - \frac{1}{2}L_0 + \frac{1}{2}G_{1/2}G_{-1/2} \] (6.4.4)
\[ N = 2 : \quad 2C^2 = L^2 - L_0 + \frac{1}{2}G^i_{1/2}G^i_{-1/2} + \frac{1}{4}B^2 \]

In [84] the quantization of the bosonic classical generators was obtained by a canonical coordinate transformation:

\[ q = \frac{p}{2x} \quad y = x^2, \] (6.4.5)

and setting \( f(u) = xp \). The same procedure applied to the \( N = 2 \) generators (6.3.12) gives

\[ G^i_r = \psi^i y^{\frac{r}{2} + r} q^{\frac{1}{2}} \]
\[ L_n = y^{1+n}q \] (6.4.6)
\[ B_k = y^kB \]

There are two problems we have to address before quantization. The first is how to quantize \( \sqrt{q} \) and the second is how to order the operators. For the square root of the momentum we define the covariant derivative \( D^i = \frac{\partial}{\partial \theta^i} + \theta^i q \) which obey \( \{D^i, D^j\} = 2\delta^{ij}q \) and since the generators are linear in momentum, ordering \( y^{n+1}q \) might add a term proportional to \( y^n \). This motivate the following quantum ansatz

\[ \sqrt{i}G^i_r = y^{r+\frac{1}{2}} D^i - i(r + \frac{1}{2})y^{r-\frac{1}{2}} \theta^i T \]
\[ iL_n = y^{1+n}q - i\frac{(n + 1)}{2}y^n T \] (6.4.7)
\[ iB_k = y^kB \]
which satisfied the $\mathcal{N} = 2$ super-Virasoro algebra for any $n, k \in \mathbb{Z}$ and $r \in \mathbb{Z} + 1/2$ provided

\begin{align*}
T &= \theta i \frac{\partial}{\partial \theta^i} \\
B &= \theta^1 \partial_2 - \theta^2 \partial_1
\end{align*}

(6.4.8)

and which can be truncated to the $\mathcal{N} = 1$ algebra generators:

\begin{align*}
\sqrt{i} G_r &= y^{r+\frac{1}{2}} \mathcal{D} \\
i L_n &= y^{1+n} q - i \frac{(n+1)}{2} y^n T,
\end{align*}

(6.4.9)

where $T = \frac{\partial}{\partial \theta}$ and $\mathcal{D} = \frac{\partial}{\partial \theta} + \theta q$.

In [85] a general construction of two half-Virasoro algebras was given provided $L_1$ and $L_{-1}$ are invertible. The conditions for combining them into a single Virasoro algebra were

\begin{align*}
L_1 &= L_0 L_{-1} L_0 \\
L_{-1} &= L_0 L_{1}^{-1} L_0 .
\end{align*}

(6.4.10)

These conditions are equivalent, when $L_{\pm 1}$ are invertible, to the vanishing of the quadratic Casimir of the $sl(2, \mathbb{R})$ algebra

\begin{align*}
L^2 = L_0^2 + L_0 - L_1 L_{-1} = 0 .
\end{align*}

(6.4.11)

Moreover if we consider the whole tower of $sl(2, \mathbb{R})$ subalgebras contained in the Virasoro algebra, i.e \( \{ \frac{1}{n} L_n, \frac{1}{n} L_0, \frac{1}{n} L_{-n} \} \) for $n > 0$ with the solution found in [85]

\begin{align*}
L_n &= (L_0 L_{-1}^{-1})^n L_0 \\
L_{-n} &= (L_0 L_1^{-1})^n L_0
\end{align*}

(6.4.12)

we can easily verify that all the quadratic Casimir vanish

\begin{align*}
L^2(n) = \frac{1}{n^2} L_0^2 + \frac{1}{n} L_0 - \frac{1}{n^2} L_n L_{-n} = 0
\end{align*}

(6.4.13)

We would like to find a similar construction for a representation of the super-Virasoro generators which is independent of the specific super conformal system.
for the $\mathcal{N} = 1$ case (6.4.2). For this we first note that the quadratic Casimir can be written as

$$1C^2 = L_0^2 + \frac{1}{2}L_0 - L_1L_{-1} + \frac{1}{2}G_{1/2}G_{-1/2}$$

(6.4.16)

$$\equiv \left[G_{1/2}G_{-1/2} - (L_0 + \frac{1}{4})\right]^2 - \frac{1}{16}$$

and that the realization of the generators in (6.4.10) can be thought of as a change of variables which we can invert to obtain

$$q = iL_{-1}$$
$$T = 2(G_{1/2}G_{-1/2} - L_0)$$
$$y = G_{1/2}G_{-1/2}L_{-1}^{-1}$$
$$D = \sqrt{i}G_{-1/2}$$
$$\theta = \frac{2}{\sqrt{i}}(G_{1/2}G_{-1/2} - L_0)G_{-1/2}^{-1/2}$$
$$\frac{\partial}{\partial \theta} = -2\sqrt{i}G_{-1/2}(G_{1/2}G_{-1/2} - L_0),$$

with $1C^2 = 0 \iff \theta^2 = 0$. This inversion allows us to rewrite the generators of half the $\mathcal{N} = 1$ Virasoro algebra as functions of only the (super) $sl(2, \mathbb{R})$ subalgebra generators:

$$L_n = (G_{1/2}G_{-1/2}L_{-1}^{-1})^{n+1}L_{-1} - (n + 1) \left(G_{1/2}G_{-1/2}L_{-1}^{-1}\right)^n(G_{1/2}G_{-1/2} - L_0)$$
$$G_r = (G_{1/2}G_{-1/2}L_{-1}^{-1})^{r+1/2}G_{-1/2}^{-1/2}$$

(6.4.18)

where $n \geq -1$. If $L_1$ is also invertible then this solution is valid for $n < -1$ and obeys the full algebra. The solution reduces to (6.4.13) for $G_{1/2}G_{-1/2} \sim L_0$.

In a similar way to the $\mathcal{N} = 0$ case, we can compute the quadratic Casimir for all the (super) $sl(2, \mathbb{R})$ subalgebras generators

$$\frac{1}{2k + 1}L_{2k+1}, \quad \frac{1}{2k + 1}L_0, \quad \frac{1}{2k + 1}L_{-2k-1}, \quad \frac{1}{\sqrt{2k + 1}}G_{k+1/2}, \quad \frac{1}{\sqrt{2k + 1}}G_{-k-1/2}$$

(6.4.19)

The quadratic Casimir of the subalgebras are fix, i.e $L^2(k) = L^2$ and $1C_2(k) = 1C_2 = 0$. 
6.5 Conclusions

We construct the $\mathcal{N} = 1,2$ super-Virasoro algebra out of the superconformal generators at the classical level of Poisson brackets. The generators are ordered and quantized. These quantum generators define new coordinates that are inverted and used to construct half of the super $\mathcal{N} = 1$ Virasoro algebra provided that the quadratic Casimir vanish, and the Hamiltonian $H = L_{-1}$ is invertible. This condition amounts to the requirement that $H$ has no ground state at $E = 0$ as is the case in [72]. Equivalently, one can demand that supersymmetry is broken and there are no states which are annihilated by the supersymmetric generators. The full super-Virasoro algebra is obtained when the special conformal generator $K = L_1$ is also invertible.

It would be interesting to find out what are the restrictions that higher supersymmetry [91] put on such constructions and if there are modifications that will account for central charges, since in the $\mathcal{N} = 1$ case the representation is not restricted only to quantum mechanics in which one would expect these extensions to vanish.
Appendix A

Bogoliubov transformation

In the appendix we set our conventions and collect some facts about Bogoliubov transformations and squeezed states.

A.1 Transformation of operators

A Bogoliubov transformation is a linear canonical transformation which mixes creation and annihilation operators. It is given by

$$ b_n = W_{nm} a_m + U_{nm} a^\dagger_m , \quad b^\dagger_n = W^*_{nm} a^\dagger_m + U^*_{nm} a_m , $$

(A.1.1)

where $a, a^\dagger$ are the original annihilation and creation operators, and $b, b^\dagger$ are the new ones. For this transformation to be canonical one has to impose

$$ [b_n, b_m] = [b^\dagger_n, b^\dagger_m] = 0 \quad (A.1.2) $$

$$ [b_n, b^\dagger_m] = \delta_{n,m} . \quad (A.1.3) $$

These restrictions imply that the matrix $W$ is invertible, the matrix $S \equiv W^{-1} U$ is symmetric, and that

$$ (1 - SS^*) = (W^\dagger W)^{-1} . \quad (A.1.4) $$

The normalized vacuum state with respect to the $b$ operators is given by

$$ |0\rangle_b = \det(1 - SS^*)^{1/4} \exp\left(-\frac{1}{2} a^\dagger S a\right) |0\rangle_a , \quad (A.1.5) $$

where by $a^\dagger S a$ we mean the quadratic form $a^\dagger_n S_{nm} a_m$. The new vacuum is a squeezed state. One can therefore think of the Bogoliubov transformation as a transformation to a basis where a given squeezed state plays the role of the vacuum.
Both eq. (A.1.4), (A.1.5) make sense provided that the eigenvalues of $SS^*$ (which are necessarily real and positive) are less than 1. In practice we will have to deal with the case where this inequality is saturated. This is the case for the three-vertex, for which all eigenvalues are $\pm 1$, as well as for projectors, which have eigenvalues $\pm 1$ for $\kappa = 0$.

Given $S$, eq. (A.1.4) determines $W$ up to a unitary transformation, which does not alter the vacuum. Therefore, we can choose $W = W_0$ Hermitian using the Taylor expansion

$$W_0 = (1 - SS^*)^{-\frac{1}{2}} = 1 + \frac{1}{2}SS^* + \frac{3}{8}(SS^*)^2 + \ldots. \quad (A.1.6)$$

For a real $S$ we have

$$[W_0, S] = 0, \quad (A.1.7)$$

and $W_0$ is real and symmetric.

Bogoliubov transformations can be composed and inverted. The inverse of transformation (A.1.1) is

$$a = W^\dagger b - U^T b^\dagger, \quad (A.1.8)$$

and given two Bogoliubov transformations

$$b = W_1 a + U_1 a^\dagger, \quad c = W_2 b + U_2 b^\dagger, \quad (A.1.9)$$

we can write

$$c = (W_2 W_1 + U_2 U_1^*) a + (W_2 U_1 + U_2 W_1^*) a^\dagger. \quad (A.1.10)$$

It is straightforward to see that this action is associative, and that the inverse is two sided. Thus, Bogoliubov transformations form a group with inverse given by

$$(W, U)^{-1} = (W^\dagger, -U^T), \quad (A.1.11)$$

multiplication rule

$$(W_2, U_2) \cdot (W_1, U_1) = (W_2 W_1 + U_2 U_1^*, W_2 U_1 + U_2 W_1^*), \quad (A.1.12)$$

and identity element $(1, 0)$. 
Another useful transformation is to a basis where a given coherent state plays the role of the vacuum.

\[
|0\rangle_b = e^{-\frac{1}{2} \mu^\dagger \mu} e^{\mu^\dagger b^\dagger} |0\rangle_a ,
\]

(A.1.13)

where \( \mu \) is a given constant vector, and \( |0\rangle_b \) is normalized. It is straightforward to see that

\[
b = a - \mu , \quad b^\dagger = a^\dagger - \mu^* \tag{A.1.14}
\]

are the canonical operators for which \( |0\rangle_b \) is the vacuum. The inverse transformations is

\[
a = b + \mu , \quad a^\dagger = b^\dagger + \mu^* , \tag{A.1.15}
\]

\[
|0\rangle_a = e^{-\frac{1}{2} \mu^\dagger \mu} e^{-\mu^T b^\dagger} |0\rangle_b . \tag{A.1.16}
\]

The transformations (A.1.1),(A.1.14) can be combined to form a generalized Bogoliubov transformation whose vacuum is a shifted squeezed state

\[
b = W(a + Sa^\dagger - \mu) \equiv Wa + Ua^\dagger - \sigma , \tag{A.1.17}
\]

\[
|0\rangle_b = \det(1 - SS^*)^{1/4} e^{\frac{1}{4} \Re(\mu^\dagger(1 - SS^*)^{-1}S\mu^*) - \frac{1}{2} \mu^\dagger(1 - SS^*)^{-1} \mu} e^{\mu^T a^\dagger - \frac{1}{2} a^\dagger Sa^\dagger} |0\rangle_a . \tag{A.1.18}
\]

The composition rule now is

\[
(W_2, U_2, \sigma_2) \cdot (W_1, U_1, \sigma_1) = (W_2 W_1 + U_2 U_1^*, W_2 U_1 + U_2 W_1^*, W_2 \sigma_1 + U_2 \sigma_1^* + \sigma_2) , \tag{A.1.19}
\]

the identity is \((1, 0, 0)\), and the inverse is

\[
(W, U, \sigma)^{-1} = (W^\dagger, -U^T, U^T \sigma^* - W^\dagger \sigma) . \tag{A.1.20}
\]

### A.2 Transformation of states

After specifying the transformation of the creation and annihilation operators, and that of the vacuum state we want to use these results to calculate the transformation properties of (shifted) squeezed states. Suppose we are given a shifted squeezed state

\[
|V, \mu\rangle = e^{\mu^T a^\dagger - \frac{1}{2} a^\dagger Va^\dagger} |0\rangle_a , \tag{A.2.1}
\]
and want to describe it in the basis of \( b = Wa + Ua^\dagger - \sigma \). We can think of this state as a ground state (up to the normalization) after a Bogoliubov transformation \( c = W_V(a + Va^\dagger - \mu) \), that is

\[
|V, \mu\rangle = \det(1 - VV^\ast)^{-1/4} e^{-\frac{i}{2} \Re(\mu^\dagger (1-VV^\ast)^{-1}V^\ast\mu^\ast) + \frac{i}{2} \mu^\dagger (1-VV^\ast)^{-1}\mu} |0\rangle_c .
\]  

(A.2.2)

To describe this state in the \( b \) basis we have to compose the transformation from \( a \) to \( c \) on the inverse of the transformation from \( a \) to \( b \), that is to write

\[
|0\rangle_c = \det(1 - \hat{V}V^\ast)^{1/4} e^{\frac{i}{2} \Re(\hat{\mu}^\dagger (1-\hat{V}\hat{V}^\ast)^{-1}\hat{V}\hat{\mu}^\ast) - \frac{i}{2} \hat{\mu}^\dagger (1-\hat{V}\hat{V}^\ast)^{-1}\hat{\mu} e^{\hat{\mu}^T b^\dagger - \frac{1}{2} b^\dagger \hat{\mu} b} |0\rangle_b .
\]  

(A.2.3)

where, using (A.1.19),(A.1.20), we find that

\[
\hat{V} = (W^\dagger - VU^\dagger)^{-1} (VW^T - U^T) ,
\]

\[
\hat{\mu} = (W^\dagger - VU^\dagger)^{-1} (VU^\dagger - W^\dagger) \sigma + (U^T - VW^T) \sigma^* + \mu .
\]  

(A.2.4)

The expression for \( |V, \mu\rangle \) is given by combining eq. (A.2.2),(A.2.3),(A.2.4). We see that it does not depend on the (somewhat arbitrary) choice of \( W_V \).

### A.3 Transformation of coordinates and momenta

We can represent the effect of the Bogoliubov transformation on the canonical coordinates. Given the transformation (A.1.17) we define

\[
p_a = \frac{1}{\sqrt{2}} (a + a^\dagger) \quad p_b = \frac{1}{\sqrt{2}} (b + b^\dagger)
\]

\[
x_a = \frac{i}{\sqrt{2}} (a - a^\dagger) \quad x_b = \frac{i}{\sqrt{2}} (b - b^\dagger) .
\]  

(A.3.1)

We could in principle allow for \( \omega \) dependence here, but that can be thought of as a composition of yet another Bogoliubov transformation. On the canonical variables the transformation acts as a linear transformation.

\[
\begin{pmatrix}
  x_b \\
  p_b
\end{pmatrix} = \begin{pmatrix}
  \Re(W - U) & -\Im(W + U) \\
  \Im(W - U) & \Re(W + U)
\end{pmatrix} \begin{pmatrix}
  x_a \\
  p_a
\end{pmatrix} + \sqrt{2} \begin{pmatrix}
  \Im(\sigma) \\
  -\Re(\sigma)
\end{pmatrix} .
\]  

(A.3.2)

The non-homogeneous part of it is given by \( \sigma \). This is the most general real linear canonical transformation.
Notes

1.1 The statement about the consistency of string theory in 10 dimensions refer only to the supersymmetric string. In contrast the bosonic string, which is the subject of this thesis, is consistent only in 26 dimensions. This theory is perturbatively unstable, since the bosonic string contains a tachyonic mode. For a discussion on its nonperturbative vacuum see section 2.3.

1.2 $M$ theory is known by its low energy effective action which is 11 dimensional supergravity and as the infinite coupling limit of type IIA string theory.

2.1 The Euler number is a topological invariant that is equal to $2(1 - g)$ for a Riemann surface of genus $g$. For example, for the torus and sphere $\chi = 0$ and 2 respectively.

2.2 For the ghosts fields we use the doubling trick

$$c(z) = \bar{c}(\bar{z}') , \quad b(z) = \bar{b}(\bar{z}') \quad \text{for} \quad \text{Im}(z) \leq 0 , \quad z' = \bar{z} . \quad (A.3.3)$$

and work with a single set of Laurent modes for each field.

3.1 There are different approaches in defining half string coordinates according to whether or not the midpoint $x(\pi/2)$ is subtracted [22, 50]. This determines the half string boundary conditions and is related to the associativity anomaly [92] of the star product.

5.1 We have to be careful in our normalization since the three-vertex in the functional basis is not dimensionless. In the continuous half-string basis it will be evident that the normalization between the three-vertex state and functional can be naturally set to 1 by taking $\omega^l_\kappa = \omega^r_\kappa$, and in any case the normalization is
dimensionless since the dimensions of the three-vertex are contained in the $\delta$-functions. This pairing of modes fails for $\kappa = 0$, but of course, dependence on $\omega_0$ should disappear in all final results.

6.1 For this system to be (super) conformal invariant one need to make diffeomorphism transformations on the background to compensate on the conformal transformations.

6.2 We rescaled the fermionic charges by a factor $\sqrt{2}$ relative to [81].
Bibliography


