

8. Continuous groups

The discussion so far has focused on finite groups, containing a number of discrete elements. There are also groups with infinitely many elements. As a very simple example, the set of all integers forms a group with respect to addition. The elements of the group are denumerable, but there are infinitely many of them.

Another simple example is the set of rotations through some angle about a fixed axis, which form a group with respect to consecutive action. The elements of the group are labelled by the continuous variable θ , the angle of rotation. There are infinitely many of them and they are not denumerable. This is an example of a continuous group.

In general, the elements of a continuous group may be labelled by a number of continuous variables. For instance, the group of rotations in three dimensions with a fixed point is labelled by three angles — two to define the direction in space of the axis of rotation through the fixed point, the third to specify the angle of rotation.

For rotation groups, the range of the continuous variables is generally finite. An example of a continuous group with a variable of infinite range is the group of translations along a line by a distance x , where $-\infty < x < \infty$.

The continuous variables labelling the group elements are generally taken to be real. The composition rules relating the labels of a product to the labels of its factors may be either simple or complicated.

[For rotations R_θ through an angle θ about a fixed axis, $R_\theta R_\phi = R_{\theta+\phi}$. For translations T_x by a distance x along a fixed line, $T_x T_y = T_{x+y}$. Non-zero complex numbers form a group under the usual rule of multiplication, the two real labels of an element being its real and imaginary parts. Denoting $x + iy$ by $C_{x,y}$, the composition rule is $C_{x,y} C_{v,w} = C_{xv-yw, xw+yv}$.]

Square matrices over a field are closed under matrix multiplication. A unit matrix exists, and non-singular matrices may be inverted, so matrix groups can be defined, labelled by their entries. Different matrix groups may be defined by imposing various restrictions on the entries in the matrices.

The set of non-singular $n \times n$ matrices with complex entries constitutes the *general linear group* $\text{GL}(n, \mathbb{C})$ and has $2n^2$ real parameters. Restricting it to real entries produces the real general linear group $\text{GL}(n, \mathbb{R})$, with n^2 real

parameters. The subset of unitary matrices $[U^\dagger U = \mathbf{1}]$ with complex entries is closed under multiplication and defines the *unitary group* $U(n)$ with n^2 independent real parameters.

[n^2 complex entries means $2n^2$ real parameters. The unitarity condition imposes n real conditions from the diagonal entries and $n(n-1)/2$ complex conditions from the off-diagonal entries. There remain $2n^2 - n - 2 \cdot n(n-1)/2 = n^2$ independent parameters.]

Unitary matrices, interpreted as transformations of complex vectors, preserve the quadratic form $\sum_{i=1}^n z_i^* z_i$. From unitarity, $|\det U|^2 = 1 \implies \det U = e^{i\phi}$, where ϕ is a real function of the matrix parameters. The subset of $U(n)$ for which $\det U = 1$ comprises the *special unitary group* (or *unimodular unitary group*) $SU(n)$, with $n^2 - 1$ independent real parameters (since $\phi = 0$ imposes an additional condition on the n^2 independent parameters). It is an invariant subgroup of $U(n)$.

Related to the unitary matrices are those matrices which preserve the more general quadratic form $\sum_{i=1}^m z_i^* z_i - \sum_{i=m+1}^{m+n} z_i^* z_i$. They, too, form a group $U(m, n)$. In terms of the modified unit matrix $\mathcal{J}_{m,n} = \begin{pmatrix} \mathbf{1}_m & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_n \end{pmatrix}$, where $\mathbf{1}_m$ is an $m \times m$ unit matrix, each such matrix satisfies $U_{m,n}^\dagger \mathcal{J}_{m,n} U_{m,n} = \mathcal{J}_{m,n}$, so $|\det U_{m,n}|^2 = 1$. The group $SU(m, n)$ satisfies, in addition, the condition $\det U_{m,n} = 1$. From the definition it follows that $U(m, n) = U(n, m)$ and $U(n, 0) = U(0, n) = U(n)$.

Analogous to the unitary matrices is the subset of orthogonal matrices $[O^{\text{tr}} O = \mathbf{1}$, where A^{tr} is the transpose of the matrix A] with real entries, which is also closed under multiplication and defines the *orthogonal group* $O(n)$ with $n(n-1)/2$ real parameters.

[n^2 real entries, n conditions from the diagonal entries of the orthogonality condition, $n(n-1)/2$ conditions from the off-diagonal entries.]

Orthogonal matrices preserve the real quadratic form $\sum_{i=1}^n x_i^2$. From orthogonality, $(\det O)^2 = 1 \implies \det O = \pm 1$. The subset of $O(n)$ for which $\det O = 1$ comprises the *special orthogonal group* (or *unimodular orthogonal group*) $SO(n)$, an invariant subgroup of $O(n)$. Choosing $\det O = 1$ is a process of selection, which does not impose an additional condition on the parameters. There remain $n(n-1)/2$ independent real parameters.

There also occur the groups $O(m, n)$, which preserve the quadratic form $\sum_{i=1}^m x_i^2 - \sum_{i=m+1}^{m+n} x_i^2$ and for which $O(m, n) = O(n, m)$ and $O(n, 0) = O(0, n) = O(n)$. Similarly, there are unimodular groups $SO(m, n)$ in which every matrix has unit determinant.

Also of interest are the $2n$ -dimensional matrices which preserve the skew-symmetric quadratic form $\sum_{i,j=1}^{2n} x_i \mathcal{J}_{ij} y_j = \sum_{i=1}^n (x_i y_{i+n} - x_{i+n} y_i)$, where $\mathcal{J} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}$ is a $2n \times 2n$ matrix made of $n \times n$ unit and zero matrices. These $2n$ -dimensional *symplectic* matrices constitute the *symplectic group* $Sp(2n)$ with $n(2n + 1)$ real parameters. It can be shown that the symplectic matrices automatically have unit determinant, i.e. they are unimodular.

[It should be noted that alternative notations are sometimes used. For instance, the group denoted here by $Sp(2n)$ is sometimes denoted $Sp(n, \mathbb{R})$.]

Continuous groups are often parametrized in such a way that the identity element is characterized by vanishing parameters. Infinitesimal values of the parameters then characterize elements which are in some sense close to the identity. Other elements can then be obtained by frequent application of elements close to the identity.

As a concrete example, consider again rotations about some axis. Parametrising them by the angle of rotation θ , the identity has $\theta = 0$ and infinitesimally small values of θ correspond to infinitesimally small rotations. A large enough number of infinitesimal rotations will produce any desired finite rotation. There is therefore considerable interest in the nature and properties of group elements close to the identity.

[Note that not all group elements need be accessible by repeated application of elements close to the identity. In the case of the orthogonal group $O(n)$, for instance, those elements having a determinant of -1 cannot be obtained by continuous variation from the identity, which has determinant $+1$.]

Consider an element of a continuous group which is infinitesimally close to the identity, $\mathbf{1} + \epsilon M$, where ϵ is an infinitesimal. The defining properties of the group (unitary, orthogonal, symplectic, ...) will constrain M . For example, the condition of unitarity, to lowest order in infinitesimals, is $(\mathbf{1} + \epsilon M)^\dagger (\mathbf{1} + \epsilon M) = \mathbf{1} \implies M^\dagger = -M$, i.e. M must be anti-Hermitian. If, in addition, the group element is required to have unit determinant, M must have a vanishing trace.

[The $n \times n$ Hermitian matrix iM has n real eigenvalues. In the basis of its eigenvectors, it has only diagonal elements, say α_k . In this basis, the determinant of $(\mathbf{1} + \epsilon M)$ is $\prod_{k=1}^n (1 - i\epsilon\alpha_k) = 1 - i\epsilon \sum_{k=1}^n \alpha_k + O(\epsilon^2) = 1 \implies \sum_{k=1}^n \alpha_k = 0$. The result is true in general, because the determinant and trace of a matrix are independent of the basis.]

The group $\text{SU}(n)$ of unimodular unitary $n \times n$ matrices can be generated by the traceless anti-Hermitian $n \times n$ matrices. There are $n^2 - 1$ independent such matrices, so $\text{SU}(n)$ has $n^2 - 1$ *infinitesimal generators*. In the same way, the other matrix groups described above can be assigned finite sets of infinitesimal generators. In terms of its infinitesimal generators $\{G_i\}$, the elements of a continuous group can be represented as $\exp(\sum_i \alpha_i G_i)$, where the numbers $\{\alpha_i\}$ could supply a convenient parametrization of the group.

For such a form of the group elements to be acceptable, the product of two such elements must also take the same form, or else the group will not be closed under multiplication. But the Baker-Campbell-Hausdorff formula asserts that $e^X e^Y = e^Z$, where X, Y, Z are matrices and Z is given by the expansion $Z = X + Y + [X, Y]/2 + ([[X, Y], Y] + [X, [X, Y]])/12 + \dots$. All the terms in the expansion, from the third one on, are expressed purely in terms of multiple commutators of X and Y . So requiring $e^{\sum_i \alpha_i M_i} e^{\sum_i \beta_i M_i} = e^{\sum_i \gamma_i M_i}$ for some appropriate $\gamma_i(\{\alpha\}, \{\beta\})$ means requiring that $[M_i, M_j]$ is a linear combination of the $\{M_i\}$.

The continuous groups of interest in physics are *Lie groups*, whose elements are analytic functions of the continuous parameters. They can be expressed in terms of infinitesimal generators defined by derivatives of group elements, with respect to the parameters, close to the identity. The requirement that the commutator of any two infinitesimal generators is a linear combination of the infinitesimal generators confers on the set of generators a mathematical structure called a *Lie algebra*. These algebras will be the subject of the rest of the course. The above discussion indicates that any Lie group can be associated with a Lie algebra. Lie proved that any Lie algebra can be associated with a Lie group.

Examples

1. The group of unimodular, unitary 2×2 matrices is $\text{SU}(2)$. Each matrix has four complex entries, so eight real parameters are required to specify it. The condition of unitarity imposes four real restrictions (two real diagonal restrictions and one complex off-diagonal restriction) and the condition that

the determinant be +1 imposes one more. There remain three independent real parameters. This is a three-parameter continuous group. The infinitesimal generators of the group are traceless, anti-Hermitian 2×2 matrices. The most general such matrix takes the form

$$\begin{aligned} & \begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix} \\ &= a \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ &= aX_1 + bX_2 + cX_3, \end{aligned}$$

where X_1, X_2, X_3 generate the associated Lie algebra $\mathfrak{su}(2)$. (It is a widely-used convention to denote a group and its associated algebra by the same name, with the group name in upper-case letters and the algebra name in lower-case letters. This convention will be adopted here.) The commutators of the generators may be evaluated straightforwardly:

$$[X_1, X_2] = 2X_3, \quad [X_2, X_3] = 2X_1, \quad [X_3, X_1] = 2X_2.$$

2. The group of unimodular, orthogonal 3×3 matrices is $\mathbf{SO}(3)$. Each matrix has nine real entries, but the condition of orthogonality imposes three diagonal and three (symmetric) off-diagonal conditions, so there are three independent real parameters. The set of orthogonal 3×3 matrices divides into two subsets — one of matrices with determinant -1 , the other of matrices with determinant $+1$. The latter subset is relevant here, and constitutes a three-parameter continuous group. Near the identity, the condition of orthogonality is $(\mathbf{1} + \epsilon M)^{\text{tr}}(\mathbf{1} + \epsilon M) = \mathbf{1}$, where A^{tr} is the transpose of the matrix A and ϵ is an infinitesimal. To lowest order in infinitesimals, this becomes $M^{\text{tr}} = -M$, so the infinitesimal generators of the group are real, antisymmetric 3×3 matrices. The most general such matrix takes the form

$$\begin{aligned} & \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \\ &= a \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ &= aY_1 + bY_2 + cY_3, \end{aligned}$$

where Y_1, Y_2, Y_3 generate the associated Lie algebra $\mathfrak{so}(3)$. By straightforward evaluation, the commutators of the generators are:

$$[Y_1, Y_2] = -Y_3, \quad [Y_2, Y_3] = -Y_1, \quad [Y_3, Y_1] = -Y_2.$$

3. The group of unimodular, unitary 3×3 matrices is $\mathbf{SU}(3)$. Each matrix has nine complex entries. Unitarity imposes three real diagonal conditions and three complex off-diagonal conditions, while the requirement of unit determinant imposes one further real condition. There remain eight independent real parameters. The infinitesimal generators are traceless, anti-Hermitian 3×3 matrices. The most general such matrix takes the form

$$\begin{aligned} & \begin{pmatrix} ia & b+ic & d+ie \\ -b+ic & if & g+ih \\ -d+ie & -g+ih & -ia-if \end{pmatrix} \\ &= a \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} + b \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ &+ e \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} + f \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} + g \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \\ &= aZ_1 + bZ_2 + cZ_3 + dZ_4 + eZ_5 + fZ_6 + gZ_7 + hZ_8, \end{aligned}$$

where Z_1, Z_2, \dots, Z_8 generate the associated Lie algebra $\mathfrak{su}(3)$. The commutators of the generators are found to be:

$$\begin{array}{llll} [Z_1, Z_2] = Z_3 & [Z_1, Z_3] = -Z_2 & [Z_1, Z_4] = 2Z_5 & [Z_1, Z_5] = -2Z_4 \\ [Z_1, Z_6] = 0 & [Z_1, Z_7] = Z_8 & [Z_1, Z_8] = -Z_7 & [Z_2, Z_3] = 2Z_1 - 2Z_6 \\ [Z_2, Z_4] = -Z_7 & [Z_2, Z_5] = -Z_8 & [Z_2, Z_6] = Z_3 & [Z_2, Z_7] = Z_4 \\ [Z_2, Z_8] = Z_5 & [Z_3, Z_4] = Z_8 & [Z_3, Z_5] = -Z_7 & [Z_3, Z_6] = -Z_2 \\ [Z_3, Z_7] = Z_5 & [Z_3, Z_8] = -Z_4 & [Z_4, Z_5] = 2Z_1 & [Z_4, Z_6] = -Z_5 \\ [Z_4, Z_7] = -Z_2 & [Z_4, Z_8] = Z_3 & [Z_5, Z_6] = Z_4 & [Z_5, Z_7] = -Z_3 \\ [Z_5, Z_8] = -Z_2 & [Z_6, Z_7] = 2Z_8 & [Z_6, Z_8] = -2Z_7 & [Z_7, Z_8] = 2Z_6 \end{array}$$