Conjugate of a cycle

Consider a group of permutations of n objects. Let $T = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \lambda'_1 & \lambda'_2 & \dots & \lambda'_n \end{pmatrix}$ be a generic member of the group and let $C = (a_1 \quad a_2 \quad a_3 \dots a_m)$ be an m-cycle in the group (where generally $m < n$). Written in full,

$$C = \begin{pmatrix} a_1 & a_2 & \dots & a_m & b_1 & b_2 & \dots & b_{n-m} \\ a_2 & a_3 & \dots & a_1 & b_1 & b_2 & \dots & b_{n-m} \end{pmatrix},$$

so the cycle divides the objects being permuted into two distinct non-overlapping sets — those whose labels belong to the cycle, denoted $\{a_i\}$, and those whose labels do not belong to the cycle, denoted $\{b_i\}$.

[Comment: each of the sets of labels $\{a_i\}$, $\{b_i\}$, $\{\lambda_i\}$, $\{\lambda'_i\}$ and $\{a'_i\}$ is simply the set of natural numbers $\{1, 2, 3, \dots, n\}$ or an appropriate subset of it.]

Now consider the conjugate of the cycle, TCT^{-1}. The effect of T on the generic label λ depends on whether $\lambda \in \{a_i\}$ or $\lambda \notin \{a_i\} \Longrightarrow \lambda \in \{b_i\}$. Take first the case $\lambda \notin \{a_i\} \Longrightarrow \lambda' \in \{b_i\}$. The inverse T^{-1} changes λ' into λ, then the cycle C leaves λ unchanged, and finally the permutation T changes λ back into λ'. The labels belonging to the set $\{b_i\}$ are unaffected by the conjugate.

Next take the case $\lambda \in \{a_i\}$ i.e. $\lambda = a_k$ for some k. Once again, the inverse T^{-1} changes λ' into $\lambda = a_k$, then the cycle C changes a_k into a_{k+1}, and finally T changes a_{k+1} into a'_{k+1}. The net effect of the conjugate permutation is

$$TCT^{-1} = \begin{pmatrix} a'_1 & a'_2 & \dots & a'_m & b_1 & b_2 & \dots & b_{n-m} \\ a'_2 & a'_3 & \dots & a'_1 & b_1 & b_2 & \dots & b_{n-m} \end{pmatrix}.$$

This is an m-cycle $(a'_1 \quad a'_2 \quad \dots a'_m)$, where the entries are the images of the original m-cycle under the permutation T.

It may be concluded that conjugate permutations have the same cycle structure. The converse is true for the full permutation group $\mathcal{S}_n$, since the above discussion ensures that two permutations with the same cycle structure can be conjugated with the assistance of the permutation whose rows are the list of cycles, appropriately paired,

$$T = \begin{pmatrix} C_1 & C_2 & \dots & C_r \\ C'_1 & C'_2 & \dots & C'_r \end{pmatrix}$$

where C_i and C'_i are of the same degree. [The converse need not hold for a general group of permutations, necessarily a subgroup of $\mathcal{S}_n$, since it may not contain the permutation required to ensure that the two permutations belong to the same class.]

A concrete example

Consider two arbitrarily chosen permutations with the same cycle structure, $\nu_1 = 2$, $\nu_2 = 3$, $\nu_3 = 2$, $\nu_4 = 1$. (The total number of permuted objects is $\sum_r r\nu_r = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 2 + 1 \cdot 4 = 18$.)

$$P_1 = (12)(14)(1,6)(4,9)(7,10)(2,5,8)(3,11,17)(13,15,16,18)$$

$$P_2 = (6)(9)(2,18)(5,7)(4,10)(1,3,11)(8,15,16)(12,13,14,17)$$

These may be rewritten as two-row permutation symbols

$$P_1 = \begin{pmatrix} 12 & 14 & 1 & 6 & 4 & 9 & 7 & 10 & 2 & 5 & 8 & 3 & 11 & 17 & 13 & 15 & 16 & 18 \\ 12 & 14 & 6 & 1 & 9 & 4 & 10 & 7 & 5 & 8 & 2 & 11 & 17 & 3 & 15 & 16 & 18 & 13 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 6 & 9 & 2 & 18 & 5 & 7 & 4 & 10 & 1 & 3 & 11 & 8 & 15 & 16 & 12 & 13 & 14 & 17 \\ 6 & 9 & 18 & 2 & 7 & 5 & 10 & 4 & 3 & 11 & 1 & 15 & 16 & 8 & 13 & 14 & 17 & 12 \end{pmatrix}$$

and should be conjugates of one another under

$$T = \begin{pmatrix} 12 & 14 & 1 & 6 & 4 & 9 & 7 & 10 & 2 & 5 & 8 & 3 & 11 & 17 & 13 & 15 & 16 & 18 \\ 6 & 9 & 2 & 18 & 5 & 7 & 4 & 10 & 1 & 3 & 11 & 8 & 15 & 16 & 12 & 13 & 14 & 17 \end{pmatrix}.$$

Pay attention to the way T is constructed. The upper row is the list of cycles in the decomposition of P_1, with the brackets separating the cycles omitted; the lower row is the same, but for the permutation P_2. The cycle decomposition of T is

$$T = (1 \quad 2)(3 \quad 8 \quad 11 \quad 15 \quad 13 \quad 12 \quad 6 \quad 18 \quad 17 \quad 16 \quad 14 \quad 9 \quad 7 \quad 4 \quad 5)(10)$$

[Note the very different cycle structure of the permutation T: $\nu_1 = 1$, $\nu_2 = 1$, $, \nu_{15} = 1$.] The relevant product is

$$\begin{aligned} TP_1T^{-1} &= \begin{pmatrix} 12 & 14 & 1 & 6 & 4 & 9 & 7 & 10 & 2 & 5 & 8 & 3 & 11 & 17 & 13 & 15 & 16 & 18 \\ 6 & 9 & 2 & 18 & 5 & 7 & 4 & 10 & 1 & 3 & 11 & 8 & 15 & 16 & 12 & 13 & 14 & 17 \end{pmatrix} \\ &\quad \times \begin{pmatrix} 12 & 14 & 1 & 6 & 4 & 9 & 7 & 10 & 2 & 5 & 8 & 3 & 11 & 17 & 13 & 15 & 16 & 18 \\ 12 & 14 & 6 & 1 & 9 & 4 & 10 & 7 & 5 & 8 & 2 & 11 & 17 & 3 & 15 & 16 & 18 & 13 \end{pmatrix} \\ &\quad \times \begin{pmatrix} 6 & 9 & 2 & 18 & 5 & 7 & 4 & 10 & 1 & 3 & 11 & 8 & 15 & 16 & 12 & 13 & 14 & 17 \\ 12 & 14 & 1 & 6 & 4 & 9 & 7 & 10 & 2 & 5 & 8 & 3 & 11 & 17 & 13 & 15 & 16 & 18 \end{pmatrix} \\ &= P_2 \end{aligned}$$

exactly as it should be. [The multiplication was carried out by use of the cancellation method.]