Introductory Algebra — Exercise no. 5
Due Thursday, 29 December, 2016

1. Show that the inverses of the elements of the k^{th} class K_k of a group $\mathcal{G}$ form a class, denoted $\mathsf{K}_{k'}$. Prove that the characters satisfy

$$\sum_k g_k \chi_k^{(\mu)} \chi_{k'}^{(\nu)} = g\delta_{\mu\nu},$$

where g_k is the number of elements in the class K_k (and $\mathsf{K}_{k'}$) and g is the order of $\mathcal{G}$.

2. Consider the quaternion group $\mathcal{Q}$, of order 8. How many different irreps does it have and what are their dimensions? What are the characters of the regular representation of the group and of the unit representation?

Show that the vector space of quaternions (i.e. the space of all linear combinations of the four quaternion units, with complex coefficients) carries a four-dimensional reducible representation of the group. Prove that this representation does not contain the unit representation, deduce from this that it must contain a two-dimensional irrep twice and find the characters of the two-dimensional irrep.

Compare the multiplication rules of the Pauli matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with those of the quaternion units $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and write down the representation matrices of the two-dimensional irrep.

Complete the character table of the group.