# Introductory Algebra — Exercise no. 2
# Due Thursday, 24 November, 2016

1. Given a group $\mathcal{G}$ and an element $X \in \mathcal{G}$. Show that

$$G_i \longmapsto X G_i X^{-1}, \text{ for all } G_i \in \mathcal{G}$$

is an isomorphism from $\mathcal{G}$ to $\mathcal{G}$ (called an inner automorphism).

2. Quaternions are a generalisation of complex numbers. A quaternion can be written in the form $a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where $a, b, c, d$ are real numbers and the quaternion units $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ have the properties

$$\mathbf{1}^2 = \mathbf{1}; \quad \mathbf{i}^2 = -\mathbf{1}; \quad \mathbf{j}^2 = -\mathbf{1}; \quad \mathbf{k}^2 = -\mathbf{1}$$

$$\mathbf{1i} = \mathbf{i} = \mathbf{i1}; \quad \mathbf{1j} = \mathbf{j} = \mathbf{j1}; \quad \mathbf{1k} = \mathbf{k} = \mathbf{k1}$$

$$\mathbf{ij} = \mathbf{k}; \quad \mathbf{ik} = -\mathbf{j}; \quad \mathbf{ji} = -\mathbf{k}; \quad \mathbf{jk} = \mathbf{i}; \quad \mathbf{ki} = \mathbf{j}; \quad \mathbf{kj} = -\mathbf{i}$$

where $-\mathbf{u}$ is interpreted as $-1\mathbf{u}$, for any quaternion unit $\mathbf{u}$. The quaternion units commute with all real numbers.

Show that the 8 elements $\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$ form a group under the multiplication defined by the above properties. Find the order of each element and decompose the group into conjugacy classes. Find an invariant subgroup of order 2 and an invariant subgroup of order 4.