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A contribution to the special feature 'Perspectives in astrophysical and geophysical fluids'.

# Interaction between mean flow and turbulence in two dimensions

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This short note is written to call attention to an analytic approach to the interaction of developed turbulence with mean flows of simple geometry (jets and vortices). It is instructive to compare cases in two and three dimensions and see why the former are solvable and the latter are not (yet). We present the analytical solutions for two-dimensional mean flows generated by an inverse turbulent cascade on a sphere and in planar domains of different aspect ratios. These solutions are obtained in the limit of small friction when the flow is strong while turbulence can be considered weak and treated perturbatively. I then discuss when these simple solutions can be realized and when more complicated flows may appear instead. The next step of describing turbulence statistics inside a flow and directions of possible future progress are briefly discussed at the end.

## 1. Introduction

On the face of it, it seems straightforward to describe the interaction of turbulence and a mean flow of a given simple geometry. In the region where the flow gradient is large, one can assume turbulence suppressed and account for it perturbatively. Conversely, in the region where the flow gradient is small, like near velocity maxima or minima, one can assume almost isotropic and homogeneous turbulence and describe turbulence distortions perturbatively. One then attempts to match the results of the two descriptions in some intermediate regions, lines or surfaces and obtains as a result both the mean flow profile and the correlation functions

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of turbulence. The two approaches were applied non-systematically and separately: the first one goes back to the rapid distortion theory [1], the second one was used for phenomenological prediction of the anisotropy correction [2] and modifications of weak turbulence spectra [3]. While the systematic development of the second approach is now ongoing with the use of the powerful field-theoretical formalism of the operator product expansion (OPE) [4], the work on the first approach is mainly done by numerical computations [5–9].

Despite almost 100 years of effort, one cannot yet derive, for instance, the mean flow profile of three-dimensional turbulent boundary layer from any self-consistent theory. One can readily recognize that the principal source of difficulty in describing the strong-flow–weak-turbulence limit in three dimensions is the usual problem of closure. Indeed, two fundamental conservation laws provide for two balance equations: one for momentum, another for energy. The momentum equation away from walls contains two quantities: the mean flow $U(\boldsymbol{r})$ and the turbulence momentum flux $\tau(\boldsymbol{r}) = \langle uv \rangle$, where $u, v$ are the fluctuating velocity components: $u \parallel U$ and $v \parallel \nabla U$. The energy exchange, between turbulence and the flow, $\tau U'$, is also expressed via these two quantities. However, the energy balance equation also involves the turbulence dissipation rate $\epsilon(\boldsymbol{r})$ whose dependence on the coordinates is *a priori* unknown. To get this dependence, one needs to develop a theory of the direct cascade inside the inhomogeneous mean flow and relate $\epsilon(\boldsymbol{r})$ to $U(\boldsymbol{r})$. In the absence of such a theory, one resorts to hand-waving as described in the next section.

It is very much different for two-dimensional turbulence generated by a small-scale force. When the inverse cascade reaches the box scale, it produces a box-size mean flow sometimes called condensate, as was predicted by Kraichnan and observed in numerous settings [10–21]. Two types of the mean flow were observed. First, an inverse cascade in a square box on a plane produces a vortex, either accompanied by an opposite-sign vortex in a double-periodic domain in numerics or by four counter-rotating small vortices in the box corners in experiments. Second, it was observed in numerics that when the aspect ratio of the torus changed away from unity, the mean flow topology changed from dipole to unidirectional flow of two counter-propagating jets (admittedly, for not very developed inverse cascade) [22]. In all these cases, the flow is fed by the turbulence, and the feeding rate is equal to the force input rate, which again can be called $\epsilon(\boldsymbol{r})$, but which is now set externally, that is known. Assuming turbulence weak comparing with the flow, one can obtain the closed system of equation on the two-point correlation functions of turbulence, as described briefly in §6. Because it involves the pressure contributions, those are integral equations, which till now were solved numerically [5–8,14]. In a circular geometry of a strong vortex, some analytic progress is possible, as described in §6. Moreover, it was noted in [23] and argued below that while the correlation functions of the pressure gradients with respective velocity components play a substantial role in redistributing the energy between the velocity components (to put it simply, the role of pressure is to restore isotropy), neglecting the correction function of the pressure itself with velocity in the total energy balance leads to predictions which quantitatively agree with direct numerical simulations (DNS). After such neglect, one obtains a closed equation on $U, \tau$ which can be solved analytically. Particularly, simple is the case of a spatially uniform forcing, which allows elegant analytic solutions for $U(\boldsymbol{r})$ and $\tau(\boldsymbol{r})$ as described in subsequent subsections. While the initial theoretical work has been motivated by laboratory experiments in the simplest settings (thin layers, solenoidal forcing, etc. [12,15,16,18–20]), the further development of the formalism may be of interest for geophysics and astrophysics, as manifested by amazingly simple analytic solutions for turbulence–flow interaction on a sphere, reported here for the first time.

Yet there are situations when the mean flow does not have a simple geometry. One then may need to account for three, rather than two, entities (such as turbulence, jet and vortex or turbulence, jet and waves), which requires analysing the third conservation law—that of enstrophy. Because the enstrophy is mainly dissipated in the direct cascade, then its dissipation rate is *a priori* unknown like the energy dissipation rate in three dimensions. Still, the enstrophy cascade in two dimensions is simpler than the energy cascade in three dimensions, so some constructive use of the enstrophy balance is possible.

## 2. Three-dimensional boundary layer

Let us describe the problem one encounters in trying to find the mean profile of a turbulent boundary layer in three dimensions. Consider the flow set in motion by an infinite plane boundary at $y=0$ moving along $x: \boldsymbol{v}=(U(y)+u(\boldsymbol{r},t), v(\boldsymbol{r},t), w(\boldsymbol{r},t))$. Averaging the Navier–Stokes equation

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \nu\Delta\boldsymbol{v} - \nabla p, \tag{2.1}$$

we get the momentum balance, assuming homogeneity in $x$ and $z$ and unit density

$$\frac{\partial\langle uv\rangle}{\partial y} - \nu\frac{\partial^2 U}{\partial y^2} = 0 \tag{2.2}$$

and

$$\partial_y\left\langle p+v^2\right\rangle = \partial_y\langle wv\rangle = 0. \tag{2.3}$$

The momentum flux is constant: $\langle uv\rangle - \nu U' = \text{const}$. Outside of the viscous layer, $\langle uv\rangle = \tau = \text{const}$. The sign of it is evidently towards lower values of $U$, so that $\tau \mathrm{d}U/\mathrm{d}y = \tau U' < 0$. For the energy balance, one obtains in a steady state

$$\frac{\partial\left\langle v\left[p+\left(u^2+v^2+w^2\right)/2\right]\right\rangle}{\partial y} + \epsilon + U'\tau = 0. \tag{2.4}$$

The first term is the spatial divergence of turbulence energy flux, the second term, $\epsilon = -\nu\langle \boldsymbol{v}\Delta\boldsymbol{v}\rangle$, is the viscous energy dissipation rate. The last term must be negative in three dimensions and describe the rate of the energy transfer from the mean flow to the fluctuations. Now, even if one assumes turbulence weak [24,25], neglects $\partial_y\langle v[p+(u^2+v^2+w^2)/2]\rangle$ and obtains

$$\epsilon(y) \approx -U'(y)\tau, \tag{2.5}$$

this simple equation is not closed. To close it, one needs to find how the energy dissipation rate depends on the mean flow profile. For that, one needs to develop a theory of the direct energy cascade inside a mean flow. While such theory is absent, several hand waving arguments were suggested [26,27]. For example, one *assumes* that the turbulence level $v^2$ is independent of $y$, then the energy flux, $\epsilon(y) \simeq v^3/\ell(y)$, depends on $y$ only via the typical scale of turbulence, which can be estimated as $\ell \simeq U'/U''$. One substitutes that into (2.5) and obtains $U'' \propto -U'^2$, which gives $U' \propto 1/y$ and a logarithmic profile $U \propto \ln y$. If, in addition, one estimates $v^2 \simeq \tau$, then $U \simeq \sqrt{\tau}\ln y$, up to a dimensionless constant.

Another way to present this argument is to say that the Galilean invariant quantity $U'$ must be determined only by the momentum flux $\tau$ and $y$, which necessarily gives $U' \simeq \sqrt{\tau}/y$. Note the analogy between this argument and the Kolmogorov–Obukhov 1941 assumption that the energy flux and distance between two points completely determine the statistics of the velocity differences in homogeneous isotropic turbulence. The statistics thus depends neither on the system size $L$ nor on the viscous scale $\eta$. The Kolmogorov–Obukhov assumption, however, is now known to be incorrect, because all the moments of the velocity difference, except the third one, depend on the system size via the so-called anomalous scaling [28]. Similarly, turbulence properties and the mean flow in the boundary layer at given $\tau, y$ may depend on either the width $\eta$ of the viscous layer or the distance $L$ to the second wall (or pipe radius) even when $\eta \ll y \ll L$. This is less far-fetched than it might seem to Prandtl and Karman, who did not know about what we now call very large-scale motions (vortices), having a scale exceeding $y$ [29,30]. Those vortices are almost parallel to the wall, so Townsend called them 'inactive' because they do not participate in the momentum transfer to the wall [24]. However, such planar vortices, in principle, can be part and product of an inverse energy cascade [31]; if true that would mean that at least part of what is given to turbulence returns to the mean flow via an inverse cascade, so that (2.5) may be not entirely correct. Vortices attached to the wall may also bring influence of viscosity. Note also that the estimate for the turbulence velocity, $v \simeq \sqrt{\tau}$, if true, would mean $\langle uv\rangle \simeq \langle u^2+v^2\rangle$, which also may be incorrect, at least logarithmically [24,25]. With all this in mind, consistent theory is still ahead of us.

## 3. Inverse cascade on a sphere

Let us describe now the case where the closed solution is possible. Consider the forced Euler equation with a bottom friction on a rotating sphere, where one introduces the angular coordinates, latitude $\theta \in [-\pi/2, \pi/2]$ and longitude $\phi \in [0, 2\pi]$. Then, the equations of motion for $u = v_\phi$ and $v = v_\theta$ take the form

$$\frac{\partial v}{\partial t} + \frac{v}{R}\frac{\partial v}{\partial \theta} + \frac{U+u}{R\cos\theta}\frac{\partial v}{\partial \phi} + \frac{(U+u)^2}{R\cot\theta} + 2\Omega(U+u)\sin\theta = f_\theta - \alpha v - \frac{\partial p}{R\partial \theta} \tag{3.1}$$

and

$$\frac{\partial u}{\partial t} + \frac{v}{R}\frac{\partial (U+u)}{\partial \theta} + \frac{U+u}{R\cos\theta}\frac{\partial u}{\partial \phi} - \frac{(U+u)v}{R\cot\theta} - 2\Omega v\sin\theta = f_\phi - \alpha(U+u) - \frac{1}{R\cos\theta}\frac{\partial p}{\partial \phi}. \tag{3.2}$$

Here, $\Omega$ is the angular frequency of rotation and $\alpha$ is the friction coefficient. The external force $\boldsymbol{f} = (f_\phi, f_\theta)$, exciting turbulence, is assumed to be random function, short-correlated in time and with the correlation length $l_{\mathrm{f}}$ much less than the sphere radius $R$. The energy production rate per unit mass is $\epsilon = \langle \boldsymbol{f}\boldsymbol{v}\rangle$ (strictly speaking, that must be the net energy production rate $\epsilon = \langle \boldsymbol{f}\boldsymbol{v}\rangle - \nu\langle|\nabla \boldsymbol{v}|^2\rangle$, but we consider the limit $\nu \to 0$ when the viscous energy dissipation is negligible). There is an extensive literature on DNS of the system (3.1) and (3.2) [32,33]. We can also write it for the vorticity

$$\omega = \frac{1}{R\cos\theta}\frac{\partial v}{\partial \phi} - \frac{1}{R\cos\theta}\frac{\partial}{\partial \theta}u\cos\theta \tag{3.3}$$

and

$$\frac{\partial \omega}{\partial t} + \frac{U+u}{R\cos\theta}\frac{\partial \omega}{\partial \phi} + \frac{v}{R}\frac{\partial \omega}{\partial \theta} - \frac{v}{R}\frac{\partial}{\partial \theta}\frac{1}{\cos\theta}\frac{\partial}{\partial \theta}U\cos\theta + 2\Omega v\cos\theta = -\alpha\omega + \nabla \times \boldsymbol{f}. \tag{3.4}$$

To have a global flow, $U$, appearing out of turbulent inverse cascade, one needs the friction coefficient to be small enough, so that the decay time is much larger than the turnover time on the scale $R$:

$$\delta = \alpha R^{2/3}\epsilon^{-1/3} \ll 1. \tag{3.5}$$

We assume axial symmetry, i.e. all the average quantities, $U, \epsilon, \tau = \langle uv\rangle$, are independent of $\phi$. Averaging (3.2), one obtains for the angular momentum

$$\alpha RU = 2\tau\tan\theta - \frac{\partial \tau}{\partial \theta} = -\frac{1}{\cos^2\theta}\frac{\partial}{\partial \theta}\tau\cos^2\theta = \langle v\omega\rangle R. \tag{3.6}$$

Here, we used the incompressibility relation, $\partial_\phi u + \partial_\theta v\cos\theta = 0$. We now multiply (3.1) by $v$ and (3.2) by $U+u$ to obtain the energy relation

$$(\epsilon - \alpha U^2)R = \frac{1}{\cos\theta}\frac{\partial}{\partial \theta}\cos\theta\left\langle v\left\{p + \left[(U+u)^2 + v^2\right]/2\right\}\right\rangle. \tag{3.7}$$

Now, we again assume weak turbulence and neglect the energy flux owing to turbulence, $\langle v[p + (u^2+v^2)/2]\rangle$, which is supposed to be small in the limit $\alpha \to 0$. The most delicate part here is neglecting the pressure–velocity correlation term $\langle vp\rangle$. Indeed, the mean pressure is of order $U^2$ as seen from (3.17) below, whereas the pressure fluctuations can be expected to be of order $Uu$, so that its correlation with $v$ can give the terms of the same order as $U\tau$. It is empirically known that neglecting the pressure term for a circular vortex on a plane gives a quantitatively correct answer [23], but it is unclear at the moment whether it is true in other cases. That can be related to the anomaly of the momentum flux term, so that the single-point value $\langle uv\rangle$ exceeds the two-point correlation function $\langle u((\boldsymbol{r}_1)v(\boldsymbol{r}_2)\rangle$ (which determines the $\langle vp\rangle$ term) for the distances $r_{12}$ exceeding the force scale. That, however, requires the development of the theory sketched in §6, which is ahead of us. So we neglect the pressure term in (3.7), remembering, however, that in considering the energy of the components

$$2(U\tan\theta + \Omega\sin\theta)\tau - \epsilon R/2 + \langle v\partial_\theta p\rangle = 0 \tag{3.8}$$

and

$$\tau\partial_\theta U - (U\tan\theta + 2\Omega\sin\theta)\tau - \epsilon R/2 + \langle u\partial_\phi p\rangle/\cos\theta = 0, \tag{3.9}$$

the separate pressure terms cannot be neglected. The pressure does work redistributing energy between the velocity components (to decrease anisotropy) but does not contribute to the overall energy balance in this order (similarly to three dimensions, see e.g. section 6.2 in [27]). In other words, we assume that the pressure fluctuation does not correlate with the velocity fluctuation at a point, but it does correlate with the convergence/divergence of the longitudinal velocity. Equivalently, the velocity is not correlated with the pressure but is correlated with the respective component of the pressure gradient.

The energy balance in this approximation is as follows

$$(\epsilon - \alpha U^2)R = \partial_\theta U\tau - U\tau\tan\theta. \tag{3.10}$$

Alternatively, one can sum (3.8) and (3.9) and obtain $U\tau\tan\theta + \tau\partial_\theta U = \epsilon R$. One can exclude $U$ from (3.6) and (3.10) and obtain a closed equation

$$\alpha R^2\epsilon(\theta) = 2\tau^2\frac{\sin^2\theta + 1}{\cos^2\theta} + \tau\tau'\tan\theta - \tau\tau''. \tag{3.11}$$

For a constant $\epsilon$, the solution of (3.6) and (3.10) is particularly simple

$$\tau = \langle uv\rangle = \pm\sqrt{\alpha\epsilon/3}R\cos\theta \quad \text{and} \quad U = \pm\sqrt{3\epsilon/\alpha}\sin\theta. \tag{3.12}$$

This simple solution, to the best of my knowledge, has never been reported before. It describes well the profile obtained by DNS for a non-rotating sphere, see, for instance, fig. 4 in [33] (where the angular momentum $M = U\cos\theta$ is plotted). The velocity profile is concave, smoothly going from linear to a constant as $\theta$ changes from zero to $\pi/2$, that is the solution describes both the zonal flow near the equator and the polar vortex. That polar vortex has a limiting form $U = \sqrt{3\epsilon/\alpha}$ that has been found analytically and confirmed by DNS on a plane before [23]. Within the framework of (3.6)–(3.11) the solution (3.12) is regular, but if one plugs it into the viscous term, it gives a pole singularity. Viscosity then must make the polar vortex core finite.

A turbulence source localized near equator generates a mean flow with maximum at mid-latitude

$$\epsilon(\theta) = \epsilon_0\cos^4\theta \Rightarrow \tau = \pm\sqrt{\alpha\epsilon_0/4}\cos^2\theta \quad \text{and} \quad U = \pm\sqrt{\epsilon_0/\alpha}\sin 2\theta. \tag{3.13}$$

There is a solid-body rotation around the poles, and thus no polar vortex core in this case.

One can also find an analytic solution for a source concentrated in mid-latitudes

$$\left.\begin{aligned} &\epsilon(\theta) = 25\epsilon_0\sin\theta(2\sin\theta + 3\sin 3\theta + \sin 5\theta) = 100\epsilon_0\sin^2\theta\cos^4\theta \Rightarrow \\ \text{and} \quad &\tau = \pm\sqrt{\frac{\alpha\epsilon_0}{2}}4\sin\theta\cos^2\theta, \quad U = \pm\sqrt{\frac{\epsilon_0}{\alpha}}(\cos\theta - 5\cos 3\theta), \\ &= \pm 4\sqrt{\frac{\epsilon_0}{\alpha}}\cos\theta(4 - 5\cos^2\theta). \end{aligned}\right\} \tag{3.14}$$

The respective flow has three jets. The velocity changes sign at $\theta = \pm\arctan(1/2)$ (the dihedral angle of great dodecahedron). Note that the empirical fit $U \propto a\cos\theta - \cos 3\theta$, with $a \simeq 0.4$–$0.6$ works well as a parametrization of the Uranus flow [34]; the Neptune flow has a similar three-jet structure [35], whereas flows on Jupiter and Saturn are multi-jet. Further analysis is needed to see whether the turbulence production profile $\epsilon(\theta)$ can, indeed, explain some features of these flows.

The global structure of surface zonal winds in the Earth atmosphere has five jets—westerlies at mid-latitudes and easterlies near equator and poles [36,37]. Needless to say that our simple two-dimensional theory that neglects temperature gradients and vertical circulation cannot be a realistic model even for the mean circulation patterns of the Earth atmosphere. For example, vertical recirculation owing to Hadley cell allows for a non-zero north–south mean tropical flow at the ground with respective Coriolis force changing the balance (3.6). Such patterns can hopefully be included into a more realistic theory together with turbulence.

Let us stress that the global flow profiles and the turbulent momentum fluxes (3.12)–(3.14) have been derived from the energy–momentum balance and not from any entropy argument, suggested for the description of large-scale flows appearing from turbulence decay [38,39]. Of course, for such symmetric flows, one can always relate vorticity $\Omega = -\Delta\psi$ and stream function $\psi$: $\Delta\psi = -\psi + 1/\psi$, $\Delta\psi = -1 - \psi$ and $\Delta\psi = -\psi$ written in dimensionless units for (3.12)–(3.14), respectively. Those relations can also be trivially presented as extrema of some functionals, which, however, do not look like entropy and depend on the profile of the energy input.

Remark that even though we assume weak fluctuations, we find the mean flow profile and the momentum flux self-consistently, that is our approach is different from consideration of small perturbations on the background of a given flow [5].

Formally, the function $\epsilon(\theta)$ does not determine $\tau(\theta), U(\theta)$ unambiguously, because there exist two one-parametric families of homogeneous solutions, which turn the right-hand side of (3.11) into zero. The first family corresponds to the constant momentum flux $J$ from pole to pole: $\tau = J/\cos^2\theta$ and $U = 0$. The second one corresponds to the divergence of the turbulence flux exactly compensating the friction loss: $\tau = A\sin\theta(1 + 2/\cos^2\theta)$ and $U = -3A\cos\theta$. Both have pole singularities in $\tau$, acting as point sources and sinks of momentum and energy, respectively. One can also find the two-parametric family of solutions with a constant energy flux $F = U\tau\cos\theta$ and $C \geq 2$:

$$g = \pm\sqrt{C - \sin\theta(2 + \cos^2\theta)}, \quad \tau = \sqrt{\frac{2F\alpha}{3}}\frac{g(\theta)}{\cos^2\theta}, \quad U = \sqrt{\frac{3F}{2\alpha}}\frac{\cos\theta}{g(\theta)} \quad \text{and} \quad \epsilon = \alpha U^2. \tag{3.15}$$

The solutions from all the three families have $\tau$ singular at the poles. The last two families have a non-zero angular momentum

$$\int_{-\pi/2}^{\pi/2} U(\theta)\cos^2\theta \,d\theta = \tau(\theta)\cos^2\theta \,|_{-\pi/2}^{\pi/2}. \tag{3.16}$$

All smooth $\tau(\theta)$, as well as those having the pole singularities of the flux weaker than $\tau(\theta) \propto 1/\cos^2\theta$, have zero total angular momentum. Note that singularities in the turbulent momentum flux cannot be ruled out on physical grounds, because we have taken the limit of zero forcing scale. Accounting for a finite forcing scale most likely will regularize the flux.

Averaging (3.1), one expresses the mean pressure via the mean velocity profile

$$U^2\tan\theta + 2\Omega RU\sin\theta = -\frac{\partial p}{\partial\theta}. \tag{3.17}$$

For the solution (3.12), the pressure has logarithmic singularities at the poles. Note that rotation influences only the equation for the mean pressure distribution. The relation between $U$ and $\tau$ as well as the energy balance are independent of $\Omega$ in this approximation. It would be wrong, however, to assume that the approximation works uniformly for any $\Omega$ and any $\epsilon(\theta)$. A revealing insight into the validity of approximation and possibilities of more complicated flows is provided by the analysis of higher vorticity invariants in the next section.

## 4. Validity limits of the strong-flow–weak-turbulence approximation

Our approach is expected to work when $\alpha$ is small. Here, we want to establish whether our approximation requires the dimensionless parameter $\delta = \alpha R^{2/3}\epsilon^{-1/3}$ to be smaller not only than unity, but also than other dimensionless parameters and whether our approximation works uniformly over the whole domain.

Let us briefly recapitulate the salient points of the analysis in the preceding section, focusing on the breakdown of time reversibility. We ask how the magnitudes of different reversibility-breaking moments depend on $\alpha$. Appearance of the mean flow is an example of spontaneous symmetry breaking, that is the direction of the flow in a given point is chosen randomly (absent rotation). The amplitude of the flow diverges as $\alpha \to 0$, so that the energy dissipation rate by the coherent part of the flow has a finite limit. With the direction of the mean flow given, different turbulent fluxes across the flow also break time reversibility, most importantly the turbulent momentum flux, $\langle uv \rangle$, which changes sign upon $t \to -t, U \to -U$. The magnitude of that flux, however, goes to zero as $\alpha \to 0$. That may be related to the fact that $\alpha$ is the only factor breaking time reversibility. The flux of vorticity is expressed via the momentum flux, $\langle v\omega \rangle = -r^{-2}\partial_r r^2 \langle uv \rangle$, and also goes to zero as $\alpha \to 0$. We now want to check if the fluxes of higher powers of vorticity can be also neglected for the solutions we obtained. For that end, we need to account for viscosity which is responsible for most of the dissipation of higher powers of vorticity. Adding $\nu \Delta \omega$ to (3.4), multiplying by $\omega^n$ and averaging, we obtain

$$\langle v\omega^n \rangle \left( 2\Omega \cos\theta - \frac{\partial}{\partial\theta} \frac{1}{\cos\theta} \frac{\partial}{\partial\theta} U \cos\theta \right) = Q_n - \frac{1}{(n+1)\cos\theta} \frac{\partial}{\partial\theta} \cos\theta \left\langle v\omega^{n+1} \right\rangle, \tag{4.1}$$

where $Q_n(\theta) = \langle \omega^n \nabla \times \boldsymbol{f} \rangle + \nu \langle \omega^n \Delta\omega \rangle - \alpha \langle \omega^{n+1} \rangle$ is a net input rate. Taking $n=1$ and substituting $\langle v\omega \rangle = \alpha U$, we obtain the balance of squared vorticity called enstrophy (in geophysical literature, it is often called the Eliassen–Palm pseudo-momentum [36,37])

$$\alpha U \left( 2\Omega \cos\theta - \frac{\partial}{\partial\theta} \frac{1}{\cos\theta} \frac{\partial}{\partial\theta} U \cos\theta \right) = Q(\theta) - \frac{1}{\cos\theta} \frac{\partial}{\partial\theta} \cos\theta \langle v\omega^2 \rangle /2. \tag{4.2}$$

In the net enstrophy input rate, $Q(\theta) = \langle \omega \nabla \times \boldsymbol{f} \rangle - \nu \langle |\nabla\omega|^2 \rangle - \alpha \langle \omega^2 \rangle$, the first term can be fixed and known, as is the case of white in time forcing, for instance. However, the dissipation terms are determined by the nature of the inverse and direct cascades in an inhomogeneous flow, that is *a priori* $Q(r)$ is unknown as was $\epsilon(r)$ in §2. Fortunately, the physics of the direct cascade in two dimensions is much simpler, and there is a scale separation between the mean flow and the forcing scale, so one can make at least qualitative statements on the relation between $Q(\theta)$ and $U(\theta)$. In an infinite space or in a uniform flow, we expect $Q=0$ i.e. all the enstrophy is burnt by turbulence. The same is likely to be true in a uniform shear which is also a scale-free situation. It is only when the external flow has a finite scale $\ell$ that the energy absorption rate $\epsilon$ owing to the inverse cascade up to that scale is accompanied by the respective enstrophy absorption rate $\epsilon/\ell^2$ whose space average is $Q$. In other words, finite-scale mean flow can absorb some enstrophy.

Again, we assume that the enstrophy flux in space, $\langle v\omega^2 \rangle$, like $\langle uv \rangle$ and every moment odd in $v$, is non-zero only by virtue of friction which breaks time-reversibility, so it must go to zero as $\alpha \to 0$ and can be neglected in (4.2). Let us check if neglecting it in (4.2) can bring any inconsistencies. For that end, we substitute the solutions (3.12)–(3.14) into the left-hand side of (4.2) and see what $Q(\theta)$ it requires, assuming first $\Omega = 0$. Let us denote the angular momentum $M = U\cos\theta$ and $\lambda = \sin\theta$ and calculate the angular dependence of the turbulence–flow enstrophy transfer rate, respectively, for flows generated by constant, equator and mid-latitude sources of turbulence

$$-\alpha U \frac{\partial}{\partial\theta} \frac{1}{\cos\theta} \frac{\partial}{\partial\theta} U\cos\theta = -\alpha M \frac{\partial^2 M}{\partial\lambda^2} \propto \begin{cases} \tan^2\theta(1+2\cos^2\theta) & \text{for (3.12)} \\ \sin^2 2\theta & \text{for (3.13)} \\ \cos^2\theta(1-5\sin^2\theta)^2 & \text{for (3.14)} \end{cases} \tag{4.3}$$

Note first that it is everywhere non-negative as we expect from the net enstrophy input rate $Q$. Let us now address the uniformity of the approximation over $\theta$. One may think that the polar–vortex divergence at $\theta \to \pi/2$ in the first line invalidates the solution. On the contrary: as explained above, one expects $Q$ to be locally determined by the largest scale, which, in this case, is the distance to the vortex centre $R\cos\theta$, so that $Q(\theta) \simeq \epsilon(\theta)/\cos^2\theta$. Indeed, numerics done for the vortex show that the moments odd in $v$ are small and our approximation works quantitatively well. As one goes away from the pole, the typical scale increases and $Q$ must drop as, indeed, predicted by the first line of (4.3). On the equator, it predicts $Q \to 0$, as one would have in a constant shear in an infinite space. Because our sphere is finite, that could be that in reality $Q$ is finite everywhere including $\theta = 0$, so that our approximation breaks there. Existing data of DNS do not show any increase of fluctuations on the equator [33], further studies may be needed to resolve this issue. The same can be said about the second-line case with the only difference that it does not give any pole singularity, because the energy flux turns into zero there. The third line of (4.3) corresponds to a three-jet flow produced by a mid-latitude source of turbulence; here, we have zero at $\theta = \arctan\left(\frac{1}{2}\right)$, where our approximation likely breaks down and fluctuations are not small. Note also that the second derivative of momentum changes sign on the respective lines ($\theta = 0$ for the first two solutions and $\theta = \arctan\left(\frac{1}{2}\right)$ for the third one), so the flow alone would be unstable according to the Rayleigh criterion, yet turbulence may stabilize it.

Let us consider what changes are brought by rotation. Non-zero $\Omega$ and zero total momentum, $\int M(\theta)\,d\theta = 0$, require that the left-hand side of (4.2), $\alpha M(2\Omega - M'')$, changes sign. In the regions where $\alpha M(2\Omega - M'')$ is negative, in our approximation, the mean flow must transfer enstrophy to turbulence (while absorbing energy from it). This suggests that the approximation must break there.

Indeed, it is known from DNS that rotation plays quite a dramatic role as the number of easterly and westerly jets increases with $\Omega$ [33]. That does not mean that cubic term is always important, because multi-jet configurations were also obtained numerically solving the evolution equations on the mean flow and second moments within a quasi-linear approximation [6]; apparently, $Q$ must be negative for jets with $M < 0$, which are then likely to be more turbulent than jets with $M > 0$. The width of a jet can be hypothesized to be the Rhines scale $\ell = R\sqrt{U/\Omega R}$, where the jet velocity $U$ is of the order of the phase speed of Rossby wave [40]. Recall that the sign change of $2\Omega - M''$ corresponds to the so-called Rayleigh–Kuo necessary instability condition, and another way to define the Rhines scale is to require $\Omega \simeq M'' \simeq M/\ell^2$, i.e. that the flow is on the verge of stability.

Whether our approach can be generalized to describe analytically the multi-jet regime remains to be seen. Note, however, that the requirement $\ell < R$ bounds friction from below, so that a multi-jet state can be realized in the intermediate interval of friction values: $\epsilon^{1/3}R^{-2/3} > \alpha > \epsilon/\Omega^2R^2$. The ultimate state of the flow at $\alpha \to 0$ must be the solution of (3.6) and (3.10), which in addition to (3.5) requires

$$\alpha \ll \epsilon/\Omega^2 R^2. \tag{4.4}$$

It is equivalent to $U \gg \Omega R$—when the friction gets weaker under the constant energy input, the mean flow is getting so strong that planet rotation adds little.

It seems appropriate to finish this section with a general remark: fluid mechanics in curved spaces are far from being sufficiently explored. It is even unclear (at least to me) what such a basic fact as absence of Galilean invariance in this case means for turbulence statistics. Another interesting question is how turbulent inverse cascade looks on a curved surface which are much larger than its curvature radius (which can take place only for negative curvature). A somewhat mathematical but illuminating example is the hyperbolic plane which is an infinite surface of constant negative curvature; the inverse cascade there is predicted to create vortical rings of larger and larger diameters but of width comparable to the curvature radius [41], checking this prediction by DNS is a challenge.

## 5. Flows on a plane

Let us explore whether an inverse cascade on a plane can create a mean flow with straight streamlines (zonal flow) in the limit of vanishing friction. Such flows were observed, in particular, in a rectangular box with double-periodic boundary conditions (torus) [22]. We assume that flow is along $x$ and that $U, \tau, \epsilon, Q$ all depend only on $y$. Let us see if the zonal average (integration

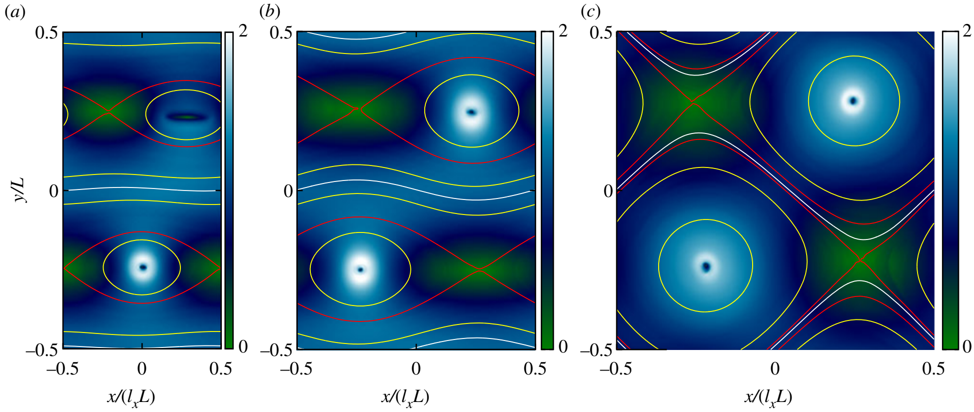


**Figure 1.** Coexistence of jets and vortices generated by an inverse turbulent cascade in double-periodic domains of different aspect ratios. Shown are the maps of the scaled speed ($\sqrt{\alpha/\epsilon}|\boldsymbol{v}|$) averaged over the mean flow turnover time $\tau_m = \sqrt{\alpha L^2/\epsilon}$ for three different aspect ratios: (*a*) $\frac{1}{2}$, (*b*) $\frac{3}{4}$ and (*c*) 1. Dimensionless friction rate is $\delta = 10^{-4}$. Overlaid are streamlines of the flow with red denoting the separatrices and white the $\psi = 0$ curve. Adapted from [42]. (Online version in colour.)

over $x$) can capture the main part of the flow. Upon such average, the energy balance has the form (2.4). Total zero momentum requires that the profile $U(y)$ changes sign and has extrema. Where $U' = 0$, the turbulence–flow energy exchange $U'\tau$ is zero and one cannot satisfy the local energy balance without requiring that the divergence of the turbulence flux is equal to $\epsilon$, i.e. finite and independent of $\alpha$. Let us show that fluctuations cannot be small also where $U = 0$. Consider the balance of enstrophy

$$-\langle v\omega\rangle U'' = -\alpha UU'' = Q - \partial_y\langle v\omega^2\rangle/2. \tag{5.1}$$

The viscous term was neglected in the vorticity balance equation but must be kept in $Q$. Similar to the energy balance, the left-hand side of (5.1), which is the enstrophy transfer rate from turbulence to the zonal flow, is independent of $\alpha$. However, it turns into zero on the lines where $U = 0$ and $U'' = 0$. There is no reason to expect that the net enstrophy input $Q$ disappears there, because it can be estimated by $\epsilon/L^2$, where $L$ is the jet size. That means that we cannot now neglect the turbulent enstrophy flux in space, $\langle v\omega^2\rangle$, as we did for a vortex. Combining insights from energy and enstrophy, we thus see that fluctuations must be strong both at the flow zeroes and maxima, i.e. everywhere. In other words, apart from the zonal flow, there must be some other strong jet-size coherent patterns pumped by an inverse cascade, which contribute energy and enstrophy fluxes but not the vorticity flux $\langle v\omega\rangle$ into a zonal average. Those coherent patterns were found to be vortices in DNS of an inverse cascade in a double-periodic domain [42]. Figure 1 shows that zonal flow appears when the aspect ratio differs from unity yet vortices remain. Further increasing the aspect ratio increases the number of vortices (2,3,4, etc., not shown). Because the vortices wander around, one can make a long-time average that gives unidirectional flow and satisfies (5.1). That, however, masks the true nature of the flow, which can be seen on any snapshot or average over medium time (larger than vortex turnover time but smaller than wandering time). Such medium-time mean flow consists of vortices embedded into jets; it is not translation invariant yet the fluctuations around that mean might be small at $\alpha \to 0$.

The findings of Frishman *et al.* [42] that strong coherent vortices always exist contradict those of Bouchet & Simonnet [22], where it was reported that vortex dipole existing for a square box is replaced by a unidirectional flow upon the change of the aspect ratio. While no details of the forcing spectra were given in [22], it is likely that the forcing scale was comparable to the box size that is no pronounced inverse cascade was present. The resulting mean flow may thus depend on the choice of forcing. On the contrary, the focus here and in [42] is on the universal limit of small-scale forcing and developed inverse cascade.

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Even though mean flows with straight lines cannot appear out of inverse cascade, they can be driven by external forces, for instance, in pipes and channels. Here, we can turn the tables and ask whether such a flow can produce turbulence without any small-scale forcing. The peculiarity of two dimensions is manifested no less dramatically in this case: contrary to three dimensions, not any large-scale flow can produce turbulence in two dimensions, no matter how small are viscosity and friction. Let us illustrate that by considering the simplest geometry of an infinite channel with two parallel walls placed at $y=\pm L$. Here, we can have either Couette flow set in motion by moving walls or Poiseuille flow driven by a pressure gradient. This could be relevant for thin fluid layers and soap-film flows. In the latter case, the linear friction could be made irrelevant, so we consider the frictionless case where dramatic difference from the three-dimensional case is most apparent. The extra conservation law of vorticity again plays prominent role. We first write the momentum balance expressing the divergence of the momentum flux via the vorticity flux:

$$-\partial_y\langle uv\rangle=\langle v\omega\rangle+\nu U''=\frac{\partial p}{\partial x}=-g. \tag{5.2}$$

Poiseuille flow is produced by a non-zero pressure gradient per unit mass $g$ (having dimensionality of acceleration), whereas Couette flow is produced by the motion of the walls and corresponds to $g=0$.

We now write the steady-state equation for the mean squared vorticity (enstrophy):

$$\langle v\omega\rangle\partial_y\Omega+\nu\langle|\nabla\omega|^2\rangle=\nu\Delta\langle\omega^2\rangle/2-\partial_y\langle v\omega^2\rangle/2 \tag{5.3}$$

and

$$\nu(\partial_y\Omega)^2+\nu\langle|\nabla\omega|^2\rangle=g\partial_y\Omega+\partial_y[\nu\partial_y\langle\omega^2\rangle-\langle v\omega^2\rangle]/2. \tag{5.4}$$

In the enstrophy balance (5.4), the left-hand side describes dissipation, the right-hand side describes production and flux. Integrating over $y$ from wall to wall, we see that the flux term turns into zero. Indeed, near the wall, $u$ turns into zero linearly, then incompressibility dictates that $v$ turns into zero quadratically, so that $\partial_y\omega$ must also turn into zero. We thus see that only the flow with a pressure gradient can sustain turbulence; two-dimensional plane Couette flow must have a linear profile with $\partial_y\Omega=0$ and no turbulence in a steady state at, however, small $\nu$. On the contrary, Poiseuille flow can generate enstrophy with the rate $-2gU'(L)/L=g^2/\nu$.

## 6. Describing turbulence statistics

In the previous sections, we have obtained a close description of a strong flow and weak fluctuations at the level of one-point averages by making an extra assumption on smallness of the $\langle vp\rangle$-term in the total energy balance. Without this extra assumption, one can use a regular perturbation theory to derive closed equations for the two-point correlation functions of turbulence. The equations are essentially equivalent to those written and computed numerically in [6] except that we consider scales exceeding the forcing correlation length $l_f$. Here, I describe briefly how this scheme works inside the pole vortex, where analytical approach seems to be feasible, see [43]. For example, assuming a constant velocity profile $U=\text{const.}$ and a steady state, $\partial_t\langle v(\boldsymbol{r}_1)v(\boldsymbol{r}_2)\rangle=0$ gives the following equation (after acting twice by Laplacian):

$$\begin{aligned} r_1^2\,\Delta_1\,r_1r_2^2\,\Delta_2\,(\langle v_1v_2\rangle-2\langle u_1u_2\rangle)+r_1^2\,\Delta_1\,r_2^2\,\Delta_2\,(r_1-r_2)r_1\partial_{r_1}\langle v_2v_1\rangle\\ +2r_2^2\,\Delta_2\,r_2r_1\partial_{r_1}\partial_{\phi_1}\langle v_2u_1\rangle+2r_1^2\,\Delta_1\,r_1(r_2\partial_{r_2})^2\langle u_1u_2\rangle=0. \end{aligned} \tag{6.1}$$

Expanding into Fourier modes $\langle v_1v_2\rangle=\sum_m V_m(r_1,r_2)\mathrm{e}^{im(\phi_1-\phi_2)}$, using incompressibility and changing variables to $x=\ln r_1$ and $y=\ln r_2$, one obtains a beautiful equation

$$(\partial_x^2-m^2)\partial_x(\partial_y^2+2\partial_y+2-m^2)\mathrm{e}^xV_m(x,y)=(\partial_y^2-m^2)\partial_x(\partial_x^2+2\partial_x+2-m^2)\mathrm{e}^yV_m(x,y).$$

The analysis of the solutions can be found in [43]. For $r_2 > r_1$, one defines $\lambda_m = \sqrt{m^2-1}$ and obtains

$$\langle v_1 v_2\rangle = A_1\cos\phi_{12} - B_1\sin\phi_{12} + \sum_{m>1} A_m\left(\frac{r_1}{r_2}\right)^{\lambda_m}\left(\frac{1+2\lambda_m}{r_1}+\frac{1-2\lambda_m}{r_2}\right)\cos m\phi_{12},$$

and the expression with $1\leftrightarrow 2$ for $r_1 > r_2$. The constants $A_1, B_1, A_m$ are undetermined in this approach. Note that if the constant $B_1$ is non-zero then there is a jump at $r_1 = r_2$ and $\phi_{12} = \phi_1 - \phi_2 \neq 0$. Such solutions can describe the structure functions of turbulence at large enough distances $r_{12}$, where the turbulent self-interaction is weaker than the shear stretching by the mean flow. When two points are very close, one must account for cubic terms, which is needed, in particular, to give the dependence of turbulence level on the coordinates. The constants then are determined by matching different limits.

The consideration of the opposite limit of short distances down to coinciding points can be obtained for the $m=0$ mode, which is non-zero only for $\langle u_1u_2\rangle$ owing to incompressibility. Integrating the equation for $\partial_t\langle u_1u_2\rangle$ over $\phi_1$, we see that all the main-order terms cancel and the second moment is related to the integral of the third one [43]

$$\int \mathrm{d}\phi_1\left(r_1^{-2}\partial_{r_1}\left(r_1^2\langle v_1u_1u_2\rangle\right)+r_2^{-2}\partial_{r_2}\left(r_2^2\langle v_2u_1u_2\rangle\right)\right) = -2\alpha\langle u_0(r_1)u_0(r_2)\rangle. \tag{6.2}$$

Here, $u_0(r_1)$ is the zeroth Fourier mode of $u(r,\phi)$, and we used the fact that $\int v\,\mathrm{d}\phi = 0$ owing to incompressibility and the homogeneity in the $\phi$-direction. We know the third moment for the distances small enough, where we can use the Kolmogorov flux law, $-\langle (v_1^l\nabla_1^l v_1)_\phi u_2\rangle = \epsilon/2$. This estimate works up to the distance where the rate of shearing by the mean flow is equal to the eddy turn over time: $U/r_1 = \epsilon^{1/3}(|\delta\boldsymbol{\rho}|)^{-2/3}$, using the result $U=\sqrt{\epsilon/\alpha}$, we arrive at the estimate $|\delta\boldsymbol{\rho}| = \epsilon^{-1/4}r_1^{3/2}\alpha^{3/4}$. Because $\Delta r = r_2 - r_1$ and $\phi_{12}$ are assumed small, we can express as $|\delta\boldsymbol{\rho}| = \sqrt{(\Delta r)^2 + r_1^2\phi_{12}^2}$. In particular, it is necessary to assume that $l_f \ll \Delta r \ll \alpha^{3/4}r_1^{3/2}\epsilon^{-1/4}$, so that the limit on $\phi_{12}$ is as follows

$$\phi_{12} < \frac{\alpha^{3/4}\sqrt{r_1}}{\epsilon^{1/4}} - \frac{1}{2}\frac{(\Delta r)^2\epsilon^{1/2}}{\alpha^{3/2}r_1^3}. \tag{6.3}$$

For larger distances, turbulence is suppressed by the mean flow, so we expect the correlation function to decay rapidly. Thus, we can estimate the leading contribution to the integral in (6.2) using the estimate for the range $\phi_{12}$. For $\Delta r \ll \alpha^{3/4}r_1^{3/2}\epsilon^{-1/4}$, we obtain following [43]

$$\langle u_0(r_1)u_0(r_2)\rangle \simeq \frac{\sqrt{r_1}\epsilon^{3/4}}{2\alpha^{1/4}} - \frac{1}{4}\frac{(\Delta r)^2\epsilon^{3/2}}{\alpha^{5/2}r_1^3}. \tag{6.4}$$

In particular, for $r_1 = r_2 = r$, we get the energy in the zeroth harmonic is

$$\langle u_0(r)^2\rangle = \frac{\sqrt{r}\epsilon^{3/4}}{2\alpha^{1/4}}. \tag{6.5}$$

Unlike the previous results based on spatial fluxes, this formula is an expression of the spectral flux of energy, i.e. of the small-scale physics. This prediction needs to be compared with the DNS.

At very small scales, weak distortions of isotropic homogeneous turbulence can possibly be described by the complementary approach, built on the very general field-theoretical formalism of OPE adapted for turbulence in [4]. That approach presumes that for any anisotropic inhomogeneous developed turbulence the structure functions at small scales must be the same as in isotropic turbulence, only expectation values differ. That approach allows one to reduce, in principle, any multi-point correlation function to the series of single-point expectation values multiplied by the universal coordinate-dependent functions. Specifically, for any fluctuating quantities $O_i$, we have (in the plane case)

$$\langle O_i(\mathbf{r}_1)O_j(\mathbf{r}_2)\rangle = \sum \langle O_k((y_1+y_2)/2)\rangle C^k_{ij}(\mathbf{r}_{12}). \tag{6.6}$$

Here, the structure functions $C^k_{ij}(\mathbf{r}_{12})$ are those of isotropic homogeneous turbulence while the expectation values depend on $y$, the form of the large-scale flow, boundary conditions, etc. In particular, the symmetry $C^k_{ij}(\mathbf{r}_{12}) = C^k_{ji}(\mathbf{r}_{21}) = C^k_{ji}(\mathbf{r}_{12}) =$ means that $\langle O_i(\mathbf{r}_1)O_j(\mathbf{r}_2)\rangle = \langle O_j(\mathbf{r}_1)O_i(\mathbf{r}_2)\rangle$ in any turbulence. The approach presents a general scheme of building such an expansion proceeding from the symmetries of the system.

For two velocities in arbitrary dimensionality $d$, the OPE terms with the lowest powers of $r_{12}$ must look as follows [4]

$$v^i(x_1)v^j(x_2) = v^i v^j + D^{ij}(x_{12})\epsilon_2 + \frac{x^k_{12}}{2}\left[\nabla_k(v^i v^j)\left(v^i\nabla_k v^j - v^j\nabla_k v^i\right)\right] + G^{ij}_{kl}(x_{12})\left(\nabla_k v^l + \nabla_l v^k\right). \tag{6.7}$$

where all fields $v$ and $\epsilon_2$ in the r.h.s. are taken at the point $x_2$, and we omitted terms that involve $x_{12}$ in powers higher than 4/3. Here, the structure functions are

$$D^{ij}(x) = \left[\delta^{ij} - \frac{\zeta_2}{\zeta_2+d-1}\frac{x^i x^j}{x^2}\right]|x_{12}|^{\zeta_2} \tag{6.8}$$

and

$$\begin{aligned} G^{ij}_{kl}(x) = B(x^2)^{a-2}\Big\{&\left[2\delta^{ij}\delta^{kl} - \delta^{jk}\delta^{il} - \delta^{ik}\delta^{jl}\right]x^4 \\ &+ \left(\tfrac{4}{3}\right)\left[(d+1+4/3)\delta^{ij}x^k x^l - \delta^{jk}x^i x^l - \delta^{jl}x^i x^k - \delta^{ik}x^j x^l - \delta^{jl}x^i x^k\right]x^2 + \left(\tfrac{8}{9}\right)x^i x^j x^k x^l\Big\}, \end{aligned} \tag{6.9}$$

In particular, that means that

$$\begin{aligned} \langle u_2 v_3\rangle = \langle u_2 v_2\rangle - \epsilon_2(y_2)x_{23}y_{23}r_{23}^{\zeta_2-2} + y_{23}\partial_{y_2}\langle u_2 v_2\rangle \\ - c_2 U'(y_2)\left[1 + 4x_{23}y_{23}r_{23}^{-2}/3 - 8x_{23}^2 y_{23}^2 r_{23}^{-4}\right]r_{23}^{4/3}. \end{aligned} \tag{6.10}$$

It is important to stress that this approach in general and (6.10) in particular are, at this point, nothing but a bold assumption. Such predictions must be confronted with experiments and DNS

## 7. Perspectives

To conclude, two directions of analytic theory of the interaction of developed turbulence with a mean flow have been briefly described. It is reassuring that one can obtain simple analytic solutions, such as (3.12)–(3.14), where only numerics existed before. The relation of those solutions to reality remains to be clarified, but their very existence provides for a good starting point. One apparent direction is to try to generalize the present approach, so that instead of two entities (flow of a simple geometry and turbulence whose statistics has the same symmetries) it may contain three. For example, coexistence of jets, coherent vortices and turbulence in a strip as described in §5. Another case of three entities comprises global flow, set of Rossby waves and solenoidal turbulence on a rotating sphere or beta-plane, as discussed in §4. Moving into three dimensions, the theory natural development are the cases where inverse cascade has been observed: thick fluid layers [31] and rapidly rotating columns [44], both forced at small scales.

I believe that further analytic progress is definitely possible in two dimensions, and it also may inspire new ideas for the description of turbulence–flow interaction in three dimensions and in less idealized settings.

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